1. (a) Let $\xi \in \partial\Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset\subset \Omega$: (i) w is superharmonic in Ω_1; (ii) $w > 0$ in $\bar{\Omega}_1 \backslash \{\xi\}$; $w(\xi) = 0$. Show that w can be extended to a barrier in Ω. (b) Let $\Omega = \{x^2 + y^2 < 1\}\backslash\{-1 \le x \le 0, y = 0\}$. Show that the function $w := -Re(\frac{1}{Ln(z)}) = -\frac{\log r}{(\log r)^2 + \theta^2}$ is a local barrier at $\xi = 0$.

2. Let
$$\Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that there exists a generic constant C_n such that
$$\Phi(x - y, t) \le C_n |x - y|^{-n}$$

Hint: maximize the function in t.

(b) Let $n = 1$ and $f(x)$ be a function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that
$$\lim_{t \to 0} \int_R \Phi(x - x_0, t) f(y) dy = \frac{1}{2}(f(x_0-) + f(x_0+))$$

(c) Let u satisfy
$$u_t = \Delta u, x \in \mathbb{R}^n, t > 0; u(x, 0) = f(x)$$

Suppose that f is continuous and has compact support. Show that $\lim_{t \to +\infty} u(x, t) = 0$ for all x.

3. Derive a solution formula for
$$u_t = \Delta u + cu + f(x, t), t > 0; u(x, 0) = g(x)$$

4. Consider the following general parabolic equation
$$L[u] = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t$$

where
$$0 < C_1 < a(x, t) < C_2, |b(x, t)| \le C_3, c(x, t) \le C_4$$

(a) Show that $L[u] \ge 0$, then
$$\max_{\Omega_T} u \le e^{C_4 T} \max_{\partial' \Omega_T} u^+$$

Here $\Omega_T = (0, L) \times (0, T), \partial\Omega_T = \partial\Omega_T \backslash ((0, L) \times \{T\})$ and $u^+ = \max(u, 0)$.

Hint: consider the function $v := u e^{C_4 t}$

(b) Prove the uniqueness of the initial value problem
$$\begin{cases} Lu(x, t) = f(, t), \text{ in}\Omega_T; \\ u(x, 0) = \phi(x), x \in \Omega \\ u(x, t) = g(x, t), \ x \in \partial\Omega, t \in (0, T] \end{cases}$$

5. (a) Use d'Alembert's formula to show that Maximum Principle does not hold for wave equation, i.e.,
$$u_{tt} = c^2 u_{xx}, 0 < x < L, 0 < t < T$$
$$u(x, 0) = f(x), \ u_t(x, 0) = g(x)$$
$$\max_{U_T} u(x, t) > \max_{\partial' U_T} u(x, t)$$

Hint: Let $f = 0$ and $g = 1$, $U = (-1, 1)$ and choose T large.

(b) Let u solve the initial value problem for the wave equation in one dimension
$$\begin{cases} u_{tt} = u_{xx} \text{ in } R \times (0, +\infty) \\ u(x, 0) = f(x), \ u_t(x, 0) = g(x) \end{cases}$$

where f and g have compact support in R. Let $k(t) = \frac{1}{2} \int u_t(x, t)^2 dt$ be the potential energy and $p(t) = \frac{1}{2} u_x^2(x, t) dx$ be the potential energy. Show that

(a) $k(t) + p(t)$ is constant in t; (b) $k(t) = p(t)$ for all large enough time t.

6. Find the explicit formula for the following wave equation
$$u_{tt} - \Delta u = 0, x \in R^3, t > 0; u(x, 0) = 0, u_t(x, 0) = \begin{cases} 1, \text{ for } |x| < 1; \\ 0, \text{ for } |x| > 1 \end{cases}$$