TRANSFORMS: BRIEF OUTLINE

FOURIER TRANSFORMS ARE USUAL FOR CONSTANT COEFFICIENT PDE'S ON INFINITE RANGE OF THE INDEPENDENT VARIABLE.

ASSUME THAT $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$ AND $f(x)$ PIECEWISE CONTINUOUS. THEN THE FOURIER TRANSFORM PAIR IS

$$\hat{F}(k) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad f(x) = \mathcal{F}^{-1}(\hat{F}(k)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(k) e^{ikx} dk$$

THE CALCULATION OF FOURIER TRANSFORMS TYPICALLY INVOLVES RESIDUE THEORY. WE RECALL SOME BASIC PROPERTIES:

(i) $\mathcal{F}(f^{(n)}(x)) = (ik)^n \mathcal{F}(f(x))$ WITH $f^{(n)}(x) = n^{th}$ derivative, AND $f^{(n)} \to 0$ AS $|x| \to \infty$

(ii) $\mathcal{F}(f(x/b)) = b \hat{F}(bk) \longrightarrow \mathcal{F}^{-1}[\hat{F}(b/k)] = \frac{1}{b} f(x/b)$

(iii) $\mathcal{F}\left(\frac{1}{1+x^2}\right) = \pi e^{-|k|}$; $\mathcal{F}(e^{-|x|}) = \frac{2}{1+k^2}$

(iv) $\mathcal{F}(\delta(x)) = 1$; $\mathcal{F}\left(e^{-x^2/2\sigma^2}\right) = \sqrt{2\pi}\, \sigma\, e^{-\sigma^2 k^2/2}$

(v) **CONVOLUTION**: $\mathcal{F}(f * g) = \hat{G}(k) \hat{F}(k)$ $\quad \hat{G}(k) = \mathcal{F}(g(x)), \ \hat{F}(k) = \mathcal{F}(f(x))$.

WHERE $f * g = \int_{-\infty}^{\infty} f(x-x') g(x') dx' = \int_{-\infty}^{\infty} f(x') g(x-x') dx'$

(vi) $\mathcal{F}(f(x+a)) = e^{ika} \mathcal{F}(f(x))$

EXAMPLES CALCULATE THE FOLLOWING:

(a) $\mathcal{F}(e^{-b|x|})$. WE USE (ii) AND (iii) RECALL $\mathcal{F}(e^{-|x|}) = 2/1+k^2 = \hat{F}(k)$

$$\mathcal{F}(e^{-b|x|}) = \frac{1}{b} \hat{F}(k/b) = \frac{1}{b} \frac{2}{1+k^2/b^2} = \frac{2b}{k^2+b^2}.$$

(b) $\mathcal{F}\left(\frac{1}{\omega^2+x^2}\right) = \frac{1}{\omega^2} \mathcal{F}\left(\frac{1}{1+x^2/\omega^2}\right) = \frac{1}{\omega^2}\left(\omega \hat{F}(\omega k)\right)$ WITH $\hat{F}(k) = \pi e^{-|k|}$

HENCE $\mathcal{F}\left(\frac{1}{\omega^2+x^2}\right) = \frac{\pi}{\omega} e^{-\omega|k|}$. HERE WE USED (ii) AND (iii)

(c) $\mathcal{F}^{-1}\left[\frac{1}{K^2+w^2}\right]$.

NOW $\mathcal{F}^{-1}\left(\frac{1}{K^2+w^2}\right) = \frac{1}{2w^2}\mathcal{F}^{-1}\left(\frac{2}{1+K^2/w^2}\right) = \frac{1}{2w^2} \; w \, f(wx) = \frac{1}{2w} e^{-w|x|}$.

PROBLEM 1 SOLVE FOR $u(x)$ IN

$$u'' - w^2 u = -f(x), \quad -\infty < x < \infty$$

$$u(\pm\infty) : \text{ BOUNDED} \quad \text{AND} \quad \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

THEN TAKE FOURIER TRANSFORMS:

$$\mathcal{F}(u'') - w^2 \mathcal{F}(u) = -\hat{F}(K) \longrightarrow (iK)^2 \mathcal{F}(u) - w^2 \mathcal{F}(u) = -\hat{F}(K)$$

LET $\hat{u}(K) = \mathcal{F}(u(x))$. THEN

$$\hat{u}(K) = \hat{F}(K)\,\hat{G}(K) \qquad \hat{G}(K) = (K^2+w^2)^{-1}.$$

BUT $g(x) = \mathcal{F}^{-1}\left(\frac{1}{K^2+w^2}\right) = \frac{1}{2w} e^{-w|x|}$ BY (c) ABOVE.

SO BY CONVOLUTION PROPERTY (V),

$$u(x) = \frac{1}{2w}\int_{-\infty}^{\infty} e^{-w|x-s|} f(s)\, ds.$$

PROBLEM 2 SOLVE THE HEAT EQUATION

$$u_t = D\, u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x) \qquad u \text{ BOUNDED AS } |x| \to \infty$$ (3)

THEN LET $\bar{u}(k,t) = \mathcal{F}(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx.$

WE OBTAIN

$$\mathcal{F}(u_t) = D\, \mathcal{F}(u_{xx})$$

SO $\bar{u}_t = D(ik)^2 \bar{u}$

$$\begin{cases} \bar{u}_t = -Dk^2 \bar{u} \\ \bar{u}(k,0) = \hat{F}(k) \end{cases} \qquad \hat{F}(k) = \mathcal{F}(f(x))$$

THEN

$$\bar{u}(k,t) = \hat{F}(k)\, e^{-Dk^2 t}$$

NOW USE CONVOLUTION.

RECALL $\mathcal{F}^{-1}\left[e^{-\sigma^2 k^2/2}\right] = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2}$ BY PROPERTY (iii)

CHOOSE $\sigma^2/2 = Dt \rightarrow \sigma = \sqrt{2Dt}$

$$\mathcal{F}^{-1}\left[e^{-Dk^2 t}\right] = \frac{1}{2\sqrt{D\pi t}}\, e^{-x^2/4Dt}$$

THEN USING CONVOLUTION,

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4Dt}\, f(s)\, ds.$$

IF $f(s) = \delta(s)$ DIRAC MASS, THEN $u(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}$

PROBLEM 3 SOLVE LAPLACE'S EQUATION IN A 1/2 PLANE GIVEN BY

$$U_{xx} + U_{yy} = 0, \quad -\infty < x < \infty, \; y > 0$$

$$U(x, 0) = h(x)$$

U BOUNDED AS $y \to \infty$ AND AS $|x| \to \infty$.

NOW LET $\bar{U}(k, y) = \int_{-\infty}^{\infty} U(x, y)\, e^{-ikx}\, dx$.

THEN $\hat{F}(U_{xx}) + \hat{F}(U_{yy}) = 0$

SO $\bar{U}_{yy} - k^2 \bar{U} = 0$

$\bar{U}(k, 0) = \hat{H}(k)$ WITH $\hat{H}(k) = \hat{F}(h(x))$.

WE SOLVE TO OBTAIN

$$\bar{U}(k, y) = \hat{H}(k)\, e^{-|k|y}.$$

NOW $\hat{F}^{-1}\left(e^{-|k|y}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|y + ikx}\, dk$

LET $s = ky$ THEN $\hat{F}^{-1}\left(e^{-|k|y}\right) = \frac{1}{2\pi y}\int_{-\infty}^{\infty} e^{-|s| + is(x/y)}\, ds$

BY (i) $= \frac{1}{y} \frac{1}{\pi\left(1 + x^2/y^2\right)} = \frac{y}{\pi(x^2 + y^2)}$

THEN USING CONVOLUTION, WE OBTAIN THE SOLUTION

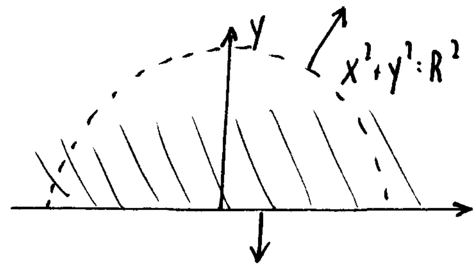
$$U(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{[(x-s)^2 + y^2]}\, h(s)\, ds.$$

NOW IF $h(s) = \delta(s)$ THEN $U(x, y) = \frac{y}{\pi[x^2 + y^2]}$

PROBLEM 4 WE NOW CONSIDER THE NEUMANN PROBLEM

$$U_{xx} + U_{yy} = 0 \quad \text{IN} \quad y \geq 0, \; -\infty < x < \infty$$

$$U_y(x,0) = F(x)$$



THIS PROBLEM REQUIRES SOME CARE SINCE $U \not\to 0$ AS $x^2+y^2 \to \infty$ AND SO THE FOURIER TRANSFORM IS NOT immediately Applicable. TO SEE THIS WE PUT $U \sim A \log|\underline{x}|$ AS $|\underline{x}| \equiv (x^2+y^2)^{1/2} \to \infty$ FOR SOME A. USING THE DIVERGENCE THEOREM OVER THE LARGE SEMI-CIRCLE AS SHOWN WE CALCULATE WITH $r = |\underline{x}|$ THAT

$$\int_{\Omega} \Delta u \, d\underline{x} = \lim_{R\to\infty} \int_{C_R} \frac{\partial u}{\partial r} \, ds - \int_{-\infty}^{\infty} \frac{\partial u}{\partial y}(x,0) \, dx = 0$$

THIS GIVES
$$0 = \pi \frac{Ar}{r}\Big|_{r=R} - \int_{-\infty}^{\infty} \frac{\partial u}{\partial y}(x,0) \, dx$$

SO
$$A = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x) \, dx \quad \text{AND} \quad U \sim A \log|\underline{x}| \quad \text{AS} \quad x^2+y^2 \to \infty.$$

IN ADDITION THE SOLUTION IS NOT UNIQUE SINCE WE CAN ALWAYS ADD AN ARBITRARY CONSTANT C TO ANY SOLUTION.

LET'S TRY TO TAKE FT IN A STRAIGHTFORWARD WAY

WE LET $\bar{U}(k,y) = \hat{\mathcal{F}}(U(x,y))$. THEN,

$$\left.\begin{array}{l} -k^2 \bar{U} + \bar{U}_{yy} = 0, \quad y \geq 0 \\ \bar{U}_y = \hat{F} \;\text{ ON }\; y = 0 \end{array}\right\} \longrightarrow \bar{U}(k,y) = \hat{F}(k)\frac{e^{-|k|y}}{|k|}.$$

THIS HAS A SINGULARITY ON THE PATH OF INTEGRATION WHEN CALCULATING THE INVERSE TRANSFORM.

INSTEAD LET $U_y = V$ FOR $|X| < \infty$ AND $y \geq 0$.

THEN
$$V_{xx} + V_{yy} = 0 \quad \text{IN} \quad -\infty < X < \infty,\ y > 0$$
$$V = F(x) \quad \text{ON} \quad y = 0$$

THIS PROBLEM WAS SOLVED IN PROBLEM 3 TO GET

$$U_y = V(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{[(x-\xi)^2 + y^2]} \cdot (\xi)\, d\xi .$$

NOW INTEGRATE WRT y TO OBTAIN U:

$$U = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) \log [\, y^2 + (x-\xi)^2]\, d\xi \; + \; C$$

NOTICE AS $x^2 + y^2 \to \infty$ $\quad y^2 + (x-\xi)^2 \longrightarrow y^2 + x^2$

$$\text{SO} \quad U \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) \log [\, x^2 + y^2]\; d\xi \sim \frac{1}{\pi} \log |x| \int_{-\infty}^{\infty} F(\xi) d\xi$$

WHICH AGREES WITH WHAT WE CALCULATED VIA THE DIVERGENCE THEOREM.

LAPLACE TRANSFORMS

THIS IS VERY USEFUL FOR AN INITIAL VALUE PROBLEM WITH INITIAL CONDITIONS GIVEN AT $t=0$. LET $f(t)$ BE PIECEWISE CONTINUOUS WITH

$$|f(t)| \leq K e^{\alpha t} \quad \forall\, t \quad (\text{possibly } \alpha > 0).$$

THEN DEFINE $$\hat{F}(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

WE HAVE THAT $\hat{F}(s)$ IS <u>ANALYTIC</u> FOR $\mathrm{Re}(s) > \alpha$. WE RECALL SOME BASIC PROPERTIES:

(i) $\mathcal{L}(f'') = s^2 \mathcal{L}(f) - f'(0) - s f(0)$, $\quad \mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

(ii) $\mathcal{L}(f(t) e^{at}) = F(s-a)$ WITH $F(s) = \mathcal{L}(f(t))$.

(iii) $\mathcal{L}(t\, f(t)) = -F'(s)$ WITH $F(s) = \mathcal{L}(f(t))$

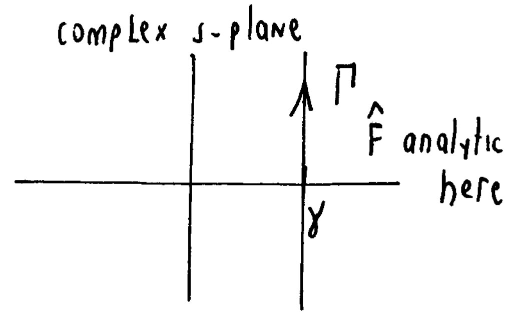
(iv) $\mathcal{L}(H_\lambda(t)) = \dfrac{e^{-s\lambda}}{s}$ WITH $H_\lambda(t) = \begin{cases} 0, & 0 \leq t < \lambda \\ 1, & t \geq \lambda \end{cases}$

(v) <u>CONVOLUTION</u> $\mathcal{L}[g * h] = \hat{G}(s)\hat{H}(s)$ $\quad \hat{G}(s) = \mathcal{L}(g(t))$, $\hat{H}(s) = \mathcal{L}(h(t))$

AND $$g * h = \int_0^t g(t-\tau)\, h(\tau)\, d\tau = \int_0^t g(\tau) h(t-\tau)\, d\tau.$$

(vi) <u>INVERSION FORMULA</u>

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{F}(s) e^{st}\, ds$$



CALCULATING INVERSE LAPLACE TRANSFORMS REQUIRES COMPLEX VARIABLE THEORY $\longrightarrow$ M301

SOME SIMPLE LAPLACE TRANSFORMS

(i) $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$ (iii) $\mathcal{L}(\cos \omega t) = \dfrac{s}{s^2+\omega^2}$

(ii) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ (iv) $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}$

THE FOLLOWING TRANSFORM PAIRS ARE COMMON IN ANALYZING DIFFUSION PROBLEMS

(iv) $\mathcal{L}\left[\operatorname{ERFC}\left(\lambda/2\sqrt{t}\right)\right] = \dfrac{e^{-\lambda\sqrt{s}}}{s}, \quad \lambda \geq 0$

where $\operatorname{erfc}(y) \equiv \dfrac{2}{\sqrt{\pi}} \displaystyle\int_y^\infty e^{-\xi^2}\, d\xi.$ $\operatorname{ERFC}(\infty) = 0$, $\operatorname{ERFC}(0) = 1$

(v) $\mathcal{L}\left[\dfrac{\lambda}{2\sqrt{\pi}\, t^{3/2}}\, e^{-\lambda^2/4t}\right] = e^{-\lambda\sqrt{s}}, \quad \lambda > 0$

(vi) $\mathcal{L}\left[\dfrac{1}{\sqrt{t}}\, e^{-\lambda^2/4t}\right] = \left(\dfrac{\pi}{s}\right)^{1/2} e^{-\lambda\sqrt{s}}, \quad \lambda > 0.$

WE NOW SOLVE A FEW BASIC PROBLEMS WITH THIS APPROACH.

PROBLEM 1 FIND THE SOLUTION TO

$$U_t = D U_{xx}, \quad 0 < x < \infty, \quad t > 0 \qquad D > 0 \text{ CONSTANT}$$

$$U(0,t) = h(t), \quad U(x,0) = 0, \quad U \to 0 \text{ AS } x \to \infty, \ t \text{ FIXED}$$

USING LAPLACE TRANSFORMS.

NOW LET $\hat{U}(x,s) = \mathcal{L}\{u(x,t)\}$. THEN USING PROPERTY (i) ON P.7

WE GET

$$s\,\hat{U} - u(x,0) = D\,\hat{U}_{xx}$$

SO

$$\hat{U}_{xx} - \frac{s}{D}\hat{U} = 0, \quad 0 < x < \infty$$

$$\hat{U}(\quad) \quad \hat{H}(s) \qquad \hat{H}(s) = \mathcal{L}\{h(t)\}$$

THE SOLUTION THAT IS BOUNDED AS $x \to \infty$ WHEN $s > 0$ IS

$$\hat{U} = A e^{-\sqrt{s/D}\,x} + B e^{\sqrt{s/D}\,x} \qquad B = 0$$

SO

$$\hat{U} = \hat{H}(s)\, e^{-\sqrt{s/D}\,x}$$

NOW INVERT THE TRANSFORM $\mathcal{L}^{-1}\left[e^{-\sqrt{s/D}\,x}\right] = \dfrac{x}{2\sqrt{\pi D}\, t^{3/2}}\, e^{-x^2/4Dt}$

USING (V) ON P. 8. HENCE USING CONVOLUTION

$$u(x,t) = \int_0^t \frac{h(t-\tau)\; x}{2\sqrt{\pi D}\;\tau^{3/2}}\, e^{-x^2/4D\tau}\, d\tau.$$

EQUIVALENTLY WE COULD HAVE WRITTEN

$$\hat{U} = \left(s\,\hat{H}(s)\right) \frac{e^{-\sqrt{s/D}\,x}}{s}, \qquad \mathcal{L}^{-1}\left(\frac{e^{-\sqrt{s/D}\,x}}{s}\right) = \mathrm{ERFC}\left(\frac{x}{2\sqrt{Dt}}\right)$$

AND

$$\left[s\,\hat{H}(s)\right] = \mathcal{L}\{h'(t)\} + h(0)$$

NOW USE CONVOLUTION THEOREM,

$$u(x,t) = \int_0^t h'(t-\tau)\, \mathrm{erfc}\left(\frac{x}{2\sqrt{D\tau}}\right) d\tau + h(0)\int_0^t \mathrm{ERFC}\left(\frac{x}{2\sqrt{D\tau}}\right) d\tau.$$

PROBLEM 2 WE CONSIDER

$$u_t = D u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad D > 0 \text{ CONSTANT}$$

$$u(x,0) = \delta(x), \quad u \to 0 \text{ AS } |x| \to \infty.$$

WE TAKE THE LAPLACE TRANSFORM:

$$U(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt$$

THEN U SOLVES:

$$D U_{xx} - s U = -\delta(x), \quad -\infty < x < \infty$$

$$U \text{ BOUNDED AS } |x| \to \infty.$$

THE CONTINUITY AND JUMP CONDITIONS ARE $U(x_0+, s) = U(x_0-, s)$

$$D\left(U_x(x_0+, s) - U_x(x_0-, s)\right) = -1$$

HENCE

$$U(x,s) = \begin{cases} A e^{\sqrt{s/D}\, x}, & \text{IF } x < 0 \\ A e^{-\sqrt{s/D}\, x}, & \text{IF } x > 0 \end{cases}$$

THE JUMP CONDITION YIELDS $-2\sqrt{\frac{s}{D}}\, A = -\frac{1}{D} \rightarrow A = \frac{1}{2\sqrt{sD}}$.

THUS

$$U(x,s) = \frac{1}{2\sqrt{sD}}\, e^{-\sqrt{s/D}\, |x|}$$

NOW RECALL

$$\mathcal{L}^{-1}\left(\frac{e^{-\lambda\sqrt{s}}}{\sqrt{s}}\right) = \frac{1}{\sqrt{\pi t}}\, e^{-\lambda^2/4t}$$

THIS YIELDS,

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}$$

GAUSSIAN PROFILE

"SOURCE" SOLUTION TO HEAT EQUATION.

REMARK IF $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ THEN

large s behavior of $F(s) \longleftrightarrow$ small t behavior of $f(t)$.

WE NOW USE THIS QUALITATIVE FACT IN AN EXAMPLE.

PROBLEM 3 CONSIDER

$$U_t = D\,U_{xx}, \quad 0 < x < 1,\ t > 0,\ D > 0 \text{ CONSTANT}$$
$$U_x(0,t) = 0, \quad U(1,t) = 1, \quad U(x,0) = 0$$

A STRAIGHTFORWARD SEPARATION OF VARIABLES SOLUTION GIVES

$$(*) \quad U(x,t) = 1 + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1/2} \cos\left[\left(k+\tfrac{1}{2}\right)\pi x\right] e^{-D(k+1/2)^2\pi^2 t}$$

NOTICE THAT IF t IS LARGE ONLY A FEW TERMS IN THIS INFINITE SERIES ARE NEEDED TO APPROXIMATE $U(x,t)$ RATHER ACCURATELY, HOWEVER, IF t IS SMALL WE NEED MANY TERMS IN THE INFINITE SUM IN $(*)$. CAN WE GET ANOTHER REPRESENTATION FOR $U(x,t)$ THAT CONVERGES QUICKLY WHEN t IS SMALL? YES, USE LAPLACE TRANSFORMS.

DEFINE $\quad \mathcal{U}(x,s) = \mathcal{L}(U(x,t))$.

THEN $\quad \mathcal{U}_{xx} - \frac{s}{D}\mathcal{U} = 0, \quad 0 < x < 1$

$$\mathcal{U}_x = 0 \text{ ON } x = 0, \quad \mathcal{U} = \frac{1}{s} \text{ ON } x = 1$$

THE SOLUTION IS $\mathcal{U} = A \cosh\left[\sqrt{\tfrac{s}{D}}\, x\right]$ $\quad A = \dfrac{1}{s\cosh\left(\sqrt{\frac{s}{D}}\right)}$

THEN $$\bar{U}(X,S) = \frac{\cosh\left(\sqrt{\frac{S}{D}}\,X\right)}{S\cosh\left(\sqrt{\frac{S}{D}}\right)}$$

NOW INVERT THE TRANSFORM. USE $\cosh y = (e^{y} + e^{-y})/2$.

SO $$\bar{U}(X,S) = \frac{e^{\sqrt{S/D}\,X} + e^{-\sqrt{S/D}\,X}}{S\left[e^{\sqrt{S/D}} + e^{-\sqrt{S/D}}\right]} = \frac{1}{S}\,e^{-\sqrt{S/D}\,(1-X)}\left(\frac{1+e^{-2\sqrt{S/D}\,X}}{1+e^{-2\sqrt{S/D}}}\right)$$

NOW FOR $|Z|<1$ $$\frac{1}{1+Z} = 1 - Z + Z^2 - Z^3 \dots = \sum_{n=0}^{\infty}(-1)^n Z^n.$$

LET $Z = e^{-2\sqrt{S/D}}$ SO THAT $$\frac{1}{1+e^{-2\sqrt{S/D}}} = \sum_{n=0}^{\infty}(-1)^n e^{-2n\sqrt{S/D}}$$

THEN $$\bar{U}(X,S) = \frac{e^{-\sqrt{S/D}\,(1-X)}}{S}\left(1 + e^{-2\sqrt{S/D}\,X}\right)\sum_{n=0}^{\infty}(-1)^n e^{-2n\sqrt{S/D}}$$

WE WRITE THIS AS
$$\bar{U}(X,S) = \frac{1}{S}\left(\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{S}{D}}\,(2n+1-X)} + \sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{\frac{S}{D}}\,(1+2n+X)}\right)$$

NOW INVERT EACH TERM USING $\mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{S}}}{S}\right] = \operatorname{ERFC}\left(a/2\sqrt{t}\right)$.

THIS GIVES
$$(+)\qquad u(x,t) = \sum_{n=0}^{\infty}(-1)^n\left[\operatorname{ERFC}\left(\frac{2n+1-X}{2\sqrt{Dt}}\right) + \operatorname{ERFC}\left(\frac{2n+1+X}{2\sqrt{Dt}}\right)\right]$$

THIS IS AN EXACT SOLUTION AND THUS SINCE THE SOLUTION IS UNIQUE, (+) AND (*) (EIGENFUNCTION SOLUTION) MUST BE EQUIVALENT. SINCE $\operatorname{ERFC}(y) \to 0$ AS $y \to \infty$ THEN ONLY A FEW TERMS IN (+) ARE NEEDED TO CALCULATE $u(x,t)$ ACCURATELY WHEN t IS SMALL.

NOTICE THAT THE LEADING TERM IN (+) IS TO TAKE $n=0$ AND THEN TAKE THE ERFC WHOSE ARGUMENT COULD VANISH AT $X=1$.

HENCE, $$u(x,t) \approx \operatorname{erfc}\left(\frac{1-x}{2\sqrt{Dt}}\right) \quad \text{FOR } t \text{ SMALL.}$$

THIS APPROXIMATION CAN BE OBTAINED VERY QUICKLY FROM

$$\bar{U}(x,s) = \frac{\cosh\left(\sqrt{\frac{s}{D}}\,x\right)}{s\cosh\left(\sqrt{\frac{s}{D}}\right)}.$$

LET $s \to \infty$ (t SMALL) TO OBTAIN

$$\bar{U}(x,s) \approx \frac{1}{s}\,\frac{e^{\sqrt{s/D}\,x}}{e^{+\sqrt{s/D}}} = \frac{1}{s}\,e^{-\sqrt{s/D}\,(1-x)}$$

LET $\lambda = \dfrac{(1-x)}{\sqrt{D}} \geq 0$ ON $0<x<1$. THEN WITH $\mathcal{L}^{-1}\left[\dfrac{e^{-\lambda\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\dfrac{\lambda}{2\sqrt{t}}\right)$

WE OBTAIN $$u(x,t) \approx \operatorname{erfc}\left(\frac{1-x}{2\sqrt{Dt}}\right) \quad \text{FOR } t \text{ SMALL.}$$

SIMILARITY SOLUTIONS

WHEN THERE IS NO NATURAL LENGTH SCALE IT CAN BE POSSIBLE TO REDUCE THE SOLUTION TO A PDE TO ONE INVOLVING ONLY AN ODE. THIS APPROACH CAN BE USED FOR BOTH LINEAR AND NONLINEAR PROBLEMS.

PROBLEM 1 FIND A SIMILARITY SOLUTION TO

$$(*) \quad U_t = D\,U_{xx}, \qquad -\infty < x < \infty,\ t > 0$$

$$U(x,0) = \delta(x), \quad U \to 0 \text{ AS } |x| \to \infty$$

WE FIRST NOTE THAT $\int_{-\infty}^{\infty} U(x,t)\,dx = \int_{-\infty}^{\infty} U(x,0)\,dx = 1.$

WE TRY A SOLUTION OF THE FORM

$$(1) \quad U = t^{-1/2} F(x/t^{1/2})$$

THEN $$\int_{-\infty}^{\infty} t^{-1/2} F(x/t^{1/2})\,dx = \int_{-\infty}^{\infty} F(z)\,dz = 1.$$

$$z = x/t^{1/2} \qquad dx = t^{1/2}\,dz$$

HENCE $\int_{-\infty}^{\infty} F(z)\,dz = 1$ IS NEEDED AND BY SYMMETRY $F'(0) = 0$ WITH $F(\infty) = 0$.

SUBSTITUTE (1) INTO (*) TO OBTAIN

$$U_t = -\frac{1}{2} t^{-3/2} F(x t^{-1/2}) + t^{-1/2}\left(-\frac{x}{2} t^{-3/2}\right) F'(x t^{-1/2})$$

$$U_x = t^{-1/2}\, t^{-1/2} F'(x t^{-1/2}) \qquad U_{xx} = t^{-1/2}\left(t^{-1} F''(x t^{-1/2})\right)$$

SO $$-\frac{1}{2} t^{-3/2} F(\eta) - \frac{x}{2} t^{-2} F'(\eta) = t^{-3/2} D\, F''(\eta)$$

THIS YIELDS $$D F''(\Lambda) = -\frac{1}{2} F(\Lambda) - \frac{x}{2} t^{-1/2} F'(\Lambda)$$

EQUIVALENTLY $$D F''(\Lambda) = -\frac{1}{2}\left[F(\Lambda) + \Lambda F'(\Lambda) \right]$$
$$F'(0) = 0, \quad F(\infty) = 0, \quad \int_{-\infty}^{\infty} F(\Lambda)\, d\Lambda = 1.$$

THIS IS AN ODE PROBLEM!

NOTE THAT $$D F'' = -\frac{1}{2} (\Lambda F)'$$

INTEGRATE ONCE $$D F' = -\frac{1}{2} \Lambda F + C$$

BUT $F \to 0$ AT ∞ SO $C = 0$.

NOW $$D F' = -\frac{1}{2} \Lambda F$$

WE WRITE $$F' + \frac{1}{2D} \Lambda F = 0$$

SO $$\left(F e^{\Lambda^2/4D} \right)' = 0$$
$$F = A e^{-\Lambda^2/4D}$$

WE REQUIRE THAT $\int_{-\infty}^{\infty} F(\Lambda)\, d\Lambda = 1 \rightarrow 2A \int_0^{\infty} e^{-\Lambda^2/4D}\, d\Lambda = 1.$

NOW $\int_0^{\infty} e^{-x^2/2\sigma^2}\, dx = \frac{\sqrt{2\pi}\,\sigma}{2}$. LET $\sigma^2 = 2D$ SO $\sigma = \sqrt{2D}$.

THEN $$2A\left[\frac{\sqrt{2\pi}}{2} \sqrt{2D} \right] = 1 \rightarrow A = \frac{1}{2\sqrt{\pi D}}.$$

WE CONCLUDE THAT $u(x,t) = t^{-1/2} F(x/t^{1/2})$, $F(\Lambda) = \frac{1}{2\sqrt{\pi D}} e^{-\Lambda^2/4D}$.

THIS YIELDS THE SOURCE SOLUTION FOUND EARLIER BY BOTH LAPLACE AND FOURIER TRANSFORMS $u(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt}$

THIS IDEA OF SIMILARITY SOLUTION REDUCTION CAN BE APPLIED TO OTHER PROBLEMS WITH NO NATURAL LENGTH SCALE IN EITHER THE BOUNDARY CONDITION OR INITIAL CONDITION. FOR INSTANCE;

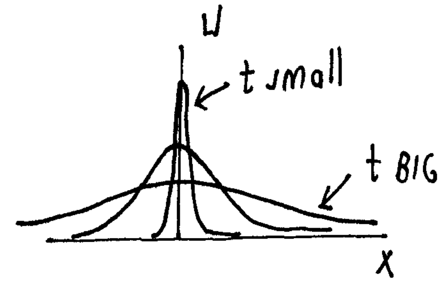
(i) $u_t = D u_{xx}$, $0 < x < \infty$, $t > 0$
$u(0,t) = 1$, $u(x,0) = 0$
$u \to 0$ AS $x \to \infty$
$\longrightarrow$ try $u(x,t) = f(x/t^{1/2})$

(ii) $u_t = D u_{xx}$, $0 < x < \infty$, $t > 0$
$u_x(0,t) = -1$, $u(x,0) = 0$
$u \to 0$ AS $x \to \infty$
$\longrightarrow$ try $u(x,t) = t^{1/2} f(x/t^{1/2})$

THE SAME IDEA WORKS WITH NONLINEAR PROBLEMS FOR WHICH EIGENFUNCTION EXPANSIONS AND TRANSFORM METHODS ARE NOT APPLICABLE. THESE LATTER METHODS REQUIRE <u>LINEARITY</u>.

CONSIDER THE NONLINEAR DIFFUSION PROBLEM

$$u_t = (u^m u_x)_x, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = \delta(x) \qquad m \geq 0.$$



IF $m = 0$, THEN $u(x,t) = \dfrac{1}{2\sqrt{\pi D t}} e^{-x^2/4Dt}$

THE INITIAL DATA HAS "COMPACT SUPPORT". BUT AT ANY $t > 0$ $u(x,t) \neq 0$. THE SOLUTION IS A GAUSSIAN WITH VARIANCE

$$\sigma^2 = 2Dt.$$

CAN WE GET NEW PHENOMENA IF $m>0$? THE ANSWER IS YES.

THERE ARE SIMILARILTY SOLUTIONS THAT HAVE <u>COMPACT SUPPORT</u> FOR ALL $t>0$.

WE CONSIDER
$$u_t = (u^m u_x)_x \qquad -\infty < x < \infty, \quad m>0$$
$$u(x,0) = \delta(x)$$

NOW WE TRY $u = t^\alpha f(\lambda)$ WITH $\lambda = x/t^b$ ← SIMILARITY REDUCTION

SUBSTITUTE
$$\alpha t^{\alpha-1} f(\lambda) - x b t^\alpha t^{-b-1} f'(\lambda) = t^{-2b} t^{\alpha(m+1)} (f^m f')'$$

NOW THIS GIVES
$$t^{\alpha-1}[\alpha f - b\lambda f'] = t^{\alpha(m+1)-2b}(f^m f')'$$

WE THEN CHOOSE $\quad \alpha - 1 = \alpha m + \alpha - 2b \quad$ OR $\quad \alpha = \dfrac{(2b-1)}{m}$.

THIS YIELDS
$$(f^m f')' = \alpha f - b\lambda f'.$$

NOW TO FIND b WE INTEGRATE OVER $-\infty < x < \infty$ TO OBTAIN
$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = 0 \qquad \text{THUS} \qquad \int_{-\infty}^{\infty} u\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx = 1.$$

NOW SUBSTITUTE $\quad u = t^\alpha f(\lambda)$ TO OBTAIN
$$\int_{-\infty}^{\infty} t^\alpha f(x/t^b)\, dx = t^{\alpha+b}\int_{-\infty}^{\infty} f(\lambda)\, d\lambda = 1. \qquad \text{HENCE} \quad \alpha = -b.$$

let $\lambda = x/t^b \qquad dx = t^b d\lambda$.

$$\alpha = \frac{2b-1}{m}$$
$$\rightarrow b = \frac{1}{2+m} = -\alpha$$

THUS THE SIMILARITY SOLUTION HAS THE FORM
$$u(x,t) = t^{-1/(m+2)} f\left(x/t^{1/2+m}\right)$$

WHERE f SATISFIES $\qquad (f^m f')' = \alpha(f + \lambda f')$

NOW WE CAN WRITE THIS AS

$$(f^m f')' = \alpha (\Lambda f)'$$

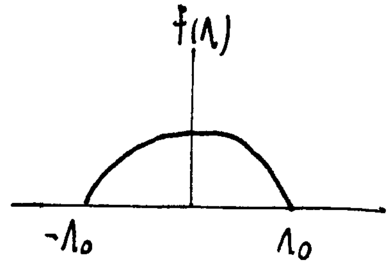
INTEGRATING BOTH SIDES $\quad f^m f' = \alpha \Lambda f + C.$

NOW WE SET $f = 0$ AT $\Lambda = \Lambda_0$ AND TAKE $f(\Lambda) = f(-\Lambda)$ (EVEN).
THIS YIELDS THAT $C = 0$ (PROVIDED THAT $f^m f' \to 0$ AS $\Lambda \to \Lambda_0$).

THEREFORE $\quad f' = \alpha \Lambda f^{1-m}.$

THIS EQUATION IS SEPARABLE AND SO WE INTEGRATE USING $f(\Lambda_0) = 0$
TO OBTAIN

$$\frac{1}{m} f^m = \frac{\alpha}{2} (\Lambda^2 - \Lambda_0^2).$$

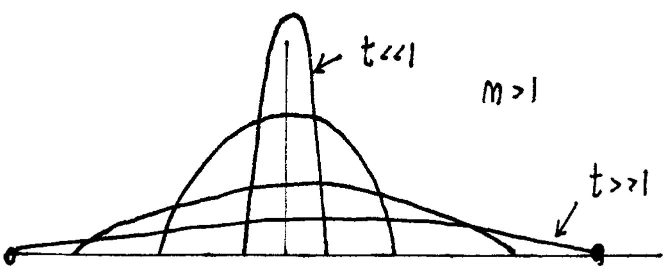


RECALLING THAT $\alpha = -1/(2+m)$, WE OBTAIN

$$f^m(\Lambda) = \begin{cases} \dfrac{m}{2(2+m)} (\Lambda_0^2 - \Lambda^2) & |\Lambda| \leq \Lambda_0 \\ 0 & |\Lambda| \geq \Lambda_0 \end{cases}$$

THE FINITE EXTENT IS DUE TO THE DEGENERACY AT $f = 0$, WHEN $m > 0$.
IN TERMS OF THE ORIGINAL VARIABLES

$$u(x,t) = t^{-1/(m+2)} \left[\frac{m}{2(2+m)} \left(\Lambda_0^2 - x^2 / t^{2/(2+m)} \right)_+ \right]^{1/m} \qquad (x)_+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$



NOW WE FINALLY DETERMINE Λ_0
BY THE CONDITION

$$\int_{-\infty}^{\infty} f(\Lambda)\, d\Lambda = 1$$

THUS $\quad 1 = \int_{-\Lambda_0}^{\Lambda_0} \left(\frac{m}{2+m}\right)^{1/m} (\Lambda_0^2 - \Lambda^2)^{1/m} d\Lambda = 2 \left(\frac{m}{2+m}\right)^{1/m} \int_0^{\Lambda_0} (\Lambda_0^2 - \Lambda^2)^{1/m} d\Lambda = 2 \left(\frac{m}{2+m}\right)^{1/m} \Lambda_0^{\frac{2}{m}+1} \int_0^{\pi/2} \cos^{\frac{m+2}{2}} \varphi\, d\varphi$

THIS YIELDS $\quad \Lambda_0 = \left[\frac{1}{2} \left(\frac{2+m}{m} \right)^{1/m} \left(\int_0^{\pi/2} \cos^{(m+2)/m} \varphi\, d\varphi \right)^{-1} \right]^{m/m+2}$