MATH 516-101, 2019-2020 Homework Five
Due Date: By 6pm on December 3rd,, 2019

10pts each unless otherwise stated.

1. Consider the following convection-diffusion problem

$$-\Delta u + \sum_{j=1}^{n} b_j \partial_j u + cu = f \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

Assume that $f \in L^2(\Omega), b_j \in C^1(\bar{\Omega}), c \in L^\infty$. Show that if $c - \frac{1}{2}\sum_{j=1}^{n} \partial_j(b_j) \geq 0$ then the above problem has a unique weak solution.

2. (20pts) Let Ω be a bounded domain in R^2. Consider the following minimization problem

$$c = \inf_{u \in H_0^1(\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega u^4 + \int_\Omega f(x)u\right)$$

Show that c can be attained and its minimizer is a weak solution

$$\Delta u = u^3 + f(x), \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

Show that the weak solution is also unique.

3. Let $u \in H^1(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$\Delta u = u^3 - f \text{ in } R^n$$

where $f \in L^2$. Prove that $\|D^2 u\|_{L^2(R^n)} \leq C\|f\|_{L^2(R^n)}$.

Hint: mimic the proof of H^2-estimates but without the cut-off function.

4. Assume that $u \in H^1(\Omega)$ is a bounded weak solution of

$$-\sum_{i,j=1}^{n} \partial_j(a^{ij}\partial_i u) = 0 \text{ in } \Omega$$

Show that $\mathbf{u^4}$ is a weak sub-solution.

5. (20pts) Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i} u) + c(x)u = f$$

where $\theta \leq (a^{ij}) \leq C_2 < +\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1), f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \leq \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \leq C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the Moser's iteration procedure.

6. Show that $u = \log|x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x)u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.

7. Let $u \in H_0^1(\Omega)$ be a weak solution of

$$-\Delta u = |u|^{q-1}u \ \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

where $q < \frac{n+2}{n-2}$. Without using Moser's iteration Lemma, use the $W^{2,p}-$ theory only to show that $u \in L^\infty$.

8. Let u be a smooth solution of $Lu = -\sum_{i,j} a^{ij} u_{x_i x_j} = 0$ in U and a^{ij} are C^1 and uniformly elliptic. Set $v := |Du|^2 + \lambda u^2$. Show that

$$Lv \leq 0 \ \text{ in } U, \ \text{ if } \lambda \text{ is large enough}$$

Deduce, by Maximum Principle that

$$\|Du\|_{L^\infty(U)} \leq C\|Du\|_{L^\infty(\partial\Omega)} + C\|u\|_{L^\infty(\partial\Omega)}$$

9. Let u be a smooth function satisfying

$$-\Delta u + u = f(x), |u| \leq 1, \ \text{ in } R^n$$

where

$$|f(x)| \leq Ce^{-|x|}$$

Deduce from maximum principle that u actually decays

$$|u(x)| \leq Ce^{-\frac{1}{2}|x|}$$