MATH 516-101 Homework Four 2019-2020

Due Date: by 6pm, November 8, 2019

1.(10pts) Let $u \in C_c^\infty(\mathbb{R}^n)$. Show that $|u(x_1, ..., x_n)| \le \frac{1}{2}\int_{-\infty}^{\infty} |\partial_1 u(x_1, ..., x_n)| dx_1$.

2. (20pts) (Gagliardo-Nirenberg interpolation inequality) Let $n \ge 2, 1 < p < n$ and $1 \le q < r < \frac{np}{n-p}$. For some $\theta \in (0,1)$ and some constant $C > 0$ we have

$$\|u\|_{L^r(R^n)} \le C\|u\|_{L^q(R^n)}^{1-\theta} \|\nabla u\|_{L^p(R^n)}^{\theta}, \forall u \in C_c^\infty(R^n)$$

(i) Use scaling to find the θ.

(ii) Prove the inequality.

Hint: Do an interpretation of L^r in terms of L^q and $L^{\frac{np}{n-p}}$ and then apply Sobolev.

3. (10pts) Fix $\alpha > 0, 1 < p < +\infty$ and let $U = B_1(0)$. Show that there exists a constant C, depending on n, p and α such that

$$\int_U u^p dx \le C \int_U |\nabla u|^p$$

provided

$$u \in W^{1,p}(U), \ |\{x \in U | u(x) = 0\}| \ge \alpha$$

4. (10pts) (a) Show that $W^{1,2}(R^N) \subset L^2(R^N)$ is not compact. (b). Let $n > 4$. Show that the embedding $W^{2,2}(U) \to L^{\frac{2n}{n-4}}(U)$ is not compact; (b) Describe the embedding of $W^{3,p}(U)$ in different dimensions. State if the embedding is continuous or compact.

5. (20pts) (a) Let $u \in W_r^{1,2} = H_r^1 = \{u \in W^{1,2}(R^n) \mid u = u(r)\}$. Show that $|u(r)| \le C\|u\|_{W^{1,2}} r^{-\frac{n-1}{2}}$. (b) Show that for $n \ge 2$, the embedding $W_r^{1,2} \subset L^p$ is compact when $2 < p < \frac{2n}{n-2}$. (c) Let $u = \mathcal{D}_r^{1,2} = \{\int |\nabla u|^2 < +\infty; u = u(r)\}$. Show that $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ and $|u(r)| \le C\|\nabla u\|_{L^2} r^{-\frac{n-2}{2}}$. However show that $D_r^{1,2} \subset L^{\frac{2n}{n-2}}$ is not compact.

6. (10pts) Let $U = (-1, 1)$. Show that the dual space of $H^1(U)$ is isomorphic to $H^{-1}(U) + E^*$ where E^* is the two dimensional subspace of $H^1(U)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$.

7. (10pts) (a). Assume that U is connected. A function $u \in W^{1,2}(U)$ is a weak solution of the Neumann problem

$$(3) \qquad -\Delta u = f \text{ in } U; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

if

$$\int_U Du \cdot Dv = \int_U fv, \quad \forall v \in W^{1,2}$$

Suppose that $f \in L^2$. Show that (3) has a weak solution if and only if

$$\int_U f = 0$$

(b). Discuss how to define a weak solution of the Poisson equation with Robin boundary conditions

$$(4) \qquad -\Delta u = f \text{ in } U; \ u + \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U$$

8. (10pts) (a) Discuss the definition of weak solutions $u \in H_0^2(\Omega)$ to

$$(*) \quad \Delta^2 u = f \text{ in } \Omega$$

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

(b) Given $f \in L^2(\Omega)$ prove that there exists a unique weak solution to (*).