MATH400-101-2020 Homework Assignment 5 (Due Date: November 6th, 2019, by 6pm)

Homework is admitted until 6pm on **November 6th**, 2019. Graded homework is placed in a cardboard box outside my office for one week for you to pick up. Afterwards unclaimed homework is moved to a drawer of a file cabinet near my office. Your assignments are organized in the alphabetic order of last names. For other people's convenience, please do not change this order when you pick up your assignment.

1. (10pts) Find a solution formula for

$$u_t = u_{xx} + u + f(x,t), t > 0$$
$$u(x,0) = \phi(x)$$

Hint: Consider $u(x,t) = e^t v(x,t)$.

2. (10pts) Consider the following diffusion equation

$$u_t = u_{xx} - e^{-x}, -\infty < x < +\infty, t > 0$$
$$u(x,0) = 0$$

(a) Use the solution formula to find a solution.

(b) Find a solution $\psi(x)$ of $\psi_{xx} - e^{-x} = 0$. Let $u(x,t) = \psi(x) + v(x,t)$ and then find a solution to $v(x,t)$.

3. (10pts) (a) Solve the following diffusion equation by the method of extension:

$$u_t = k u_{xx}, x > 0, t > 0$$
$$u(0,t) = -t$$
$$u(x,0) = 0$$

Hint: let $u(x,t) = -t + v(x,t)$. **Write the solution in terms of integrals of $\mathcal{E}rf$ function.**

(b) Use the energy method to show that the solution to (a) is unique, assuming that the solution has fast decay.

4. (10pts) Solve the following diffusion equation

$$\begin{cases} u_t = 2u_{xx}, 0 < x < \pi \\ u(x,0) = \sin(x)\cos(x) + 2\sin(20x) \\ u(0,t) = 0, u(\pi,t) = 0 \end{cases} \tag{1}$$

5. (20pts) (a) Find the eigenvalues and eigenfunctions of

$$X^{''} + \lambda X = 0, 0 < x < l, \ X^{'}(0) = 0, X(l) = 0$$

(b) Solve the following wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx}, 0 < x < 1 \\ u(x,0) = 2\cos(\frac{3\pi}{2}x), u_t(x,0) = \cos(\frac{\pi}{2}x) \\ u_x(0,t) = 0, u(1,t) = 0 \end{cases} \tag{2}$$

6. (20pts) (a) Solve

$$\begin{cases} u_t - u_{xx} = 0, 0 < x < 1, \\ u(x,0) = \phi(x), 0 < x < 1 \\ u_x(0,t) + 2u(0,t) = 0, u_x(1,t) + u(1,t) = 0 \end{cases} \tag{3}$$

by separation of variables. (b) Under what conditions on $\phi(x)$, does the solution to (3) remain bounded as $t \to +\infty$?

7. (10pts) For the following eigenvalue problems, find out :(1) how many negative eigenvalues there are (2) the algebraic equations for all positive, zero and negative eigenvalues (c) the corresponding eigenfunctions

$$(a) \quad X^{''} + \lambda X = 0, 0 < x < \frac{1}{2}, \quad 2X^{'}(0) + X(0) = 0, \quad X^{'}(\frac{1}{2}) + X(\frac{1}{2}) = 0;$$
$$(b) \quad X^{''} + \lambda X = 0, 0 < x < 1, \quad X^{'}(0) + 2X(0) = 0, \quad X^{'}(1) - 2X(1) = 0$$

$$(c) \quad X^{''} + \lambda X = 0, 0 < x < 2, \quad X^{'}(0) + 3X(0) = 0, \quad X^{'}(2) - 2X(2) = 0$$

8. (10pts) For the following eigenvalue problems, transform it into standard Sturm-Liouville eigenvalue problem as

$$(p(x)X^{'})^{'} - q(x)X(x) + \lambda w(x)X(x) = 0$$

$$(a) \quad X^{''} + x^2 X^{'} + \lambda X = 0, \quad (b) \quad X^{''} + \frac{1}{x}X^{'} - xX + \lambda X = 0$$