MATH 516-101 Homework Four 2018-2019
Due Date: by 6pm, November 9, 2018

1. Let $u \in H_0^1((0,1))$. Show that there is $w \in C([0,1])$ with $w(0) = w(1) = 0$ such that $u = w$ almost everywhere in $[0,1]$.
2. Fix $\alpha > 0, 1 < p < +\infty$ and let $U = B_1(0)$. Show that there exists a constant C, depending on n and α such that

$$\int_U u^p dx \le C \int_U |\nabla u|^p$$

provided

$$u \in W^{1,p}(U), \ |\{x \in U | u(x) = 0\}| \ge \alpha$$

3. (a) Show that $W^{1,2}(R^N) \subset L^2(R^N)$ is not compact. (b). Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{3,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

4. (a) Let $u \in W_r^{1,2} = H_r^1 = \{u \in W^{1,2}(R^n) \mid u = u(r)\}$. Show that $|u(r)| \le C\|u\|_{W^{1,2}} r^{-\frac{n-1}{2}}$. (b) Show that for $n \ge 2$, the embedding $W_r^{1,2} \subset L^p$ is compact when $2 < p < \frac{2n}{n-2}$. (c) Let $u = \mathcal{D}_r^{1,2} = \{\int |\nabla u|^2 < +\infty; u = u(r)\}$. Show that $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ and $|u(r)| \le C\|\nabla u\|_{L^2} r^{-\frac{n-2}{2}}$. However $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ is not compact.

5. Let $U = (-1,1)$. Show that the dual space of $H^1(U)$ is isomorphic to $H^{-1}(U) + E^*$ where E^* is the two dimensional subspace of $H^1(U)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$.

6. (a). Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$(3) \qquad -\Delta u = f \text{ in } U; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

if

$$\int_U Du \cdot Dv = \int_U fv, \quad \forall v \in W^{1,2}$$

Suppose that $f \in L^2$. Show that (3) has a weak solution if and only if

$$\int_U f = 0$$

(b). Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$(4) \qquad -\Delta u = f \text{ in } U; \ u + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

7. (a) Discuss the definition of weak solutions $u \in H_0^2(\Omega)$ to

$$(*) \quad \Delta^2 u = f \text{ in } \Omega$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

(b) Given $f \in L^2(\Omega)$ prove that there exists a unique weak solution to (*).