$f \in H^m(\Omega)$, $\partial\Omega \in C^{m+2}$

$$\|u\|_{H^{m+2}(\Omega)} \le C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Corollary: $-\Delta u = \mu u$, $u = 0$ on $\partial\Omega$

If $\partial\Omega \in C^\infty \Rightarrow u \in C^\infty$

Local Boundedness

Thm: $\int a_{ij} D_i u D_j \varphi + cu\varphi \le \int f\varphi$, $\varphi \ge 0$, $q > \frac{n}{2}$

Then $\sup_{B_{\frac{1}{2}}} u^+ \le C\left(2^{-n/p}\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right)$

Proof: De Giorgi's Approach

$v = (u-k)^+$, $\eta \in C_0^\infty(B_1)$

$$\int (v\eta)^2 \le c\left(\int v^2 |D\eta|^2 \, |\{v\eta \ne 0\}|^{\varepsilon} + C\, |\{v\eta \ne 0\}|^{1+\varepsilon}\right.$$

Choose $r < R < 1$, $A(k,r) = \{x \in B_r : u \ge k\}$

$$\int_{A(k,r)} (u-k)^2 \le c\left\{\frac{1}{(R-r)^2}|A(k,R)|^{\varepsilon}\int_{A(k,R)}(u-k)^2 + c\,|A(k,R)|^{1+\varepsilon}\right\}$$

We choose $h > k > k_0$ large

$$\int_{A(h,r)} (u-h)^2 \le \int_{A(k,r)} (u-k)^2$$

$$|A(h,r)| = |\{B_r \cap \{u-k > h-k\}| \le \frac{1}{(h-k)^2}\int_{A(k,r)} (u-k)^2$$

$$\int_{A(h,r)} (u-h)^2 \le C\left\{\frac{1}{(R-r)^2}\int_{A(h,R)}(u-h)^2 + (h+F)^2 |A(h,R)|\right\} |A(h,R)|^{\varepsilon}$$

$$\le C\left\{\frac{1}{(R-r)^2} + \frac{(h+F)^2}{(h-k)^2}\right\}\frac{1}{(h-k)^{2\varepsilon}}\left(\int_{A(k,R)}(u-k)^2\right)^{1+\varepsilon}$$

$$\|(u-h)^+\|_{L^2(B_r)} \le C\left(\frac{1}{R-r} + \frac{h+F}{h-k}\right)\frac{1}{(h-k)^{\varepsilon}}\|(u-k)^+\|^{1+\varepsilon}_{L^2(B_R)}$$

Set $\varphi(k,r) = \|(u-k)^+\|_{L^2(B_r)}$, For $\tau = \frac{1}{2}$, $k>0$ to be determ

$$k_\ell = k_0 + k\left(1 - \frac{1}{2^\ell}\right)$$

$$r_\ell = \tau + \frac{1}{2^\ell}(1-\tau)$$

$$k_\ell - k_{\ell-1} = \frac{k}{2^\ell}, \quad r_{\ell-1} - r_\ell = \frac{1}{2^\ell}(1-\tau)$$

$$\varphi(k_\ell, r_\ell) \le c\, C^{\ell}\,[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}$$

$$c = c(k_0, k).$$

$\ell \to +\infty \quad \varphi(k_0+k, \tau) = 0$ #

Method 2. Approach by Moser

For some $k>0$, $m>0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u}, & u<m \\ k+m, & \text{if } u \ge m \end{cases}$$

$$\varphi = \eta^2(\bar{u}_m^{\beta}\bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

$\int(\eta w)$ $\quad D\varphi = \eta^2\bar{u}_m^{\beta}(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^{\beta}\bar{u} - k^{\beta+1})$

$$\int a_{ij} D_i u D_j \varphi \geq \lambda \beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2$$
$$+ \frac{\lambda}{2} \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 - \frac{2\Lambda^2}{\lambda} \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2$$

$$\beta \int \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \int \eta^2 \bar{u}_m^\beta |D\bar{u}|^2$$
$$\leq C \left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int (|c| \eta^2 \bar{u}_m^\beta \bar{u}^2 + |f| \eta^2 \bar{u}_m^\beta \bar{u}) \right.$$
$$\leq C \left\{ \int |D\eta|^2 \bar{u}_m^\beta \bar{u}^2 + \int c_0 \eta^2 \bar{u}_m \bar{u}^2 \right\}$$
$$c_0 = |c| + \frac{|f|}{R}$$

$$w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}, \quad |Dw|^2 \leq (1+\beta) \left\{ \beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2 \right\}$$
$$\int |Dw|^2 \eta^2 \leq C \left\{ (1+\beta) \int w^2 |D\eta|^2 + (1+\beta) \int c_0 w^2 \eta^2 \right\}$$

$$\int |D(w\eta)|^2 \leq C(1+\beta)^\alpha \left( \int (|D\eta|^2 + \eta^2) w^2 \right.$$
$$\left( \int |\eta w|^{2\chi} \right)^{\frac{1}{\chi}} \leq C(1+\beta)^\alpha \left( \int |D\eta|^2 + \eta^2 w^2 \right.$$
$$\left( \int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq \frac{C(1+\beta)^\alpha}{(R-r)^2} \int_{B_R} w^2 \qquad \gamma = \beta + 2$$
$$\left( \int_{B_r} \bar{u}_m^{\gamma\chi} \right)^{\frac{1}{\chi}} \leq C \frac{(\gamma-1)^\alpha}{(R-r)^2} \int_{B_R} \bar{u}^\gamma$$

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \leq \left( \frac{C(\gamma-1)^\alpha}{(R-r)^2} \right)^{\frac{1}{\gamma}} \|\bar{u}\|_{L^\gamma(B_R)}$$

$$\gamma_i = 2\chi^i, \quad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$$
$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|_{L^2(B_1)}$$
$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C^{\sum \frac{i}{\chi^i}} \|\bar{u}\|$$

# Part V. Maximum Principles, Gradient Estimates and Harnack Inequalities

$$Lu = -a^{ij}u_{ij} + b^i u_i + c(x)u$$

## 5.1 Weak Maximum Principle

Thm 1. $c \equiv 0$. Then

(1) If $Lu \le 0$ in $\Omega$, then $\max_{\Omega} u = \max_{\partial\Omega} u$

(2) If $Lu \geq 0$ in $\Omega$, then $\min_{\Omega} u = \min_{\partial\Omega} u$

Proof: Let $v_\varepsilon(x) = u(x) + \varepsilon e^{\lambda x_1}$.

Then $Lv_\varepsilon < 0$

Then $v_\varepsilon$ can't attain its maximum inside $\Omega$

$$\max_{\Omega} v_\varepsilon = \max_{\partial\Omega} v_\varepsilon$$

Now letting $\varepsilon \to 0$.

Thm 2. $c \geq 0$. Then

(1) If $Lu \le 0$ in $\Omega$, then $\max_{\Omega} u^+ = \max_{\partial\Omega} u^+$

(2) If $Lu \geq 0$ in $\Omega$, then $\min_{\Omega} u^- = \min_{\partial\Omega} u^-$

Proof: Let $V = \{u > 0\}$. Then

$$Lu \mp c(x)u \le 0$$

$$\max_{V} u^+ = \max_{\partial V} u^+ \qquad \#$$

Rmk: 1) boundedness

$u(x) = \log |x|$, $\Omega = \{|x| >$

2) $c \geq 0$

$\Delta u + 2u = 0$, $u = \frac{\sin|x|}{|x|}$

## 5.2. Strong Maximum Principle

Lemma (Hopf boundary lemma): Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $c \equiv 0$ in $\Omega$. Assume that

$$Lu \leq 0 \quad \text{in } \Omega$$

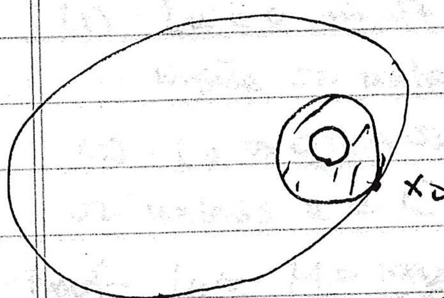
and $\exists\, x^0 \in \partial\Omega$ s.t.

$$u(x^0) > u(x) \quad \text{for all } x \in \Omega$$

Assume $x^0$ has an interior ball $B$, $x^0 \in \partial B$

(i) Then $\frac{\partial u}{\partial \nu}(x_0) > 0$

(ii) If $c \geq 0$ in $\Omega$, the same conclusion holds if $u(x^0) \geq 0$





Proof: Assume $B_r(0) \subset \Omega$, $x_0 \in (\partial B_r(0)) \cap$

We know

$$u(x_0) > u(x) \quad \text{in } B_r(0)$$

$$v(x) = e^{-\lambda |x|^2} - e^{-\lambda r^2} > 0 \quad (x \in B_r(0))$$

$$Lv = -a^{ij} v_{ij} + b^i v_i + cv$$

$$= e^{-\lambda |x|^2} \left(a^{ij}\left(-4\lambda^2 x_i x_j + 2\lambda \delta_{ij}\right)\right)$$
$$- e^{-\lambda |x|^2} b^i 2\lambda x_i + c\left(e^{-\lambda |x|^2} - e^{\lambda r^2}\right)$$

$$\leq e^{-\lambda |x|^2}\left(-4\theta \lambda^2 |x|^2 + 2\lambda \operatorname{tr} A + 2\lambda |b||x| + c\right), \quad c \geq 0$$

Consider $R = B_0(r) \setminus B_0\left(\frac{r}{2}\right)$, $\quad |x| > \frac{r}{2}$

Then on $R$, $Lv < 0$ for $\lambda$ large

Now consider $w(x) = u(x^0) - u(x) + \varepsilon v$

$$L(u) \leq -c\,u(x^0) \leq 0 \quad \text{in } R$$

on $\partial B_r(0)$ $\quad v = 0, \quad u(x) - u(x^0) \leq 0$

By weak M. P.

$$u(x) - u(x^0) + \varepsilon v \le 0 \quad \text{on } \partial R$$

$$\Rightarrow \quad \frac{\partial u}{\partial \nu}(x^0) + \varepsilon \frac{\partial v}{\partial \nu}(x^0) \ge 0.$$

$$\frac{\partial u}{\partial \nu}(x^0) \ge -\varepsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\varepsilon}{r} Dv(x^0)\cdot x^0 = 2\lambda \varepsilon r e^{-\lambda r^2} > 0$$

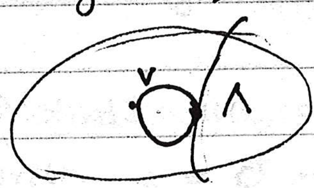
Thm3. (Strong Maximum Principle). Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c \equiv 0$ in $\Omega$. Then

(1) $Lu \le 0$ in $\Omega$ $\Rightarrow$ $u$ can't attain its maximum inside $\Omega$ unless $u \equiv C$

(2) $Lu \ge 0$ in $\Omega$ $\Rightarrow$ $u$ can't attain its minimum inside $\Omega$ unless $u \equiv C$

Proof: Let $M = \max_{\Omega} u$, $\Lambda = \{x \in \Omega \mid u(x) = M\}$. Then if $u \not\equiv M$ the set $V = \{u < M\}$ is not empty

Let $y \in V$, $\operatorname{dist}(y, \Lambda\} < \operatorname{dist}(y, \partial\Omega)$.




$x^0 \in \partial\Lambda$, $u(x^0) > u(x)$ in $B$.

By Hopf's Lemma, $\frac{\partial u}{\partial \nu}(x^0) > 0$. Then $Du(x^0) \equiv 0$

Thm4. (strong M. P. for $c \ge 0$). Assume $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $c \ge 0$ in $\Omega$

Then

(i) $Lu \le 0$ in $\Omega$ $\Rightarrow$ $u$ can't attain its non-negative maximum in $\Omega$ unless $u \equiv C$

Proof: same as before #

Corollary. (Comparison Principle). $c(x) \ge 0$. If $Lu \le Lv$, $u \le v$ on $\partial\Omega$

Then $u < v$ in $\Omega$ or $u \equiv v$ in $\partial\Omega$

<u>Corollary 2.</u> $Lu \le 0$, $c \ge 0$ : Assume that $u$ attains a nonnegative maximum at $x_0 \in \bar{\Omega}$. Then $x_0 \in \partial\Omega$ and

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Applicatons

1). $$\begin{cases} Lu = f \\ \dfrac{\partial u}{\partial \nu} + a(x)\, u = \varphi \end{cases} \qquad (*)$$

Assume that $a(x) \ge 0$, $c(x) \ge 0$. Then $(*)$ has a unique sol'n if $c \not\equiv 0$ or $\alpha \not\equiv 0$. If $c \equiv 0$, $a \equiv 0$, then $u$ is unique upto a Constant.

<u>Thm 4.</u> $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $Lu \le 0$. If $u \le 0$ in $\Omega$, then either $u < 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.

<u>Rmk</u>: No sign condition on $c(x)$ is required

<u>Proof</u>: Method 1. $u(x_0) = 0$ for some $x_0 \in \Omega$. we will prove $u \equiv 0$ in

write $c = c^+ - c^-$. Then

$$Lu = -a^{ij}u_{ij} + b_i u + c^+ u - c^- u \le 0$$

$$-a^{ij}u_{ij} + b_i u + c^+ u \le c^- u \le 0$$

by strong M. P. $\Rightarrow$ $u \equiv 0$

Method 2. Set $v = u e^{-\alpha x_1}$ for some $\alpha > 0$.

$Lu \le 0 \Rightarrow$

$$-a^{ij}v_{ij} + \tilde{b}_i v + (c - a_{11}\alpha^2 - b_1\alpha) v \le 0$$

choose $\alpha$ large so that $c - a_{11}\alpha^2 - b_1\alpha \le 0$. $\Rightarrow$

$$\ldots \le (a_{11}\alpha^2 + b_1\alpha - c)\, v \le 0$$

The next result is the generalized M. P. for $L$ with no restriction on $c(x)$.

Thm 5. Suppose $\exists\, w \in C^2(\Omega) \cap C^1(\bar\Omega)$, $w > 0$, $Lw \geq 0$ in $\Omega$.
Let $u \in C^2(\Omega) \cap C(\bar\Omega)$ satisfies $Lu \leq 0$ in $\Omega$.
Then $\frac{u}{w}$ cannot assume its nonnegative M. P. unless $\frac{u}{w} \equiv$ Constant. If in addition, $\frac{u}{w}$ assumes its nonnegative maximum point at $x_0 \in \partial\Omega$, $\frac{u}{w} \not\equiv c$, then $\frac{\partial}{\partial\nu}\left(\frac{u}{w}\right)(x_0) > 0$.

Proof. Set $v = \frac{u}{w}$. $v$ satisfies

$$-a^{ij} v_{ij} + \tilde{b}\, v_i + \left(\frac{Lw}{w}\right) v \leq 0$$

(Narrow domain principle): If $\Omega \subset \{ |(y-x)\cdot e| < d_0 \}$ for some $d_0$ small. Then $L$ satisfies M. P. in the sense of ?

Proof: $w = e^{\alpha d} - e^{\alpha x_1} > 0$

$$Lw = \cdot\,(a_{11}\alpha^2 + b_1 \alpha) e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1})$$

$$\geq a_{11}\alpha^2 + b_1\alpha + N e^{\alpha d}$$

$\alpha$ large., ($d$ small) #

(A priori estimates) $\begin{cases} Lu = f \\ u = \varphi \end{cases}$, $c(x) \geq 0$

Then $|u(x)| \leq \max_{\partial\Omega} |\varphi| + C \max_{\Omega} |f|$

Proof: $w = e^{\alpha d} - e^{\alpha x_1}$ for $\Omega \subset \{0 < x_1 < d\}$. $\alpha$ large.

# Gradient Estimates (Bernstein)

<u>Thm 1</u> Suppose $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies

$Lu = -a_{ij} u_{ij} + b_i u_i = f(x, u)$ in $\Omega$

$a_{ij}, b_i \in C^1(\bar{\Omega})$, $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then

$$\sup_{\Omega} |Du| \leq \sup_{\partial\Omega} |Du| + C$$

(global gradient estimates)

<u>Thm 2</u> (Interior gradient estimates)

$$\sup_{\Omega'} |Du| \leq C$$

<u>Proof:</u> We compute $L(|Du|^2)$ ← Bernstein

$$L(|Du|^2) \leq -\lambda |D^2u|^2 + C|Du|^2 + C$$

$$L(u^2) \leq -\lambda |Du|^2 + 2uf$$

$$L(|Du|^2 + \alpha u^2) \leq -\lambda |D^2u|^2 - |Du|^2 + C$$

$$L(|Du|^2 + \alpha u^2 + C e^{\beta x_1}) \leq 0$$, Assume that $\Omega \subset \{x_1$

By M. P. This proves global estimates

For interior, we use a cut-off function

$$w = \eta |Du|^2 + \alpha |u|^2 + e^{\beta x_1}$$

$$L(\eta |Du|^2) \leq -\lambda \eta |D^2u|^2 - 2\underline{a_{ij} u_k \eta_i u_{ki}} + C|Du|^2 + L\eta |D$$

$$|a_{ij} u_k \eta_i u_{ki}| \leq \varepsilon |D\eta|^2 |D^2u|^2 + C |Du|^2$$

In fact we can choose

$$\eta = \chi^m$$

$$|D\eta|^2 = \chi^{2(m-1)} |D\chi|^2 \leq c\chi^m \qquad m \geq 2$$