In fact we can choose

$$\eta = \chi^m$$

$$|D\eta|^2 = \chi^{2(m-1)} |D\chi|^2 \le C\chi^m \qquad m \geq 2$$

Finally we see how Bernstein estimates can be used to derive Harnack's Inequality (Li-Yau inequality an important ingredient for Perelman's proof of Poincare Conjecture)

Thm Assume $u \geq 0$ is a C^2 solution of

$$Lu = -a^{ij}u_{ij} + b^i u_i + cu = 0 \quad \text{in } \Omega$$

Then $\forall\, \Omega' \subset\subset \Omega$, $\exists\, C = C(\Omega')$ s.t.

$$\sup_{\Omega'} u \le C \inf_{\Omega'} u$$

Proof: 1. By M.P. ~~We~~ may assume that $u > 0$ in Ω

2. Set $v = \log u \Rightarrow u = e^v \Rightarrow$

$$-(a^{ij}v_{ij} + a^{ij}v_i v_j) + b^i v_i + c = 0 \qquad (1)$$

Let $w = a^{ij}v_i v_j$. Then

$$-a^{ij}v_{ij} = w - b^i v_i - c \qquad (2)$$

We now compute

$$w_{kl} = a^{ij}v_{ki}v_{lj} + 2a^{ij}v_i v_{klj} + R$$

$$|R| \le \varepsilon |D^3 v|^2 + C(\varepsilon)|Dv|^2, \quad R = a^{ij}_{kl}v_i v_j + a^{ij}_k v_{il} + a^{ij}_l v_i v_{kj}$$

Thus

$$-a^{kl} w_{kl} = 2a^{ij} v_j \left(-a^{lk} v_{ikl}\right) - 2a^{ij}a^{kl} v_{ik} v_{jl} - R$$

$$\cancel{-a^{kl} w_{kl} = w_i}$$

Note that

$$-a^{lk} v_{ikl} = w_i - (b^i v_i)_j - c_j$$

$$= w_i + R_i'$$

$$|R_i'| \le C|D^2 v| + |c|$$

$$a^{ij}a^{kl} v_{ik} v_{jl} \ge \theta^2 |D^2 v|^2$$

All together

$$-a^{kl} w_{kl} + \tilde{b}^k w_k \le -\frac{\theta^2}{2} |D^2 v|^2 + C|Dv|^2 + C$$

Let $z = \eta^2 w$

z attains it max at some point $x_0 \in \Omega$

$$2\eta\eta_i w + \eta^2 w_i = 0 \Rightarrow \eta w_i = -2\eta_i w$$

$$0 \le -a^{kl} z_{kl} + \tilde{b}^k z_k$$

$$= -a^{kl}(\eta^2 w)_{kl} + \tilde{b}^k (\eta^2 w)_k$$

$$= \eta^2\left(-a^{kl} w_{kl} + b^k w_k\right) + \hat{R}$$

$$|\hat{R}| \le C\Big(\eta^{\overset{2\eta}{2}} \underbrace{|Dw|}_{|D\eta|} + \eta|D\eta||Dw| + |D^2\eta^2| w\Big)$$

$$\eta^2\left(\frac{\theta}{2}|D^2 w|^2 + C|Dw|^2\right) \le C|D\eta|^2 w \quad +C$$

$$w \le C|D^2 v| + |Dv| + C, \quad w \ge |Dv|^2$$

$$\Rightarrow \eta^2 w^2 \le \eta^2 w + C|D\eta|^2 w + \eta^2 C, \quad \eta = \chi^2.$$

$$\Rightarrow z = \eta w \le C \Rightarrow \max_{\Omega'} w \le \max_{\Omega} z \le C$$

#

Part VI Weak Solutions to Parabolic Equations

In the last part, we consider the existence of weak solutions to

$$\begin{cases} u_t + Lu = f \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0 \\ u(x,0) = g(x) \end{cases}$$

Consider $u(t,\cdot) : [0,T] \to H_0^1(\Omega)$

Formally

$$(u_t(t), v)_{L^2} + a(u(t), v; t) = (f(t), v)_{L^2}, \quad v \in H_0^1(\Omega)$$

where

$$a(u,v;t) = \int_\Omega a^{ij}(x,t)\,\partial_i u\,\partial_j u\,dx + \int_\Omega b^i \partial_i u\, v + \int_\Omega c(x,t) u(x) v(x) dx$$

Assume that $a^{ij}, b^i, c \in L^\infty(\Omega \times (0,T))$

$$f \in L^2(0,T; H^{-1}(\Omega)), \quad g \in L^2(\Omega)$$

We know that $a : H_0^1(\Omega) \times H_0^1(\Omega) \times (0,T) \to \mathbb{R}$

$$C\|u\|^2_{H_0^1} \le a(u,u;t) + \gamma\|u\|^2_{L^2}$$

We first make sense of u_t: Let $w \in L^1(0,T; H_0^1(\Omega))$. Then $u_t = w$ if

$$\int_0^T \phi(t) u(t)\,dt = -\int_0^T \phi'(t) w(t)\,dt, \quad \forall \phi \in C_0^\infty(0,T)$$

Here the integral is vector-valued Lebesgue integrals which are defined in an analogous way to the Lebesgue integral of an integrable real-valued functions as the L^1-limit of integrals of simple functions.

Def: Let $u \in L^1(0,T; X)$. We say $w \in L^1(0,T; X)$ is the weak derivatives of u, written as

$$u' = v$$

provided $\int_0^T \phi'(t) u(t)\,dt = -\int_0^T \phi(t) v(t)\,dt, \quad \forall \phi \in C_0^\infty(0,T)$

Def: $W^{1,p}(0,T; X)$

$$\|u\|_{W^{1,p}(0,T;X)} = \left(\int_0^T (\|u(t)\|^p + \|u'(t)\|^p)\,dt\right)^{\frac{1}{p}}$$

Thm $u \in W^{1,p}(0,T; X)$, $1 \le p \le \infty$. Then

(i) $u \in C([0,T]; X)$

(ii) $u(t) = u(s) + \int_s^t u'(\tau)\,d\tau$

(iii) $\max\limits_{0 \le t \le T} \|u(t)\| \le C \|u\|_{W^{1,p}(0,T;X)}$

Proof: Same as before $u^\varepsilon = \eta_\varepsilon * u$. extend u to 0 outside $(0,T)$.

$u^\varepsilon \to u$ in $L^p(0,T; X)$

$(u^\varepsilon)' \to u'$ in $L^p_{loc}(0,T; X)$

$0 < s < t < T$

Thm2. $u \in L^2(0,T; H_0^1)$, $u' \in (0,T; H^{-1}(\Omega))$

Then (i) $u \in C([0,T]; L^2)$

(ii) $t \to \|u(t)\|^2_{L^2}$ is absolutely continuous.

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\langle u'(t), u(t)\rangle$$

(iii) $\max \|u(t)\|_{L^2} \le C(\|u\|_{L^2(0,T; H_0^1)} + \|u_t\|_{L^2(0,T; H^{-1})})$

Proof · Similarly $u^\varepsilon = \eta_\varepsilon * u$

$$\frac{d}{dt}\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} = 2\langle u^{\varepsilon\prime}(t) - u^{\delta\prime}(t), u^\varepsilon - u^\delta\rangle$$

$$\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} = \|u^\varepsilon(s) - u^\delta(s)\|^2_{L^2} + 2\int_s^t \langle u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau), u^\varepsilon(\tau) - u^\delta(\tau)\rangle d\tau$$

$$\lim_{\varepsilon,\delta\to 0} \sup_{0\le t\le T} \|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2} \le \lim_{\varepsilon,\delta\to 0} \int_0^T \Big(\|u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau)\|^2_{H^{-1}} + \|u^\varepsilon(\tau) - u^\delta(\tau)\|^2_{H_0^1}\Big) d\tau$$

$$= 0$$

uniform converge

Def: We say that a function $u \in L^2(0,T; H_0^1(\Omega))$, $u' \in L^2(0,T; H^{-1})$ is a weak sol'n to $u_t + Lu = f$ is

(i) $\langle u', v\rangle + a(u, v; t) = (f, v)_{L^2}$, $\forall v \in H_0^1(\Omega)$, a.e. $0\le t\le T$

(ii), $u(0) = g$

Existence of weak solutions: Galerkin approximations

Let w_k be a orthogonal basis of $H_0^1(\Omega)$, orthonormal basis of L^2. Take w_k to be normalized eigenfunctions of $L = -\Delta$

For $u_m : [0,T] \to H_0^1(\Omega)$ of the form

$$u_m(t) := \sum d_m^k(t) w_k$$

$$d_m^k(0) = (g, w_k)$$

$$(u_m', w_k) + a(u_m, w_k; t) = (f, w_k), \qquad k = 1, \cdots, m$$

Thm 1. For each $m = 1, 2, \cdots$, $\exists!\, u_m$ satisfying $(*)$

Proof: $(*)$ is

$$d_m^{k\,\prime}(t) + \sum \underbrace{e^{kl}(t)}_{\in L^\infty} d_m^l(t) = f^k(t) \in L^\infty$$

$\dashv$

Now we want to take $m \to +\infty$

Thm 2 (Energy Estimates):

$$\max \|u_m(t)\|_{L^2} + \|u_m\|_{L^2(0,T;H_0^1)} + \|u_m'\|_{L^2(0,T;H^{-1})}$$

$$\leq C\left(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\right)$$

Proof: By estimates before

$$\frac{d}{dt}\left(\|u_m\|_{L^2}^2\right) + 2\beta \|u_m\|_{H_0^1}^2 \leq C_1 \|u_m\|_{L^2}^2 + \|f\|_{L^2}^2$$

Then $\max \|u(t)\|^2_{L^2} \le C(\|g\|_{L^2} + \|f\|^2_{L^2(0,T;L^2)})$

Now integrating from 0 to $T \Rightarrow$

$$\|u_m\|^2_{L^2(0,T;H_0^1)} = \int_0^T \|u_m\|^2_{H_0^1} dt \le C(\|g\|^2_{L^2} + \|f\|^2_{L^2(0,T;H_0^1)}$$

Finally for u_m': For any $v \in H_0^1$, $\|v\|_{H_0^1} \le 1$. we write

$v = v^1 + v^2$, $v^1 \in \operatorname{span}\{w_k\}$, $(v^2, w_k) = 0$, $k = 1, \dots, m$

$$(u_m', v^1) + B[u_m, v^1; t) = (f, v^1)$$

$$\langle u_m', v^1 \rangle = (u_m', v) = (u_m', v^1) = (f, v^1) - B(u_m, v^1; t)$$

$$|\langle u_m', v \rangle| \le C(\|f\|_{L^2} + \|u_m\|_{H_0^1})$$

$$\|v^1\|_{H_0^1} \le 1$$

$$\Rightarrow \|u_m'\|_{H^{-1}} \le C(\|f\|_{L^2} + \|u_m\|_{H_0^1})$$

$$\int_0^T \|u_m'\|^2_{H^{-1}} \le C(\int_0^T \|f\|^2 + \int_0^T \|u_m\|^2)$$

$$\le C(\|g\|^2_{L^2} + \|f\|^2_{L^2(0,T;L^2(\Omega))})$$

Existence and uniqueness.

Thm 3. There exists a weak solution

Proof: 1. $u_m \in L^2(0,T;H_0^1)$, $u_m' \in L^2(0,T;H^{-1})$, bdd $\Rightarrow$

compact sequence

Fix an integer N, choose $v \in C^1([0,T]; H_0^1)$

$$v(t) = \sum_{k=1}^{N} d^k(t)\, w_k$$

$$\int_0^T \langle u_m', v\rangle + a(u_m, v; t) = \int_0^T (f, v)\, dt$$

For $m = m'$, taking a limit we get

$$\int_0^T \langle u', v\rangle + a(u, v; t) = \int_0^T (f, v)$$

This equality then holds for all function $v \in L^2(0,T; H_0^1)$ as functions of $C^1([0,T]; H_0^1)$ are dense.

Hence $\langle u', v\rangle + a[u, v; t] = (f, v)$, $\forall v \in H_0^1$, a.e. $0 \le t \le$

Since $u \in L^2(0,T; H_0^1)$, $u' \in L^2(0,T; H_0^1)$

$$\Rightarrow u \in C([0,T]; L^2)$$

Now we prove $u(0) = g$: By the definition of weak derivatives

$$\int_0^T -\langle v', u\rangle + a[u, v; t] = \int_0^T (f, v)\, dt + (u(0), v(0))$$

for each $v \in C^1([0,T]; H_0^1)$ with $v(T) = 0$.

Similarly for $u = u_m$ we have

$$\int_0^T -\langle v', u_m\rangle + a(u_m, v; t) = \int_0^T (f, v)\, dt + (u_m(0), v(0))$$

$$\xrightarrow{m \to +\infty} \int_0^T -\langle v', u\rangle + a[u, v; t] = \int_0^T (f, v)\, dt + (g, v(0))$$

Thm 4. Uniqueness.

Pf: $\frac{d}{dt} \|u\|_{L^2}^2 \le \gamma \|u\|_{L^2}^2$

Regularity: L^2-theory

Thm (i) Assume that $g \in H_0^1(\Omega)$, $f \in L^2(0,T;L^2)$
$u \in L^2(0,T;H_0^1)$, $u' \in L^2(0,T;H^{-1})$ is
a weak sol'n to $\begin{cases} u_t + Lu = f \\ u = 0 & \text{on } \partial\Omega \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases}$
Then

$u \in L^2(0,T;H^2) \cap L^\infty(0,T;H_0^1)$

$u' \in L^2(0,T;L^2)$

$$\sup_{0\le t\le T} \|u(t)\|_{H_0^1} + \|u\|_{L^2(0,T;H^2)} + \|u'\|_{L^2(0,T;L^2)}$$
$$\le C(\|f\|_{L^2(0,T;L^2)} + \|g\|_{H_0^1})$$

(ii) if in addition, $g \in H^2(\Omega)$, $f' \in L^2(0,T;L^2(\Omega))$
Then
$u \in L^\infty(0,T;H^2(\Omega))$, $u' \in L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1)$
$u'' \in L^2(0,T;H^{-1})$

$$\sup_{0\le t\le T}(\|u(t)\|_{H^2} + \|u'(t)\|_{L^2}) + \|u'\|_{L^2(0,T;H_0^1)}$$
$$+ \|u''\|_{L^2(0,T;H^{-1})} \le C(\|f\|_{H^1(0,T;L^2)} + \|g\|_{H^2}$$

Proof: Multiplying the weak sol'n by $d_m^{k'}$
$(u_m', u_m') + a(u_m, u_m') = (f, u_m')$

$a(u_m, u_m') = \int_\Omega a^{ij} u_{x_i} u'_{x_j} + b^i u_{m_i} u_m' + c u_m u_m'$

$\frac{d}{dt} \frac{1}{2} A[u_m, u_m]) + \varepsilon\|u_m'\|_{L^2} + \frac{C}{\varepsilon}\|u_m\|^2_{H_0^1}$

$$\|u_m'\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2} A[u_m, u_m]\right) \leq \frac{C}{\varepsilon}\left(\|u_m\|^2_{H_0^1} + \|f\|^2_{L^2}\right)$$
$$+ 2\varepsilon \|u_m'\|^2_{L^2}$$

$$\int_0^T \|u_m'\|_{L^2} + \sup_{0 \le t \le T} A[u_m, u_m]$$
$$\leq C\left[A[u_m(0), u_m(0)] + \int_0^T \|u_m\|^2_{H_0^1} + \|f\|^2_{L^2}\right.$$
$$\leq C\left(\|g\|^2_{H_0^1} + \|f\|^2_{L^2(0,T;L^2)}\right)$$

$$\Rightarrow \sup \|u_m\|^2_{H_0^1} \leq C\left(\|g\|^2_{H_0^1} + \|f\|_{L^2(0,T;L^2)}\right)$$

Now $(u', v) + B[u, v] = (f, v)$

$B[u, v] = (h, v)$

$h = f - u', \quad h \in L^2$.

$\Rightarrow u \in H^2(\Omega)$.

$$\|u_m'\|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2} A[u_m, u_m]\right) \le \frac{C}{\varepsilon}\left(\|u_m\|_{H_0^1}^2 + \|f\|_{L^2}^2\right) + 2\varepsilon\|u_m'\|_{L^2}^2$$

$$\int_0^T \|u_m'\|_{L^2} + \sup_{0\le t\le T} A[u_m, u_m]$$

$$\le C\left[A[u_m(0), u_m(0)] + \int_0^T \|u_m\|_{H_0^1}^2 + \|f\|_{L^2}^2\right]$$

$$\le C\left(\|g\|_{H_0^1}^2 + \|f\|_{L^2(0,T;L^2)}^2\right)$$

$$\Rightarrow \sup \|u_m\|_{H_0^1}^2 \le C\left(\|g\|_{H_0^1}^2 + \|f\|_{L^2(0,T;L^2)}\right)$$

Now $(u', v) + B[u, v] = (f, v)$

$B[u, v] = (h, v)$

$h = f - u', \quad h \in L^2$

$\Rightarrow u \in H^2(\Omega)$.

General regularity of weak solutions of parabolic equations

$u_t + Lu = f$

- L^2-theory : $f \in L^2(Q_T)$ Then $u \in W^{1,2}((0,T); H^2(Q_T^\delta))$

$$\|u\|_{W^{1,2}((0,T); H^2(Q_T))} \le C\left(\|u\|_{L^2((0,T); H_0^1)} + \|f\|_{L^2(Q_T)}\right)$$

- Local Boundedness : $(x^0, t_0) \in Q_T$, $Q_R = Q_R(x^0, t_0) = B_R(x^0) \times (t_0 - R^2, t_0] \subset Q_T$

$u_t - \Delta u + c(x,t)u = f(x,t)$

($\cdot$ Lu case, $\|b^2\|_{L^q(Q_T)} + \|c\|_{L^q(Q_T)}$, $q > \frac{n+2}{2}$

$$\sup_{Q_T} u \le \sup_{\partial_p Q_T} u^+ + CF_0 |\Omega|^{\frac{1}{n+2}-\frac{1}{p}}$$

$f^i \in L^p(Q_T)$, $p > n+2$, $\quad f \in L^{\frac{p(n+2)}{n+2+p}}$

$$F_0 = \sum_i \|f^i\|_{L^p} + \|f\|_{L^{\frac{p(n+2)}{n+2+p}}(Q_T)}$$

- L^p theory: $2 \le p < +\infty$

$$\|u\|_{W^{1,p}((0,T)\times W^{2,p}(\Omega))} \le c\left(\|f\|_{L^p(Q_T)} + \|u\|_{W^{1,2}((0,T)\times W^{2}}\right.$$

- Schauder estimate

$$u_t = \Delta u + f$$

$$[u]_{C^{\alpha,\frac{\alpha}{2}}} + [D^2 u]_{C^{\alpha}(Q_{\frac{R}{2}})} + [u_t]_{C^{\frac{\alpha}{2}}(Q_{\frac{R}{2}})} \le c\left(\frac{1}{R^{2+\alpha}}|u|_{L^\infty(Q_R)} + \frac{1}{R^{\alpha}}|f|_{L^\infty} + [f]_{\alpha,\frac{\alpha}{2}}\right.$$

- Sobolev embedding theorems

$Q_T = \Omega \times (0,T)$

$$W_p^{2k,k}(Q_T) = \left(\iint_{Q_T} \sum_{|\alpha|+2r \le 2k} |D^\alpha D_t^r u|^p \, dx\,dt\right)^{\frac{1}{p}}$$

- $W_p^{2,1}(Q_T) \subset L^q(Q_T)$, $1 \le q < +\infty$, $p = \frac{n+2}{2}$

  $\subset L^q(Q_T)$, $\quad 1 \le q \le \frac{(n+2)p}{n+2-2p}$

- $W_p^{2,1}(Q_T) \subset C^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})$, $0 < \alpha \le 2 - \frac{n+2}{p}$

$W_p^{2k,k} \ldots \begin{cases} L^q(Q_T), & 1 \le q \le \frac{(n+2)p}{n+2-2kp}, \quad kp < \frac{n+2}{2} \\ \ldots & kp = \frac{n+2}{2} \end{cases}$