Global approximation of $W^{k,p}(\Omega)$ by $C^\infty(\bar{\Omega})$

<u>Thm</u> Assume that Ω is bounded and $\partial\Omega \in C^1$. Then $\forall u \in W^{k,p}$

$\exists\, u_m \in C^\infty(\bar{\Omega})$ such that

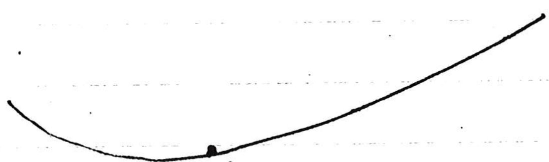
$$\|u_m - u\|_{W^{k,p}} \to 0$$

Proof Fix $x_0 \in \partial\Omega$. $\partial\Omega \in C^1 \Rightarrow \exists\, f \in C^1$, $\partial\Omega$ can be repres

$$x_n = f(x')$$

$$\Omega \cap B(x_0, r) = \{x_n > f(x')\} \cap B(x_0, r)$$

set $V = \Omega \cap B(x^0, \frac{r}{2})$



x_0

Let $x \in V$, define $x^\varepsilon = x + \lambda \varepsilon e_n$, $\varepsilon > 0$

Now define $u_\varepsilon(x) = u(x^\varepsilon)$, $x \in V$, $v^\varepsilon = \rho_\varepsilon * u_\varepsilon$

Then $v^\varepsilon \in C^\infty(\bar{V})$

Now we claim $v^\varepsilon \to u$ in $W^{k,p}(V)$

$$\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} \le \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} + \|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)}$$

The first term $\to 0$, second term $\to 0$ by L^p-continuity

In general case, we can streghten the $\partial\Omega$ by finite many deformation and then we can use a partition of unit as before. #

Extensions.

<u>Thm</u> Ω is bounded, $\partial\Omega \in C^1$. Let $\Omega_1 \supset\supset \Omega$. Then

$\exists\, E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$

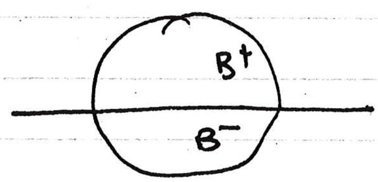
such that $\forall u \in W^{1,p}(\Omega)$

(1) $Eu = u$ a.e. in Ω

(2) Eu has support in V

(3) $\|Eu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\Omega)}$

Proof: 1) $\partial\Omega$ is half-space



$$Eu = \begin{cases} u(x) \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}) \end{cases}$$

Then $Eu \in C^1(B)$.

and $\|Eu\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}$

$Eu \notin C^2$ but

$\|Eu\|_{W^{2,p}(B)} \leq C\|u\|_{W^{2,p}(B^+)}$

2) Let $y = \Phi(x)$ be the boundary deformation straighten

$u' = u(\Psi(y))$

$W = \Phi(B)$, $\bar{u} = Eu'$

$\|\bar{u}\|_{W^{1,p}(W)} \leq C\|u\|_{W^{1,p}(\Omega)}$

3) partition of unit

In general, $W^{k,p}(\mathbb{R}^n_+)$ extension

$$E_0 u(x) = \begin{cases} u(x), & x_n > 0 \\ \sum_{i=1}^{k} c_i u(x', -\frac{x_n}{i}), & x_n < 0 \end{cases}$$

where $\sum_{i=1}^{k} c_i (-\frac{1}{i})^m = 1, \quad m = 0, \cdots, k-1$

$\partial\Omega \in C^{k-1}$,

$$Eu = u\eta_0 + \sum_{j=1}^{N} E_0 [(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j$$

Thm: Let Ω be a $C^{k-1,1}$ domain in $\mathbb{R}^n$, $k \geq 1$, Then (i) $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$, $1 \leq p < +\infty$
and (ii) $\forall \Omega' \supset\supset \Omega$, $\exists E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ s.t.

$Eu = u$ in Ω and

$$\|Eu\|_{k,p;\Omega'} \leq C\|u\|_{k,p;\Omega}$$

Proof: Let us first consider density result for $\mathbb{R}^n_+$.

$$v_h(x) = u_h(x + 2he_n) = h^{-n} \int_{y_n > 0} u(y) \rho\left(\frac{x + 2he_n - y}{h}\right) dy, \quad h > 0$$

$v_h \to u$ in $W^{k,p}_+(\mathbb{R}^n_+)$.

Trace Theorem: Assume that Ω is bounded and $\partial\Omega \in C^1$. Then there exists a bounded linear operator

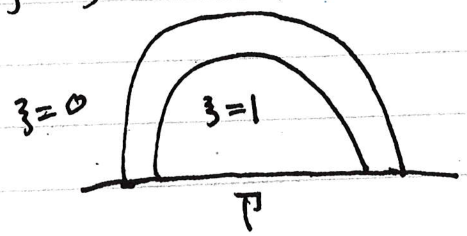
$$T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) $Tu = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$

Proof: 1) Model Problem: $u \in C^1(\bar{\Omega})$.



$$\int_\Gamma |u|^p dx' \le \int_\Gamma |u|^p \eta(x',0)\, dx' = -\int_{B^+} (\eta |u|^p)_{x_n} dx$$

$$\le C \int_{B^+} |u|^p + \int_{B^+} |Du|^p$$

2) By a change of variable, $x_0 \in \partial\Omega$

$$\int_\Gamma |u|^p \le C \int_\Omega |u|^p + \int_\Omega |Du|^p$$

3) Partitition of Unit $\quad \partial U = \bigcup_{i=1}^N \Gamma_i$

$$\|u\|_{L^p(\Gamma_i)} \le C\|u\|_{W^{1,p}(\Omega)}$$

$$Tu = u|_{\partial\Omega}, \quad \|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

4) By density Theorem
$\exists\, u_m \in C^\infty(\bar{\Omega})$, $\Rightarrow$ extension to $B_1(\bar{\Omega})$, $u_m = (Eu) * \rho_{\frac{1}{m}} \rightarrow u$ in $C^0(\bar{\Omega})$

$$\|Tu_m - Tu_\ell\|_{L^p(\partial\Omega)} \le C\|u_m - u_\ell\|_{W^{1,p}}$$

$$\{Tu_m\} \rightarrow \text{ in } L^p(\partial\Omega). \qquad \#$$

Characterization of $W_0^{1,p}(\Omega)$

Theorem 2 (Trace-zero functions in $W^{1,p}(\Omega)$). Assume that Ω is bounded and $\partial\Omega \in C^1$, $u \in W^{1,p}(\Omega)$. Then
$u \in W_0^{1,p}(\Omega)$ iff $Tu = 0$ on $\partial\Omega$

Proof: "$\Rightarrow$". $u \in W_0^{1,p}(\Omega)$, $u_m \in C_0^\infty(\Omega)$, $u_m \to u$ in $W^{1,p}(\Omega)$

$Tu_m \to Tu$

so $Tu = 0$ on $\partial\Omega$

"$\Leftarrow$" Assume that $Tu=0$ on $\partial\Omega$. Then we want to construct a $C_0^\infty(\Omega)$ function u_m, $u_m \to u$ in $W^{1,p}$

$Tu=0$ on $\partial\Omega$ $\Rightarrow$ Use partition of unit and flatten the boundary as usual,

$u \in W^{1,p}(\mathbb{R}^n_+)$, $Tu=0$ on $\partial\mathbb{R}^n_+$

u has compact support in $\overline{\mathbb{R}^n_+}$

$Tu=0 \Rightarrow \exists\, u_m \in C^1(\overline{\mathbb{R}^n_+})$

$u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$

$Tu_m \to 0$ in $L^p(\mathbb{R}^{n-1})$

$$|u_m(x',x_n)| \le |u_m(x',0)| + \int_0^{x_n} |u_{m,x_n}(x',t)|\,dt$$

$$\int_{\mathbb{R}^{n-1}} |u_m(x',x_n)|^p dx' \le \int_{\mathbb{R}^n} |u_m(x',0)|^p + \int_0^{x_n} C \int_{\mathbb{R}^{n-1}} |u_{m,x_n}(x',t)|^p)\, x_n^{p-1}\, dx'dt$$

$$\int_{\mathbb{R}^{n-1}} |u(x',x_n)|^p dx' \le x_n^{p-1} \int_0^{x_n} C \int_{\mathbb{R}^{n-1}} |\nabla u|^p dx'dt$$

Now let $\eta \in C^\infty(\mathbb{R}_+)$, $\eta \equiv 1$ on $[0,1]$, $\eta = 0$, $t>2$

$v_m(x) = u(x)(1-\zeta(mx_n))$, $|\nabla v_m| \le |\nabla u|(1-\zeta(mx_n)) + m|\nabla\zeta|\cdot|u|$

$$\int_{\mathbb{R}^n_+} |Du_m - Du|^p \le \int_{\mathbb{R}^n_+} |\zeta(mx_n)|^p |Du|^p + Cm^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx'dt$$

- $\int_{\mathbb{R}^n_+} |\zeta_m|^p |Du|^p dx \le \int_{\mathbb{R}^{n-1}\times[0,\frac{1}{m}]} |Du|^p dx \to 0$

- $m^p \int_0^{2/m}\int_{\mathbb{R}^{n-1}} |u|^p dx'dt$

$$\le m^p \left(\int_0^{2/m} t^{p-1} dt\right)\left(\int_0^{2/m}\int_{\mathbb{R}^{n-1}} |Du|^p dx'dx_n\right)$$

$$\le C\int_0^{2/m}\int_{\mathbb{R}^{n-1}} |Du|^p dx'dx_n \to 0 \text{ as } m\to+\infty$$

Next we present higher integrability of Sobolev Spaces – Embedding theorems. There are two types

1. Gagliardo-Nirenberg: If $p<n$, then $W^{1,p}(\mathbb{R}^n)\hookrightarrow L^{p^*}$, $p^*=\frac{n}{n}$

2. Morrey: If $p>n$, $u\in W^{1,p}(\mathbb{R}^n)$ is Hölder continuous (after a modification ~~off~~ a set of measure zero.

We consider G-N inequality first. Let $p^*=\frac{np}{n-p}>p$

$$\frac{1}{p^*}+\frac{1}{n}=\frac{1}{p}$$

As a preliminary, we describe a useful application of generalized Hölder inequality. Let $g_1,\cdots,g_{n-1}\in L^1(\Omega)$. Then $g_i^{\frac{1}{n-1}}\in L^{n-1}(\Omega)$

$$\int_\Omega g_1^{\frac{1}{n-1}} g_2^{\frac{1}{n-1}}\cdots g_{n-1}^{\frac{1}{n-1}} ds \le \prod_{i=1}^{n-1}\|g_i^{\frac{1}{n-1}}\|_{L^{n-1}} = \prod_{i=1}^{n-1}\|g_i\|_{L^1}^{\frac{1}{n-1}}$$

Proof of G-N inequality:

Thm: Assume $1\le p<n$. Then $\|f\|_{L^{p^*}}\le C\|\nabla f\|_{L^p(\mathbb{R}^n)}$, $\forall f\in C_c^1($

p^* = scaling invariance

$$f(x_1, \ldots, x_i, \ldots, x_n) = \int_{-\infty}^{x_i} D_{x_i} f(x_1, \cdots, s_i, \cdots, x_n)\, ds_i$$

$$|f(x_1, \cdots, x_n)| \le \int_{-\infty}^{+\infty} |D_{x_i} f(x_1, \cdots, s_i, \cdots, x_n)|\, ds_i, \qquad 1 \le i \le n$$

$$|f(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} |D_{x_i} f(x_1, \cdots, s_i, \cdots, x_n)|\, ds_i \right)^{\frac{1}{n-1}}$$

$$\int_{-\infty}^{+\infty} |f(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{-\infty}^{+\infty} |D_{x_1} f|\, ds_1 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{+\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} |D_{x_i} f|\, ds_i \right)^{\frac{1}{n-1}} dx_1 \right.$$

$$\le \left(\int_{-\infty}^{+\infty} |D_{x_1} f|\, ds_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |D_{x_i} f|\, ds_i\, dx_1 \right)^{\frac{1}{n-1}}$$

(generalized Hölder)

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x)|^{\frac{n}{n-1}} dx_1 dx_2 \le \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |D_{x_2} f|\, dx_2 dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |D_{x_1} f|\, ds_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=3}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |D_{x_i} f|\, dx_i dx_1 \right. \ dx_2$$

$$\le \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |D_{x_2} f|\, dx_2 dx_1 \right)^{\frac{1}{n-1}} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |D_{x_1} f|\, ds_1 dx_2 \right)^{\frac{1}{n-1}} \cdot \left(\prod_{i=3}^{n} \iiint \underset{ds_i dx_1 dx_2}{\uparrow} \right)^{\frac{1}{n-1}}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int |f(x)|^{\frac{n}{n-1}} dx_1 dx_2 \le \prod_{i=1}^{n} \left(\int \cdots \int |D_{x_i} f|\, dx_1 \cdots dx_n \right)^{\frac{1}{n-1}}$$

$$\le \left(\int |\nabla f| \right)^{\frac{n}{n-1}}$$

This proves $p=1$

For $p>1$, $\quad g = |f|^\beta$, $\quad \beta = \frac{p(n-1)}{n-p}$

$$\left(\int_{R^n} |f|^{\frac{\beta n}{n-1}} dx \right)^{\frac{n-1}{n}} \le \int \beta |f|^{\beta-1} |\nabla f|\, dx$$

$$\le \beta \left(\int |f|^{\frac{(\beta-1)p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int |\nabla f|^p \right)^{\frac{1}{p}}$$

$$\frac{(\beta-1)p}{p-1} = \frac{\beta n}{n-1} = \frac{np}{n-p} = p^*$$

Corollary (Embedding): Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with C^1 boundary, $1 \le p < n$. Then $\forall q \in [1, p^*]$, $p^* = \frac{np}{n-p}$

$$\|f\|_{L^q(\Omega)} \le C \|f\|_{W^{1,p}(\Omega)}$$

Proof: Let $\tilde{\Omega} = \{x \in \mathbb{R}^n \mid d(x, \Omega) < 1\}$ be an open nbhd around Ω
By extension theorem, $E: W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^n)$

$$\|f\|_{L^q(\Omega)} \le C_1 \|f\|_{L^{p^*}(\Omega)} \le C \|Ef\|_{L^{p^*}(\mathbb{R}^n)} \le C_3 \|f\|_{W^{1,p}(\Omega)}$$

Morrey's Estimate

Theorem 4. $n < p \le +\infty$, $u \in W^{1,p}(\mathbb{R}^n)$. Then

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\gamma = 1 - \frac{n}{p}$$

Proof: 1. We first claim that

$$\fint_{B(x,r)} |u(y) - u(x)|\, dy \le C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

Let $0 < s < r$

$$|u(x+s\omega) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x+t\omega)\, dt \right|$$

$$= \left| \int_0^s Du(x+t\omega) \cdot \omega\, dt \right|$$

$$\le \int_0^s |Du(x+t\omega)|\, dt$$

$$\int_{\partial B(0,1)} |u(x+s\omega) - u(x)|\, dS_\omega \le \int_0^s \int_{\partial B(0,1)} |Du(x+t\omega)|\, dS(\omega)\, dt$$

$$= \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}}\, dS_y\, dt = \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

$$\le \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

$$\int_{\partial B(x,s)} |u(z)-u(x)|\, dS_z \le s^{n-1} \int_{B(x,r)} \frac{Du(y)}{|x-y|^{n-1}}\, dy$$

$$\int_{B(x,r)} |u(y)-u(x)|\, dy \le \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

2. Fix $x \in \mathbb{R}^n$,

$$|u(x)| \le ⨍_{B(x,1)} |u(x)-u(y)|\, dy + ⨍_{B(x,1)} |u(y)|\, dy$$

$$\le c\int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy + C\|u\|_{L^p}$$

$$\le c\left(\int |Du|^p\right)\left(\int \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}} + C\|u\|_{L^p}$$

$$\le C\|u\|_{W^{1,p}}$$

3. $x \neq y$, $r = |x-y|$, $W = B(x,r) \cap B(y,r)$

$$|u(x)-u(y)| \le ⨍_W |u(x)-u(z)|\, dz + ⨍_W |u(y)-u(z)|\, dz$$

$$\le c ⨍_{B(x,r)} |u(x)-u(z)|\, dz + ⨍_W |u(y)-u(z)|\, dz$$

$$\le c ⨍_{B(x,r)} \frac{|Du|}{|x-z|^{n-1}} + \cdots$$

$$\le c\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|Du\|_{L^p} \le C r^{1-\frac{n}{p}} \|Du\|_{L^p}$$

$$|u(x)-u(y)| \le C\,|x-y|^{1-\frac{n}{p}}\,\|Du\|_{L^p}$$

4. Combining L^∞ and Hölder

$$\|u\|_{C^{0,1-\frac{n}{p}}} \le C\,\|Du\|_{L^p}$$

Theorem 5. $\partial\Omega \in C^1$, $n<p\le+\infty$, $u\in W^{1,p}(\Omega)$

Then $\exists$ an equivalent class of u, called u^* s.t.

$$\|u^*\|_{C^{0,\gamma}} \le C\,\|u\|_{W^{1,p}}$$

Proof: By extension, $Eu \in W^{1,p}(R^n)$, Eu has compact support

$u_m = (Eu)_m \in C_0^\infty(R^n)$

$$u_m \to \bar{u} \quad \text{in } W^{1,p}(R^n)$$

$$\|u_m-u_n\|_{C^{0,\gamma}} \le C\,\|u_m-u_l\|_{W^{1,p}}$$

So $\exists\, u^* \in C^{0,1-\frac{n}{p}}$ s.t

$$u_m \to u^* \text{ in } C^{0,1-\frac{n}{p}}.$$

#

General Sobolev inequalities

Thm 6. $W^{k,p} \longrightarrow$
- L^q, $\frac{1}{q}=\frac{1}{p}-\frac{k}{p}$, $kp<n$
- $C^{k-[\frac{n}{p}]-1,\gamma}$, $\gamma=[\frac{n}{p}]+1-\frac{n}{p}$, if $\frac{n}{p}$ is not an integer; $\gamma<1$, if $\frac{n}{p}$ is an integer
- $\int e^{c|u|^2} <+\infty$ Orlicz space, if $kp=n$.

$u \in W_0^{1,n}(\Omega)$, $\displaystyle\int_\Omega \exp\left(\frac{|u|}{c_1 \|Du\|_n}\right)^{\frac{n}{n-1}} \le c_2 |\Omega|$

Compactness