Rk2

Ex. 3 candidates for non compactness of bounded sequence.

1). Oscillation $f_k(x) = \sin kx$, $\|f_k\|_{L^q(0,\pi)} = \text{const}$ $\forall q$

Ex.1 $L^q([0,\pi]) \subset L^2([0,\pi])$

$\|\nabla f_k\|_{L^1(0,\pi)} \xrightarrow[k\to\infty]{} \infty$ ~~$f_k(x) \xrightarrow[k\to\infty]{} 0$ in L^q~~

2). concentration $\Omega = (-1,1)$ $f_k(x) = k^{\frac{1}{p}-1}\phi(kx)$, $\|f_k(x)\|_{L^q(\mathbb{R})} = \text{const}$

$W^{1,p}(\Omega) \hookrightarrow L^{p^*}$

$\|f_k(x)\|_{L^q(\Omega)} \leq \text{const}$ $\forall q \leq p^*$

$\|\nabla f_k(x)\|_{L^p(\Omega)} = \text{const}$

~~$f_k(x) \longrightarrow 0$ in L^q~~ $\forall q < p^*$

~~$f_k(x)$ does not converge in L^{p^*}.~~ $f_k \xrightarrow[k\to\infty]{} 0$ in L^{p^*}

3. escape to ∞ $f_k(x) = \phi(x - x_k)$ $|x_k| \to \infty$ not allowed since Ω bdd.

$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$

In particular, $W^{1,p} \hookrightarrow L^{p^*}$ is not compact.

Poincaré ineq: Ω bdd Lipschitz, $1 \leq p < \infty$,

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$$

2nd proof. using compactness by contraction.

suppose not, $\exists\, u_k \in W^{1,p}(\Omega)$, $\int_\Omega u_k = 0$, $\|\nabla u_k\|_{L^p} \leq \frac{1}{k}$

$\|u_k\|_{L^p} = 1$.

a subsequence, still denoted u_k, $\longrightarrow \bar{u}$ strongly in L^p

$\int \bar{u} = 0$

$$\int v\,\phi_{x_j} = \lim_{k\to\infty} \int v_k\,\phi_{x_j} = \lim -\int (D_{x_j} v_k)\phi = 0$$

$\therefore$ $Dv = \text{exist}, = 0.$

$\Rightarrow$ $v = 0.$ $\Rightarrow\Leftarrow$.

$$\left(\fint |u - u_B|^{n'}\right)^{\frac{1}{n'}} \leq \fint_B |u - u_B|^{n} \leq \fint_B |Du|^n$$