MATH 516-101-2019 Homework One
Due Date: September 21st, 2018
Office Hours: MWF 4:30-5:30pm

1. (20pts) This problem concerns the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y)dy \tag{1}$$

where $n \geq 3$.

a) Show that if $|f(y)| \leq \frac{C}{(1+|y|)^\alpha}$ for $\alpha \in (2, n)$. Then $|u(x)| \leq \frac{C}{|x|^{\alpha-2}}$ for $|x| > 1$

b) Show that if $|f(y)| \leq \frac{C}{(1+|y|)^n}$, then $|u(x)| \leq \frac{C}{|x|^{n-2}} \log |x|$ for $|x| > 1$

c) Show that if $|f(y)| \leq \frac{C}{(1+|y|)^\alpha}$ for $\alpha > n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}}$ for $|x| > 1$

Hint: For $|x| = R >> 1$, divide the integral into three parts

$$\int_{R^n} (...)dy = \int_{|y-x|<\frac{|x|}{2}} (...) + \int_{\frac{|x|}{2}<|y-x|<2|x|} (...) + \int_{|y-x|>2|x|} (...)$$

and estimate each parts. For example in the region $|y-x| < \frac{|x|}{2}$ we have $|y| > |x| - |x-y| > \frac{|x|}{2}$ and

$$\int_{|y-x|<\frac{|x|}{2}} \frac{1}{|x-y|^{n-2}} |f(y)|dy \leq \int_0^{\frac{|x|}{2}} \frac{r^{n-1}}{r^{n-2}} dr \frac{C}{|x|^\alpha} \leq \frac{C}{|x|^{\alpha-2}}$$

2. (10pts) Assume that $u \in C^0(\Omega)$ satisfies the Mean-Value-Property. Show that $u \in C^\infty(\Omega)$.
Hint: Take a mollifier $\rho(x)$. Show that $u(x) = \rho_\epsilon u$.

3. (20pts) This problem concerns Green's function and Green's representation formula.
a) Write the Green's function for the unit ball $B_1(0)$.
b) Use a) and reflection to find the Green's function in half ball $B^+(0,1) = B(0,1) \cap \{x_n > 0\}$.
c) Use b) and reflection to find the Green's function in a quarter ball $B_1(0) \cap \{x > 0, y > 0\}$.

4. (10pts) The Kelvin transform is defined by

$$v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$$

Suppose u satisfies $-\Delta u(x) = f(x)$. Find out the new equation for v.

5. (20pts) Let $G(x, y) = \Gamma(|x-y|) - H(x, y)$ be the Green's function in Ω and define:

$$v(x) = \int_\Omega G(x, y) f(y)dy$$

Suppose that f is bounded and integrable in Ω. Show that $\lim_{x \to x_0, x \in \Omega} v(x) = 0$ for any $x_0 \in \partial\Omega$.

6. (20pts) Let u be harmonic in $\Omega \subset \mathbb{R}^n$. Show that for all $x_0 \in \Omega$

$$|\nabla u(x_0)| \leq \frac{n}{d_0} [\sup_\Omega u - u(x_0)], d_0 = d(x, \partial\Omega).$$

Hint: Do gradient estimate for $\sup_\Omega u - u(x)$. Note that this function is nonnegative.