Sol'ns to Problem 8 of HW3 MATH516-101

Sol'n: By uniqueness, we may assume that

$$u = u(r,t).$$

Let $U = ru$. Then

$$\begin{cases} U_{tt} = U_{rr} \\ U(r,0) = 0, \ U_t(r,0) = h(r) = \begin{cases} r, & 0<r<1 \\ 0, & r>1 \end{cases} \\ U(0,t) = 0 \end{cases}$$

Odd extension of $h_{odd}(r) = \begin{cases} h(r), & r>0 \\ -h(-r), & r<0 \end{cases}$

Then $$U(r,t) = \frac{1}{2} \int_{r-ct}^{r+ct} h_{odd}(y)\,dy$$

$$u = \frac{1}{2r} \int_{r-ct}^{r+ct} h_{odd}(y)\,dy, \quad c=1$$

If $r>ct$, then $u(r,t) = \frac{1}{2r} \int_{r-t}^{r+t} h(y)\,dy$.

If $r<t$, then $u(r,t) = \frac{1}{2r} \int_{t-r}^{t+r} h(y)\,dy$.

When $r>t$, we consider

Case 1. $0<r-t<r+t<1, \Leftrightarrow t<r<1-t$

$$u = \frac{1}{2r} \left. \frac{1}{2} y^2 \right|_{r-t}^{r+t} = \frac{1}{4r}\left[(r+t)^2-(r-t)^2\right] = t$$

Case 2, $0<r-t<1<r+t \Leftrightarrow \max(t,1-t)<r<1+t$

$$u = \frac{1}{2r} \left. \frac{1}{2} y^2 \right|_{r-t}^{1} = \frac{1}{4r}\left(1-(r-t)^2\right)$$

Case3 $\quad 1<r-t \Rightarrow u=0$

when $r<t$, $\quad u=\frac{1}{2r}\int_{t-r}^{t+r} h(y)\,dy$

Case 1 $\quad 0<t-r<t+r<1 \iff r<t,\ r<1-t$

$\iff r<\min(t,1-t)$

$$u=\frac{1}{2r}\frac{1}{2}y^2\Big|_{t-r}^{t+r}=t$$

Case 2 $\quad 0<t-r<1<t+r \iff 1-t<r<t,\ t-1<r$

$\iff \max(1-t,t-1)<r<t$

$$u=\frac{1}{2r}\frac{1}{2}y^2\Big|_{t-r}^{1}=\frac{1}{4r}\left(1-(t-r)^2\right)$$

Case 3, $\quad 1<t-r<t+r \Rightarrow u\equiv 0$

Final sol'n:

Case1 $\quad |r-t|<r+t<1 \Rightarrow u=t$

Case2 $\quad |r-t|<1<r+t \Rightarrow u=\frac{1}{4r}\left(1-(r-t)^2\right)$

Case3 $\quad 1<|r-t| \Rightarrow u=0$