MATH 516-101 2019-2020 Homework THREE
Due Date: by 6pm, October 23, 2019

10 points each.

1. Derive a solution formula for the two-dimensional wave equation with source

$$u_{tt} = \Delta u + f(x,t), x \in R^2$$

$$u(x,0) = 0, u_t(x,0) = 0$$

2. Derive a solution formula for the three-dimensional wave equation with radial source

$$u_{tt} = \Delta u + f(r,t), t > 0, r > 0$$

$$u(r,0) = 0, u_t(r,0) = 0$$

3. Find the first order and second order weak derivatives for the following function $u : R \to R$, if exists:

$$(a) \quad u(x) = \begin{cases} 1 - |x|, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases} ; (b) \quad u(x) = |\sin x|.$$

4. Suppose $u : (a,b) \to R$ and the weak derivative exists and satisfies

$$Du = 0 \ a.e. \text{ in } (a,b)$$

Prove that u is constant a.e. in (a,b).

5. Let $1 < p < +\infty$. Show that if $u, v \in W^{1,p}(\Omega)$ then $\max(u,v), \min(u,v) \in W^{1,p}(\Omega)$. Show that this is not true for $W^{2,p}(\Omega)$.

6. Consider the following function

$$u(x) = \frac{1}{|x|^\gamma}$$

in $\Omega = B_1(0)$. Show that if $\gamma + 1 < n$, the weak derivatives are given by

$$\partial_j u = -\gamma \frac{x_j}{|x|^{\gamma+2}}$$

i.e., you need to show rigorously that

$$\int u \partial_j \phi = \int \phi \gamma \frac{x_j}{|x|^{\gamma+2}}$$

For the condition on γ such that $u \in W^{1,p}$ or $u \in W^{2,p}$.

7. Let $\eta(t) = 1$ for $t \le 0$ and $\eta(t) = 0$ for $t > 1$. Let $f \in W^{k,p}(R^n)$ and $f_k = f\eta(|x| - k)$. Show that $\|f_k - f\|_{W^{k,p}} \to 0$ as $k \to +\infty$. As a consequence show that $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

8. Let $u \in C^\infty(\bar{R}^n_+)$. Extend u to Eu on R^n such that

$$Eu = u, x \in R^n_+; Eu \in C^{3,1}(R^n) \cap W^{4,p}(R^n); \|Eu\|_{W^{4,p}} \leq \|u\|_{W^{4,p}}$$

Here $R^n_+ = \{(x', x_n); x_n > 0\}$ and $C^{3,1} = \{u \in C^3, D^\alpha u$ is Lipschitz, $|\alpha| = 3\}$.

9. Let $R^+ = \{x \in R; x > 0\}$ and assume that $u \in W^{2,p}(R^+)$. Define the symmetric extension of u by setting $Eu(x) = u(|x|)$. Prove that $Eu \in W^{1,p}(R)$ but $Eu \notin W^{2,p}(R)$, in general.

10. (a) If $n = 1$ and $u \in W^{1,1}(\Omega)$ then $u \in L^\infty$ and u is continuous. (b) If $n > 1$, find an example of $u \in W^{1,n}(B_1)$ and $u \notin L^\infty$.