Be sure this exam has 16 pages including the cover.

The University of British Columbia

MATH 400, Section 101, 2018-2019

Final Exam

2.5 Hours

Name ______________________ Signature ______________________

Student Number ______________________ Section ______________________

This exam consists of 10 questions. No notes. No calculators are allowed. A list of formulas is provided on the last page. Write your answer in the blank page provided.

Problem	max score	score
1.	10	
2.	20	
3.	14	
4.	6	
5.	6	
6.	8	
7.	10	
8.	6	
9.	10	
10.	10	
total	100	

(10 points) 1. This question contains two parts.

(5 points) (a) Find the general solutions to

$$u_x + 2xu_y = u.$$

(5 points) (b) Find the function $h(x)$ so that the following problem admits a solution:

$$u_x + 2xu_y = u,$$
$$u(x, x^2) = h(x), \ -\infty < x < +\infty.$$

(a) $\frac{dx}{1} = \frac{dy}{2x} \Rightarrow 2x\,dx = dy \Rightarrow x^2 - y = \xi$

characteristics

Let $x' = x$, $y' = x^2 - y$, $u = U$

Then $U_{x'} = U \Rightarrow U = f(y')e^{x'}$

$u = f(x^2 - y)e^x$

(b) From (a), $u = f(x^2 - y)e^x$

$u(x, x^2) = f(x^2 - x^2)e^x = f(0)e^x$

So $h(x)$ must be of the form

$h(x) = Be^x$

(20 points) 2. Solve the following quasi-linear partial differential equation

$$u_t + (1-u)u_x = 0, \ -\infty < x < +\infty, t > 0,$$

$$u(x,0) = \frac{1}{2}, \ -\infty < x < +\infty,$$

$$u(0-,t) = 2, \ u(0+,t) = \frac{3}{4}, \ t > 0.$$

Sol'n. $u(x,0) = \frac{1}{2} \Rightarrow \begin{cases} \frac{dt}{ds} = 1, & t(0) = 0 \\ \frac{dx}{ds} = 1-u, & x(0) = \xi \\ \frac{du}{ds} = 0, & u(0) = \frac{1}{2} \end{cases}$

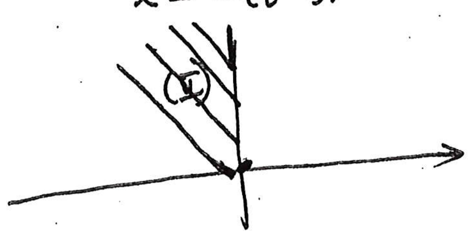
$\Rightarrow \quad x = (1-\frac{1}{2})s + \xi \quad \Rightarrow \quad x = \frac{t}{2} + \xi$

$t = s$

(I).

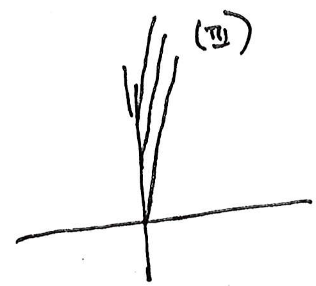
$u(0-,t) = 2, \Rightarrow \begin{cases} \frac{dt}{ds} = 1, & t(0) = \xi \\ \frac{dx}{ds} = 1-u, & x(0) = 0 \\ \frac{du}{ds} = 0, & u(0) = 2 \end{cases}$ $\Rightarrow t = s + \xi$, $\Rightarrow x = (1-2)s = -s$

$x = -(t-\xi) = -t + \xi, \quad x < 0$



$u(0+,t) = \frac{3}{4} \Rightarrow \begin{cases} \frac{dt}{ds} = 1, & t(0) = \xi \\ \frac{dx}{ds} = 1-u, & x(0) = 0 \\ \frac{du}{ds} = 0, & u(0) = \frac{3}{4} \end{cases}$ $\Rightarrow t = s + \xi$, $\Rightarrow x = \frac{1}{4}s$

$x = \frac{1}{4}(t-\xi), \quad x > 0.$ x



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Between (I) & (II), a shock curve

$$\frac{ds}{dt} = \frac{Q^+ - Q^-}{u^+ - u^-} = \frac{u^+ - \frac{1}{2}(u^+)^2 - (u^- - \frac{1}{2}(u^-)^2)}{u^+ - u^-} = 1 - \frac{1}{2}(u^+ + u^-)$$

$$= 1 - \frac{1}{2}\left(\frac{1}{2} + 2\right) = -\frac{1}{4}$$

$$s(0) = 0$$

$$s = -\frac{1}{4}t$$

Between (II) & (III), an expansion fan:

$$1 - U(\Lambda) = \Lambda \Rightarrow U(\Lambda) = 1 - \Lambda = 1 - \frac{x}{t}$$

$$u(x,t) = \begin{cases} \frac{1}{2}, & x < -\frac{1}{4}t \\ 2, & -\frac{t}{4} < x < 0 \\ \frac{3}{4}, & 0 < x < \frac{1}{4}t \\ 1 - \frac{x}{t} & \frac{t}{4} < x < \frac{t}{2} \\ \frac{1}{2} & x > \frac{t}{2} \end{cases}$$

(14 points) 3. Solve the following fully nonlinear partial differential equations

$$u_x u_y - u = 0,$$

$$u(x, -x) = 1, \quad -\infty < x < +\infty.$$

Hint: You may use the Charpit's formula listed on the last page.

Sol'n: $F = pq - u$

$(x_0(\xi), y_0(\xi), u_0(\xi)) = (\xi, -\xi, 1)$

$p_0 q_0 - \cancel{1_0} = 0$

$u_0' = 0 = p_0 x_0' + q_0 y_0' = p_0 - q_0$

$p_0^2 = 1 \Rightarrow p_0 = 1$ or -1, $q_0 = 1$ or -1

Let $p_0 = q_0 = 1$. Then

$\frac{dx}{ds} = q, \quad x(0) = \xi \quad \Rightarrow x = e^s + \xi - 1$

$\frac{dy}{ds} = p, \quad y(0) = -\xi \quad \Rightarrow y = e^s - \xi - 1$

$\frac{dp}{ds} = p, \quad p(0) = 1 \quad \Rightarrow p = e^s$

$\frac{dq}{ds} = q, \quad q(0) = 1 \quad \Rightarrow q = e^s$

$\frac{du}{ds} = 2pq, \quad u(0) = 1, \quad u = e^{2s}$

$\left.\begin{array}{l} x = e^s + \xi - 1 \\ y = e^s - \xi - 1 \end{array}\right] \Rightarrow 2e^s - 2 = x + y$

$e^s = \frac{1}{2}(x+y) + 1$

$$u = \left(\frac{1}{2}(x+y) + 1\right)^2$$

(6 points) 4. Solve the following heat equation

$$4u_t = u_{xx}, -\infty < x < +\infty, t > 0,$$

$$u(x,0) = e^{-2x}, -\infty < x < +\infty.$$

Hint: the source function is given by

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

Sol'n: $k = \frac{1}{4}$.

$$u(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{t} - 2y} \, dy.$$

$$\frac{(x-y)^2}{t} + 2y = \frac{(x-y)^2 + 2ty}{t} = \frac{y^2 + (2t - 2x)y + (t-x)^2 - (t-x)^2 + x^2}{t}$$

$$= \frac{(y+t-x)^2}{t} - \frac{(t-x)^2 - x^2}{t}$$

$$u = \frac{1}{\sqrt{\pi t}} \int e^{-\frac{(y+t-x)^2}{t}} dy \cdot e^{-\frac{-2tx+t^2}{t}}$$

$$= e^{-2x-t}$$

(6 points) 4. Solve the following heat equation

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$$\frac{(x-y)^2}{t} + 2y = \frac{(x-y)^2 + 2ty}{t} = \frac{y^2 + (2t-2x)y + (t-x)^2 - (t}{t}$$

$$= \frac{(y+t-x)^2}{t} - \frac{(t-x)^2 - x^2}{t}$$

$$u = \frac{1}{\sqrt{\pi t}} \int e^{-\frac{(y+t-x)^2}{t}} dy \cdot e^{-\frac{-2tx+t^2}{t}}$$

$$= e^{-2x-t}$$

(8 points) 6. This problem contains two parts.

(4 points) (a) Write the following eigenvalue problem into a standard Sturm-Liouville eigenvalue problem

$$X'' + 2xX' - xX + \lambda X = 0.$$

(4 points) (b) Consider the following eigenvalue problem

$$X'' + \lambda X = 0, 0 < x < 2,$$

$$X'(0) + 3X(0) = 0, X'(2) + 2X(2) = 0.$$

Determine the number of negative eigenvalues and write down the algebraic equation for the negative eigenvalues.

Hint: You can use the formula sheet at the end of the exam paper.

(a)

$$\left.\begin{array}{l} p = \mu \\ p' = \mu 2x \end{array}\right\} \Rightarrow \mu = p = e^{x^2}$$

$q = -x\mu$

$w = \mu$

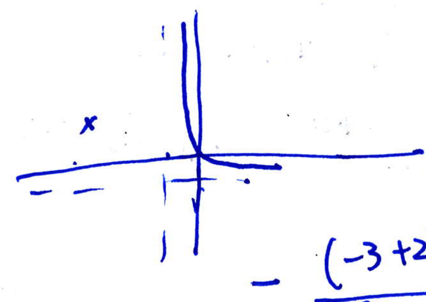
$$(e^{x^2} X')' - x e^{x^2} X + \lambda e^{x^2} X = 0.$$

(b). $a_0 = -3$, $a_l = 2$, $l = 2$

$$a_0 + a_l + a_0 \cdot a_l \cdot l = -3 + 2 + (-3)\cdot 2 \cdot 2 < 0$$

Region IV: one negative

all positive



$$\tanh(2\gamma_{-1}) = -\frac{(-3+2)\gamma_{-1}}{\gamma_{-1}^2 - 6} = \frac{\gamma_{-1}}{\gamma_{-1}^2 - 6}$$

(10 points) 7. Solve the following diffusion equation with source

$$u_t = u_{xx} + 3e^{-t}\sin(2x), 0 < x < \pi, t > 0,$$

$$u(0,t) = \pi,\ u(\pi,t) = 0,$$

$$u(x,0) = 0,\ 0 < x < \pi.$$

Hint: you may use the list of formula attached on the last page.

Sol'n: $f = 3e^{-t}\sin 2x$, $l = \pi$, $k=1$, $j(t) = \pi$, $g(t) = 0$

$u = \sum_{n=1}^{+\infty} u_n(t)\sin(nx)$, $f = \sum_{n=1}^{+\infty} f_n(t)\sin nx \Rightarrow \begin{cases} f_n = 0 \text{ fo} \\ n \\ f_2 = 3e^{-t} \end{cases}$

$$\begin{cases} u_n' + n^2 u_n = \frac{2n}{\pi}(\pi - 0) + f_n(t) \\ u_n(0) = 0 \end{cases}$$

$n \neq 2 \Rightarrow f_n = 0 \Rightarrow \begin{cases} u_n' + n^2 u_n = 2n \\ u_n(0) = 0 \end{cases}$

$$u_n = \frac{2}{n} - \frac{2}{n} e^{-n^2 t}$$

$n = 2 \Rightarrow f_2 = 3e^{-t} \Rightarrow \begin{cases} u_2' + 4u_2 = 4 + 3e^{-t} \\ u_2(0) = 0 \end{cases}$

$u_2 = 1 + e^{-t} + ce^{-4t} \Rightarrow u_2 = 1 + e^{-t} - 2e^{-4t}$

Hence

$$u = \sum_{n\neq 2}\left(\frac{2}{n} - \frac{2}{n}e^{-n^2 t}\right)\sin nx + \left(1 + e^{-t} - 2e^{-4t}\right)\sin 2x$$

(6 points) 8. Consider the following Laplace equation

$$\Delta u = f \text{ in } \Omega$$

$$u = g \text{ on } \partial\Omega$$

where Ω is a smooth and bounded domain in $\mathbb{R}^2$.

Show that the solution is unique.

sol'n: Let u_1, u_2 be two sol'ns

$$\begin{cases} \Delta u_1 = f & \text{in } \Omega \\ u_1 = g & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta u_2 = f & \text{in } \Omega \\ u_2 = g & \text{on } \partial\Omega \end{cases}$$

Let $w = u_1 - u_2$. Then

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}.$$

By Divergence Theorem,

$$0 = \int_\Omega w \Delta w = \int_{\partial\Omega} w \frac{\partial w}{\partial n} - \int_\Omega |\nabla w|^2$$

$$\Rightarrow \int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w \frac{\partial w}{\partial n} = 0.$$

$\Rightarrow$ $w \equiv$ Constant

$\Rightarrow$ $w \equiv 0$ since $w = 0$ on $\partial\Omega$

So $u_1 \equiv u_2$. The sol'n is unique.

(10 points) 9. Use the method of separation of variables to solve the following Laplace equation

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2} = 0, 1 < r < 2, 0 < \theta < \frac{\pi}{2},$$

$$u(1,\theta) = 2\cos(\theta),\ u(2,\theta) = 0,\ 0 < \theta < \frac{\pi}{2},$$

$$u_\theta(r,0) = 0,\ u(r,\frac{\pi}{2}) = 0,\ 1 < r < 2.$$

Sol'n: $u = R(r)\,\Theta(\theta)$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0$$

$$\Theta'' + \lambda\Theta = 0, \quad \Theta'(0) = 0, \quad \Theta(\tfrac{\pi}{2}) = 0$$

$\lambda = \beta^2$, $\Theta = \cos\beta\theta$, $\cos(\frac{\pi}{2}\beta) = 0 \Rightarrow \frac{\pi}{2}\beta = \frac{\pi}{2}(2n-1)$

$$\beta = (2n-1), \quad n = 1, \cdots$$

$$\lambda = (2n-1)^2$$

$$R = a\,r^{2n-1} + b\,r^{-(2n-1)}$$

$$u = \sum_{n=1}^{+\infty} \left(a_n r^{2n-1} + b_n r^{-(2n-1)}\right)\cos((2n-1)\theta)$$

$$u(1,\theta) = 2\cos\theta = \sum_{n=1}^{+\infty} (a_n + b_n)\cos((2n-1)\theta)$$

$$u(2,\theta) = 0 = \sum_{n=1}^{+\infty} \left(a_n 2^{2n-1} + b_n 2^{-(2n-1)}\right)\cos((2n-1)\theta)$$

So $a_n, b_n = 0$, for $n \geq 2$

$$\left.\begin{array}{l} a_1 + b_1 = 2 \\ a_1 2 + b_1 2^{-1} = 0 \end{array}\right\} \quad a_1 = -\frac{2}{3}, \quad b_1 = \frac{8}{3}$$

$$u(r,\theta) = \left(-\frac{2}{3}r + \frac{8}{3}r^{-1}\right)\cos\theta$$

(10 points) 10. Use the method of separation of variables to solve the following diffusion equation

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2},\ 0 \le r < 1,\ 0 \le \theta < 2\pi,$$

$$u(1, \theta, t) = 0,\ 0 \le \theta < 2\pi,$$

$$u(r, \theta, 0) = r^2 \sin 2\theta,\ 0 \le r < 1,\ 0 \le \theta < 2\pi.$$

Write your answer in terms of the Bessel function of order n:

$$J_n'' + \frac{1}{r}J_n' - \frac{n^2}{r^2}J_n + J_n = 0,\ J_n(r) \sim r^n \text{ as } r \to 0.$$

The zeroes of $J_n(r)$ are denoted as $z_{m,n}$, $m = 1, ..., +\infty$.

Sol'n:

$$u = R(r)\,\Theta(\theta)\,T(t)$$

$$R'' + \frac{1}{r}R' + \frac{\Theta''}{r^2\Theta}R = \frac{T'}{T}$$

$$\Theta'' + \lambda_1\Theta = 0,\quad \frac{T'}{T} = -\lambda_2. \qquad \lambda_1 = n^2.$$

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R + \lambda_2 R = 0.$$

$$\lambda_2 = z_{m,n}^2, \quad R = J_n(\sqrt{\lambda_2}\, r)$$

$$u = \sum_{m,n} (a_{m,n}\cos n\theta + b_{m,n}\sin n\theta)\, J_n(z_{m,n} r)\, e^{-z_{m,n}^2 t}$$

Now, $u(r,\theta,0) = r^2\sin 2\theta$

$$r^2 \sin 2\theta = \sum_{m,n} (a_{m,n}\cos n\theta + b_{m,n}\sin n\theta)\, J_n(z_{m,n} r)$$

$n \neq 2$, $a_{m,n}, b_{m,n} = 0$,

$n = 2$, $a_{m,n} = 0$, $\quad r^2 = \sum_m b_{m,2} J_2(z_{m,2} r)$.

$$b_{m,2} = \frac{\int_0^1 r^3 J_2(z_{m,n} r)\,dr}{\int_0^1 r J_2^2(z_{m,2} r)\,dr}$$

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so

$$u = \sum_{m=1}^{+\infty} b_{m,2} \sin 2\theta \, J_2(z_{m,2} r) e^{-z_{m,2}^2 t}$$

where

$$b_{m,2} = \frac{\int_0^1 r^3 J_2(z_{m,2} r) dr}{\int_0^1 r J_2^2(z_{m,2} r) dr}$$