MATH 516-101-2020 Homework Two
Due Date: by 6pm, October 7, 2019

1. (a) Let $\xi \in \partial\Omega$ and $w(x)$ be a barrier on $\Omega_1 \subset\subset \Omega$: (i) w is superharmonic in Ω_1; (ii) $w > 0$ in $\bar{\Omega}_1\backslash\{\xi\}$; $w(\xi) = 0$. Show that w can be extended to a barrier in Ω. (b) Let $\Omega = \{x^2 + y^2 < 1\}\backslash\{-1 \leq x \leq 0, y = 0\}$. Show that the function $w := -Re(\frac{1}{\mathrm{Ln}(z)}) = -\frac{\log r}{(\log r)^2+\theta^2}$ is a local barrier at $\xi = 0$.

2. (a) Show that the problem of minimizing energy

$$I[u] = \int_J x^2|u'(x)|^2 dx,$$

for $u \in C(\bar{J})$ with piecewise continuous derivatives in $J := (-1, 1)$, satisfying the boundary conditions $u(-1) = 0, u(1) = 1$, is not attained. (b) Consider the problem of minimizing the energy

$$I[u] = \int_0^1 (1 + |u'(x)|^2)^{\frac{1}{4}} dx$$

for all $u \in C^1((0, 1)) \cap C([0, 1])$ satisfying $u(0) = 0, u(1) = 1$. Show that the minimum is 1 and is not attained.

3. Discuss Dirichlet Principle for

$$\begin{cases} \Delta^2 u = f \ \text{ in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases}$$

4. Let

$$\Phi(x - y, t) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$$

(a) Show that there exists a generic constant C_n such that

$$\Phi(x - y, t) \leq C_n|x - y|^{-n}$$

Hint: maximize the function in t.

(b) Let $n = 1$ and $f(x)$ be a **bounded measurbale** function such that $f(x_0-)$ and $f(x_0+)$ exists. Show that

$$\lim_{t\to 0} \int_R \Phi(x - x_0, t) f(y)dy = \frac{1}{2}(f(x_0-) + f(x_0+))$$

(c) Let u satisfy

$$u_t = \Delta u, x \in \mathbb{R}^n, t > 0; u(x, 0) = f(x)$$

Suppose that f is continuous and has compact support. Show that $\lim_{t\to+\infty} u(x, t) = 0$ for all x.

5. Derive a solution formula for

$$u_t = \Delta u + cu + f(x, t), t > 0; u(x, 0) = g(x)$$

6. Consider the following general parabolic equation

$$L[u] = a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u - u_t$$

where

$$0 < C_1 < a(x, t) < C_2, |b(x, t)| \leq C_3, c(x, t) \leq C_4$$

(a) Show that $L[u] \geq 0$, then

$$\max_{\bar{\Omega}_T} u \leq e^{C_4 T} \max_{\partial'\Omega_T} u^+$$

Here $\Omega_T = (0, L) \times (0, T), \partial\Omega_T = \partial\Omega_T\backslash((0, L) \times \{T\})$ and $u^+ = \max(u, 0)$.

Hint: consider the function $v := ue^{-C_4 t}$

(b) Prove the uniqueness of the initial value problem

$$\begin{cases} Lu(x, t) = f(, t), \ \text{in}\Omega_T; \\ u(x, 0) = \phi(x), x \in \Omega \\ u(x, t) = g(x, t), \ x \in \partial\Omega, t \in (0, T] \end{cases}$$

7. (a) Use d'Alembert's formula to show that Maximum Principle does not hold for wave equation, i.e.,

$$u_{tt} = u_{xx}, -L < x < L, 0 < t < T$$

$$u(x, 0) = f(x), \ u_t(x, 0) = g(x)$$

$$\max_{\bar{U}_T} u(x,t) > \max_{\partial' U_T} u(x,t)$$

Hint: Let $f = 0$ and $g \in C_0^\infty([-1,1])$, $g \geq 0$ and choose T small.

(b) Let u solve the initial value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} = u_{xx} \text{ in } R \times (0, +\infty) \\ u(x,0) = f(x),\ u_t(x,0) = g(x) \end{cases}$$

where f and g have compact support in R. Let $k(t) = \frac{1}{2}\int u_t(x,t)^2 dx$ be the potential energy and $p(t) = \frac{1}{2}\int u_x^2(x,t) dx$ be the potential energy. Show that

(a) $k(t) + p(t)$ is constant in t; (b) $k(t) = p(t)$ for all large enough time t.

8. Find the explicit formula for the following wave equation

$$u_{tt} - \Delta u = 0, x \in R^3, t > 0; u(x,0) = 0, u_t(x,0) = \begin{cases} 1, \text{ for } |x| < 1; \\ 0, \text{for } |x| > 1 \end{cases}$$

Hint: u is radially symmetric.