Part II. Sobolev Space

II.1. Distributions and weak derivatives

We denote $L^1_{loc}(R^n)$ the space of locally integrable functions $f: R^n \to R$

The support $(\phi) = \overline{\{x \mid \phi(x) \neq 0\}}$

If $f \in C^1$, $\phi \in C_0^\infty$, then

$$\int Df\,\phi = -\int_R f\,D\phi$$

we note that the right-hand-size is well-define for $\phi \in C_0^\infty(R^n)$.

<u>Def</u>: The weak derivative of f, is a locally integrable function g such that

$$\int_{R^n} g(x)\phi = -\int_{R^n} f(x) D_i\phi$$

we call $g(x) = D_i f$. Note that by arbitrarily changing the function f or g in a measure zero we do not affect the weak derivatives in anyway

<u>Ex.1</u> $f(x) = \begin{cases} 0, & x \le 0 \\ x, & x > 0 \end{cases}$

$$\int f(x)\,\phi'(x)\,dx = -\int_0^\infty \phi(x) = -\int_R H(x)\,\phi(x)$$

$$H(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

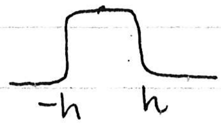
So $H(x) = f'(x)$ a.e.

<u>Ex.2</u> The weak derivative of $H(x)$ doesn't $\exists$

Suppose $g \in L^1_{loc}$ s.t. $\int g\,\phi(x) = -\int H\,\phi'(x) = -\int_0^\infty \phi'(x)\,dx = \phi(0)$

So $\int g(x)\phi(x)\,dx = -\phi(0)$, $\forall \phi \in C_0^\infty$

choose ϕ s.t



then $\int g(x)\phi(x) \le \int_{-h}^{h} |g(x)| := \int_{[h,h]} |g| \to 0$ since $g \in L^1_{loc}$

$\vdash$

Ex.3. $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 2+\sin x, & \text{if } x \text{ is irrational} \end{cases}$

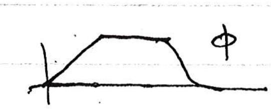
Then $Df = \cos x$.

Ex.4. The cantor function $f: \mathbb{R} \to [0,1]$, defined by

$$f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x \ge 0 \\ \frac{1}{2}, & x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{4}, & x \in [\frac{1}{9}, \frac{2}{9}] \\ \frac{3}{4}, & x \in [\frac{7}{9}, \frac{8}{9}] \end{cases}$$

f is cts but not absolute cts.

We claim that $Df = g$ doesn't $\exists$. If $g\ \exists$, then $g(x) = f'(x) = 0$ on all open intervals, So $g \equiv 0$



Let $\phi(x) = 1$ for $x \in [0,1]$. $\phi(x) = 0$ for $x > 2$. Then

$$\int g(x)\phi(x)\,dx = 0 \ne 1 = -\int f(x)\phi'(x)\,dx$$

Proof: Let $\bar{x} \in \Omega$, choose $\varepsilon < \operatorname{dist}(\bar{x}, \partial\Omega)$

$u_\varepsilon(\bar{x}) = \int u(x)\rho_\varepsilon(\bar{x}-x)\,dx = 0$

Let $\Omega' \subset\subset \Omega$ $\qquad \varepsilon \to 0$

$\Uparrow \quad \|u\|_{L^1(\Omega')} = \|u - u_\varepsilon\|_{L^1(\Omega')} \to 0$

Lemma 1: Weak derivative is unique.

Pf: If $\exists$ $\int g\phi = 0$, $\forall \phi \in C_c^\infty$, $g \in L^1$, then $g = 0$ a.e.

Lemma 2: $D^\alpha(D^\beta f) = D^\beta(D^\alpha f) = D^{\alpha+\beta} f$

Lemma 3: $f_n \to f$ in L^1_{loc}, $D^\alpha f_n \to g$ in L^1_{loc}, then $D^\alpha \cancel{f_n}\ g = D^\alpha f$

Regularizaion and Approximation by Smooth Functions

Let $u \in L^1_{loc}(\Omega)$, define regularization of u as

$$u_h(x) = h^{-n} \int_\Omega \rho\left(\frac{x-y}{h}\right) u(y)\,dy$$

$\forall\ h \in d(x, \partial\Omega)$

It is clear that $u_h \in C^\infty(\Omega')$, $\forall\ \Omega' \subset\subset \Omega$, $h < d(\Omega', \partial\Omega)$

Lemma 1. $u \in C^0(\Omega)$. Then $u_h \to u$ uniformly on $\Omega' \subset\subset \Omega$

Proof:

$$u_h(x) = \int_{|z|\le 1} \rho(z)\, u(x-hz)\,dz$$

$$\sup_{\Omega'} |u_h(x) - u(x)| \le \sup_{\Omega'} \int_{|z|\le 1} \rho(z)\, |u(x) - u(x-hz)|\,dz$$

$$\le \sup_{\Omega'} \sup_{|z|\le 1} |u(x) - u(x-hz)|$$

Since u is uniformly continuous in $B_h(\Omega')$

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Lemma 2. $u \in L^p_{loc}(\Omega)$ or $L^p(\Omega)$, Then $u_h \to u$ in L^p_{loc} or L^p

Proof: Use Hölder

$$|u_h(x)|^p \le \int_{|z|\le 1} \rho(z)\, |u(x-hz)|^p\,dz, \quad \Omega' \subset\subset \Omega,\ 2h < d(\Omega', \Omega)$$

$$\int_\Omega |u_h|^p dx \le \int_{\Omega'} \int_{|z|\le 1} \rho(z)\, |u(x-hz)|^p\,dz\,dx$$

$$= \int_{|z|\le 1} \rho(z)\,dz \int_{\Omega'} |u(x-hz)|^p\,dx$$

$$\le \int_{B_h(\Omega')} |u|^p\,dx$$

Now for $u \in L^p(\Omega)$, $\exists\ w \in C^0(\Omega)$ s.t.

$$\|u - w\|_{L^p(\Omega'_h)} \le \varepsilon$$

By Lemma 1, $\quad \|w - w_h\|_{L^p(\Omega')} < \varepsilon$

$$\|u - u_h\|_{L^p(\Omega')} \le \|u - w\|_{L^p(\Omega')} + \|w - w_h\|_{L^p(\Omega')} + \|u_h - w_h\|_{L^p(\Omega')}$$

$$\le 2\varepsilon + \|u - w\|_{L^p(\Omega'')} \le 3\varepsilon$$

for $h \le h'$.

Hence $u_h \to u$ in $L^p_{loc}(\Omega)$.

The result for $u \in L^p(\Omega)$ can then be obtained by extending u to be zero outside Ω and applying the result for $L^p_{loc}(\mathbb{R}^n)$.

Now we can define weak derivative

$$\int_\Omega g \cdot \phi = (-1)^{|\alpha|} \int f D^\alpha \phi, \quad \forall \phi \in C_0^\infty(\Omega)$$

$f, g \in L^1_{loc}$, then $g = D^\alpha f$

Lemma 3. $\quad u \in L^1_{loc}(\Omega)$, $\quad D^\alpha u$ exists, then for $d(x, \partial\Omega) > h$,

$$D^\alpha u_h(x) = (D^\alpha u)_h(x)$$

Pf: $\displaystyle D^\alpha u_h(x) = h^{-n} \int_\Omega D_x^\alpha \rho\left(\frac{x-y}{h}\right) u(y)\,dy$

$$= (-1)^{|\alpha|}\, h^{-n} \int_\Omega D_y^\alpha \rho\left(\frac{x-y}{h}\right) u(y)\,dy$$

$$= h^{-n} \int_\Omega \rho\left(\frac{x-y}{h}\right) D^\alpha u(y)\,dy$$

$$= (D^\alpha u)_h(x)$$

We then have the following basic approximation theorem

Thm 4 $u, v \in L^1_{loc}(\Omega)$. Then $v = D^\alpha u$ iff $\exists$ a sequence of $C^\infty(\Omega)$ of $\{u_m\}$ converging to u in $L^1_{loc}(\Omega)$

whose derivatives $D^\alpha u_m$ converges to v in $L^1_{loc}(\Omega)$

By Thm 4, many calculas can be extended to weak derivative, e.g. product formula

$$D(uv) = uDv + vDu \ , \quad \forall u, v; \ uv, \ uDv + vDu \in L^1_{loc}(\Omega)$$

Chain Rule

Lem 5. $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$, $u \in W^1(\Omega)$. Then $f \circ u \in W^1(\Omega)$ and $D(f \circ u) = f'(u) Du$

Proof: $u_m \in C^1(\Omega)$, $u_m, Du_m \to u, Du$ respectively.
Then $\forall \Omega' \subset\subset \Omega$,

$$\int_{\Omega'} |f(u_m) - f(u)| \le \sup|f'| \int_{\Omega'} |u_m - u| dx \to 0$$

$$\int_{\Omega'} |f'(u_m) Du_m - f'(u) Du| \le \sup|f'| \int_{\Omega'} |Du_m - Du| dx + \int_{\Omega'} |\underset{\substack{\downarrow \\ \text{DCT}}}{f'(u_m) - f'(u)}| \, |Du| dx$$

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As a consequence, $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$.
$u = u^+ + u^-$, $|u| = u^+ - u^-$

Thm 6. $u \in W^1(\Omega)$, $u^+, u^-, |u| \in W^1(\Omega)$ and

$$Du^+ = \begin{cases} Du, & \text{if } u > 0 \\ 0, & \text{if } u \le 0 \end{cases}$$

$$Du^- = \begin{cases} 0, & \text{if } u \geq 0 \\ Du, & \text{if } u < 0 \end{cases}$$

$$D|u| = \begin{cases} Du, & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -Du & \text{if } u < 0 \end{cases}$$

Proof: For $\varepsilon > 0$, define

$$f_\varepsilon(u) = \begin{cases} (u^2+\varepsilon^2)^{1/2} - \varepsilon, & \text{if } u > 0 \\ 0, & \text{if } u \le 0 \end{cases}$$

$$\int_\Omega f_\varepsilon(u)\, D\phi = -\int_{u>0} \varphi \frac{u\, Du}{(u^2+\varepsilon^2)^{1/2}}\, dx$$

$\varepsilon \to 0$.

$$\int_\Omega u^+ D\phi = -\int_{u>0} \phi\, Du\, dx$$

$$W^{k,p}(\Omega) = \{ u \in W^k(\Omega) \mid D^\alpha u \in L^p(\Omega), \ \forall |\alpha| \le k \}$$

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha| \le k} |D^\alpha u|^p dx \Big)^{1/p}.$$

Convergence in $W^{k,p}(\Omega)$, $W^{k,p}_{loc}(\Omega)$

$$W^{k,p}_0(\Omega) = \overline{C_0^\infty(\Omega)}\Big|_{W^{k,p}}$$

Examples of $W^{k,p}(\Omega)$

Ex. 1. $\Omega = (-1, 1)$, $u = |x|$

$$Du = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}, \quad u \in W^{1,p}(\Omega)$$

But $D^2 u$ doesn't $\exists$

Ex. 2. $\Omega = B_1(0)$, $u(x) = |x|^{-\alpha}$

$u \in W^{1,p}(\Omega)$ iff $\alpha < \frac{n-p}{p}$

Ex. 3. $u(x) = \sum_{k=1}^{+\infty} \frac{1}{2^k} |x - r_k|^{-\alpha}$, $\{r_k\}$ dense in Ω

Then $u \in W^{1,p}(\Omega)$, $\alpha < \frac{n-p}{p}$. Yet u is unbdd in any open set of Ω.

Thm $W^{k,p}(\Omega)$ is a Banach space.

pf: ① $\|u+v\| \le \|u\| + \|v\|$

② complete

$\|u_n - u_m\| \to 0 \Rightarrow \|u_n - u_m\|_{L^p}$, $\|D^\alpha u_n - D^\alpha u_m\|_{L^p}$

$u_n \to u$, $D^\alpha u_n \to g$

Claim $g = D^\alpha u$

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Approximation

Define $u^\varepsilon = \eta_\varepsilon * u$

Then $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$, $\Omega_\varepsilon = \{ d(x, \partial\Omega) > \varepsilon \}$

$u^\varepsilon \to u$ in $W^{k,p}_{loc}(\Omega)$

Proof: $D^\alpha u^\varepsilon = [\eta_\varepsilon * D^\alpha u](x)$

Now choose an open set $V \subset\subset U$, $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$

$$\|u^\varepsilon - u\|^p_{W^{k,p}(V)} \to 0.$$

Thm (Global Approximation by smooth functions, Density Theorem).
Assume that Ω is bdd, $u \in W^{k,p}(\Omega)$ $1 \le p < +\infty$
Then $\exists\ u_n \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ s.t.

$$u_m \to u \text{ in } W^{k,p}(\Omega)$$

Proof: $\Omega = \bigcup_{i=1}^{+\infty} \Omega_i$, $\Omega_i = \{d(x,\partial\Omega) > \frac{1}{i}\}$, $V_i = U_{i+3} - \overline{U}_{i+1}$

Choose $V_0 \subset\subset \Omega$, $\Omega = \bigcup_{i=0}^{+\infty} V_i$

Let $\{\xi_i\}$ be a partition of unit w.r.t. to V_i

$$0 \le \xi_i \le 1$$

$$\sum_{i=0}^{+\infty} \xi_i = 1$$

$$\xi_i u \in W^{k,p}(\Omega), \quad \operatorname{supp}(\xi_i u) \subset V_i$$

Fix $\delta > 0$, choose $\varepsilon_i > 0$, $u^i = \eta_{\varepsilon_i} * (\xi_i u)$

$$\|u^i - \xi_i u\|_{W^{k,p}} \le \frac{\delta}{2^{i+1}}$$

$\operatorname{supp} u^i \subset W_i$, $W_i = U_{i+4} - \overline{V}_i \supset V_i$

$v = \sum u^i$, $\quad u = \sum \xi_i u$.

$$\|v - u\|_{W^{k,p}(V)} \le \sum_{i=0}^{+\infty} \|u^i - \xi_i u\|_{W^{k,p}} \le \delta \sum \frac{1}{2^{i+1}} = \delta$$

Take sup over sets $V \subset\subset U$, $\Rightarrow \|v - u\|_{W^{k,p}(\Omega)} \le \delta$. #