MATH 516-101-2020 Homework One
Due Date: By 6pm on September 20th, 2019

1. (10pts) Let $f \in C_c^\infty(R^2)$. Show that the following function

$$v(x) = \frac{1}{2\pi} \int_{R^2} (\log|x-y|) f(y) dy$$

satisfies $\Delta v(x) = f(x), x \in R^2$. (Extra 5points) Show that $v(x) = \frac{1}{2\pi}(\int_{R^2} f(x)dx) \log|x| + C + O(\frac{1}{|x|})$ as $|x| \to +\infty$.

2. (20points) Let $\Omega \subset R^n$ be a smooth and bounded domain in $R^n, n \geq 3$ and

$$v(x) = \int_\Omega f(y) \frac{1}{|x-y|^{n-2}} dy.$$

Show that if $f \in C^1(\Omega)$ and f **is bounded**, then $v \in C^2(\Omega)$ and $\Delta v(x) = -(n-2)\omega_n f(x), x \in \Omega$.

3. (20pts) This problem concerns the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|^2)^{\frac{\alpha}{2}}} dy \tag{1}$$

where $n \geq 3$.

a) Show that for $\alpha \in (2, n)$, $|u(x)| \leq \frac{C}{|x|^{\alpha-2}}$ for $|x| > 1$

b) Show that for $\alpha > n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}}$ for $|x| > 1$

c) If $f \in C_c^\infty(R^n), n \geq 3$ and $v = \int_{R^n} \frac{1}{|x-y|^{n-2}} f(y) dy$, show that $\lim_{|x| \to +\infty} |x|^{n-2} v(x) = C$ for some constant C.

Hint: For $|x| = R >> 1$, divide the integral into three parts

$$\int_{R^n} (...) dy = \int_{|y-x|<\frac{|x|}{2}} (...) + \int_{\frac{|x|}{2}<|y-x|<2|x|} (...) + \int_{|y-x|>2|x|} (...)$$

and estimate each parts. For example in the region $|y-x| < \frac{|x|}{2}$ we have $|y| > |x| - |x-y| > \frac{|x|}{2}$ and

$$\int_{|y-x|<\frac{|x|}{2}} \frac{1}{|x-y|^{n-2}} |f(y)| dy \leq \int_0^{\frac{|x|}{2}} \frac{r^{n-1}}{r^{n-2}} dr \frac{C}{|x|^\alpha} \leq \frac{C}{|x|^{\alpha-2}}$$

4. (10pts) The Kelvin transform is defined by

$$v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$$

Suppose u satisfies $-\Delta u(x) = u^3$. Find out the new equation satisfied by v.

5. (10pts) Let u be a harmonic function in $B_1(0) \backslash \{0\} = \{0 < |x| < 1\}$ be such that $\lim_{x \to 0} |x|^{n-2} u(x) = 0$. Show that $u \in C^2(B_1(0))$. Here $n \geq 3$.

6. (20pts) Let Ω be a bounded and smooth domain in R^n and $G(x,y) = \Phi(|x-y|) - \phi^x(y)$ be the Green's function in Ω. We define:

$$v(x) = \int_\Omega G(x,y) f(y) dy.$$

Assume that $f \in C^1(\Omega) \cap C(\bar{\Omega})$. Show that v is the solution of the inhomogeneous problem

$$-\Delta v(x) = f(x), x \in \Omega; v = 0 \text{ on } \partial\Omega$$

Hint: You can use results in Problem 2.

7. (10pts) Determine the Green's function for the annulus $\{1 < |x| < 2\}$ in R^2.