If the domain $\Omega \subset I\!R^n$ is bounded, then $\mathbf{L}^q(\Omega) \subseteq \mathbf{L}^{p^*}(\Omega)$ for every $q \in [1, p^*]$. Using the Gagliardo-Nirenberg inequality we obtain

Corollary 6.2 (embedding). *Let $\Omega \subset I\!R^n$ be a bounded open domain with C^1 boundary, and assume $1 \leq p < n$. Then, for every $q \in [1, p^*]$ with $p^* \doteq \frac{np}{n-p}$, there exists a constant C such that*

$$\|f\|_{\mathbf{L}^q(\Omega)} \leq C\,\|f\|_{W^{1,p}(\Omega)} \qquad \textit{for all } \ f \in W^{1,p}(\Omega). \tag{6.27}$$

Proof. Let $\widetilde{\Omega} \doteq \{x \in I\!R^n\,;\ \ d(x, \Omega) < 1\}$ be the open neighborhood of radius one around the set Ω. By Theorem 4.3 there exists a bounded extension operator $E : W^{1,p}(\Omega) \mapsto W^{1,p}(I\!R^n)$, with the property that Ef is supported inside $\widetilde{\Omega}$, for every $f \in W^{1,p}(\Omega)$. Applying the Gagliardo-Nirenberg inequality to Ef, for suitable constants C_1, C_2, C_3 we obtain

$$\|f\|_{\mathbf{L}^q(\Omega)} \leq C_1\,\|f\|_{\mathbf{L}^{p^*}(\Omega)} \leq C_2\,\|Ef\|_{\mathbf{L}^{p^*}(I\!R^n)} \leq C_3\|f\|_{W^{1,p}(\Omega)}\,.$$

6.3 High order Sobolev estimates

Let $\Omega \subset I\!R^n$ be a bounded open set with C^1 boundary, and let $u \in W^{k,p}(\Omega)$. The number

$$k - \frac{n}{p}$$

will be called the **net smoothness** of u. As in Fig. 10, let m be the integer part and let $0 \leq \gamma < 1$ be the fractional part of this number, so that

$$k - \frac{n}{p} = m + \gamma\,. \tag{6.28}$$

In the following, we say that a Banach space X is **continuously embedded** in a Banach space Y if $X \subseteq Y$ and there exists a constant C such that

$$\|u\|_Y \leq C\,\|u\|_X \qquad \text{for all } \ u \in X\,.$$

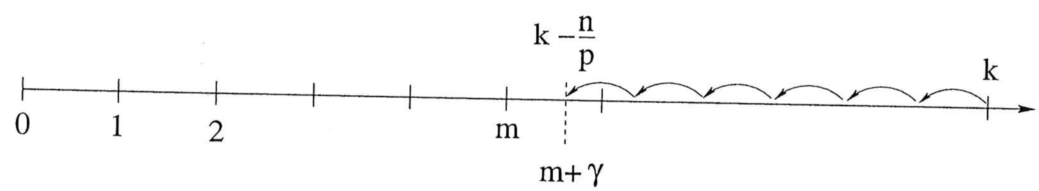

Figure 10: Computing the "net smoothness" of a function $f \in W^{k,p} \subset \mathcal{C}^{m,\gamma}$.

Theorem 6.3 (general Sobolev embeddings). *Let $\Omega \subset I\!R^n$ be a bounded open set with C^1 boundary, and consider the space $W^{k,p}(\Omega)$. Let m, γ be as in (6.28). Then the following continuous embeddings hold.*

(i) If $k - \frac{n}{p} < 0$ *then* $W^{k,p}(\Omega) \subseteq \mathbf{L}^q(\Omega)$, *with* $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} = \frac{1}{n}\left(\frac{n}{p} - k\right)$.

(ii) If $k - \frac{n}{p} = 0$, *then* $W^{k,p}(\Omega) \subseteq \mathbf{L}^q(\Omega)$ *for every* $1 \leq q < \infty$.

(iii) If $m \geq 0$ *and* $\gamma > 0$, *then* $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m,\gamma}(\Omega)$.

(iv) If $m \geq 1$ *and* $\gamma = 0$, *then for every* $0 \leq \gamma' < 1$ *one has* $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m-1,\gamma'}(\Omega)$.

Remark 6.1 Functions in a Sobolev space are only defined up to a set of measure zero. More precisely, by saying that $W^{k,p}(\Omega) \subseteq \mathcal{C}^{m,\gamma}(\Omega)$ we mean the following. For every $u \in W^{k,p}(\Omega)$ there exists a function $\tilde{u} \in \mathcal{C}^{m,\gamma}(\Omega)$ such that $\tilde{u}(x) = u(x)$ for a.e. $x \in \Omega$. Moreover, there exists a constant C, depending on k, p, m, γ but not on u, such that

$$\|u\|_{\mathcal{C}^{m,\gamma}(\Omega)} \leq C\,\|\tilde{u}\|_{W^{k,p}(\Omega)}.$$

Proof of the theorem. 1. We start by proving (i). Assume $k - \frac{n}{p} < 0$ and let $u \in W^{k,p}(\Omega)$. Since $D^\alpha u \in W^{1,p}(\Omega)$ for every $|\alpha| \leq k-1$, the Gagliardo-Nirenberg inequality yields

$$\|D^\alpha u\|_{L^{p^*}(\Omega)} \leq C\,\|u\|_{W^{k,p}(\Omega)} \qquad |\alpha| \leq k-1.$$

Therefore $u \in W^{k-1,p^*}(\Omega)$, where p^* is the Sobolev conjugate of p.

This argument can be iterated. Set $p_1 = p^*$, $p_2 \doteq p_1^*$, $\ldots$, $p_j \doteq p_{j-1}^*$. By (6.17) this means

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{n}, \qquad \ldots \qquad \frac{1}{p_j} = \frac{1}{p} - \frac{j}{n},$$

provided that $jp < n$. Using the Gagliardo-Nirenberg inequality several times, we obtain

$$W^{k,p}(\Omega) \subseteq W^{k-1,p_1}(\Omega) \subseteq W^{k-2,p_2}(\Omega) \subseteq \cdots \subseteq W^{k-j,p_j}(\Omega). \tag{6.29}$$

After k steps we find that $u \in W^{0,p_k}(\Omega) = \mathbf{L}^{p_k}(\Omega)$, with $\frac{1}{p_k} = \frac{1}{p} - \frac{k}{n} = \frac{1}{q}$. Hence $p_k = q$ and (i) is proved.

2. In the special case $kp = n$, repeating the above argument, after $k-1$ steps we find

$$\frac{1}{p_{k-1}} = \frac{1}{p} - \frac{k-1}{n} = \frac{1}{n}.$$

Therefore $p_{k-1} = n$ and

$$W^{k,p}(\Omega) \subset W^{1,n}(\Omega) \subseteq W^{1,n-\varepsilon}(\Omega)$$

for every $\varepsilon > 0$. Using the Gagliardo-Nirenberg inequality once again, we obtain

$$u \in W^{1,n-\varepsilon}(\Omega) \subseteq \mathbf{L}^q(\Omega) \qquad q = \frac{n(n-\varepsilon)}{n-(n-\varepsilon)} = \frac{n^2 - \varepsilon n}{\varepsilon}.$$

$u \in W_0^{1,n}(\Omega)$, $\displaystyle\int_\Omega \exp\left(\frac{|u|}{C_1\|Du\|_n}\right)^{\frac{n}{n-1}} \le C_2|\Omega|$

Compactness

Let X and Y be two Banach space. We say that X is compactly embedded in Y, written

$$X \subset\subset Y \text{ or } X \hookrightarrow Y$$

If (i) $\|u\|_Y \le C\|u\|_X$

(ii) ~~each~~ bdd sequence in X is ~~precompact~~ precompact in Y, i.e. $\exists$ a convergent subsequence in Y

Theorem: Assume that Ω is ~~compact~~ bounded open, $\partial\Omega \in C^1$, $1 \le p < n$

Then $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$, $\forall\ 1 \le q < p^* = \frac{pn}{n-p}$

Proof: By extension, assume that $\Omega = R^n$, u_m compact support in bounded $V \subset R^n$. $\sup_m \|u_m\|_{W^{1,p}(V)} < +\infty$

Let us consider $u_m^\varepsilon = \eta_\varepsilon * u_m$, $\varepsilon > 0$

Claim 1: $u_m^\varepsilon \to u_m$ in $L^q(V)$ as $\varepsilon \to 0$, uniformly in m

Proof: $u_m^\varepsilon(x) - u_m(x) = \displaystyle\int_{B(0,1)} \eta(y)\left(u_m(x-\varepsilon y) - u_m(x)\right)dy$

$$= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}\left(u_m(x-\varepsilon t y)\right)dt\,dy$$

$$= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon t y)\cdot y \, dt\, dy$$

$$\int_V |u_m^\varepsilon - u_m(x)|\, dx \le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\varepsilon t y)|\, dx\, dt\, dy$$

$$\le \varepsilon \int_V |Du_m(z)|\, dz$$

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon \|Du_m\|_{L^1(V)} \le \varepsilon C \|Du_m\|_{L^p(\Omega)}$$

So $u_m^\varepsilon \to u_m$ in $L^1(V)$, uniformly in m

in $L^q(V)$,

Next

Claim 2. For each fixed $\varepsilon > 0$, u_m^ε is uniformly bdd and equicontinuo

Indeed, $|u_m^\varepsilon(x)| \le \|\eta_\varepsilon\|_{L^\infty} \|u_m\|_{L^1} \le \frac{C}{\varepsilon^n} < +\infty$

$$|Du_m^\varepsilon(x)| \le \frac{C}{\varepsilon^{n+1}} < +\infty$$

Now. Fix $\delta > 0$, choose $\varepsilon > 0$ s.t.

$\|u_m^\varepsilon - u_m\|_{L^q(V)} < \delta/2$, uniformly in m by claim 1

Now $u_m^\varepsilon(x)$ have compact support in V, using claim 2.

$\exists\ u_{m_j}^\varepsilon$, converges uniformly

$$\lim_{j,k\to\infty} \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q(\Omega)} = 0$$

Hence $\lim\limits_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \delta$

Now let $\delta = \frac{1}{2^k}$, by standard diagonal process $\exists\, u_{m_\ell}$

s.t. $$\limsup_{\ell, k \to \infty} \|u_{m_\ell} - u_{m_k}\|_{L^q(\Omega)} = 0$$

Remark 1. $q = p^*$, no compactness

2. $W_0^{1,p}(U) \subset\subset L^p(U)$, even if ∂U is not C^1

Consequence of Compactness:

1) Poincare Inequality

$$\|u - \bar{u}\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)}$$

Proof: By contradiction $\Rightarrow u_k \to v$

$$\bar{v} = 0, \quad \int |\nabla v|^p = 0, \quad \|v\|_{L^p} = 1$$

Claim: $Dv = 0$ a.e., Ω connected $\Rightarrow v \equiv$ Constant

Proof: 1. $\varepsilon > 0$, $u_\varepsilon = \eta_\varepsilon * u$. $\Rightarrow Du_\varepsilon = 0$ in $\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) >$

So $u_\varepsilon \equiv C_\varepsilon$ in each connected component of Ω_ε

2. Consider $x, y \in \Omega$. Since Ω is connected, $\exists$ a polygonal path Γ joining x with y and remaining inside Ω. Let $\delta = \min_{z \in \Gamma} d(z, \partial\Omega)$. Then $\varepsilon < \delta$, Γ is in $\Omega_\varepsilon \Rightarrow u_\varepsilon(x) = u_\varepsilon(y)$

3. Call $\hat{u}(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x)$. $\hat{u}$ is a constant in Ω. Moreover $\hat{u}(x) = u(x)$ for every Lebesgue point of u, hence a.e. in Ω #

Corollary 1

$$\| u - \bar{u}_{B(x,r)} \|_{L^p} \le C r \|Du\|_{L^p(B(x,r))}$$

Scaling.

2). The space H^{-1}

$H^{-1} = (H_0^1(\Omega))^*$

$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$

$\|f\|_{H^{-1}(\Omega)} = \sup \{ \langle f, u \rangle \mid u \in H_0^1(\Omega), \ \|u\|_{H_0^1(\Omega)} \le 1 \}$

Theorem (Characterization of H^{-1}).

(i) Assume $f \in H^{-1}(\Omega)$. Then $\exists\, f^0, f^1, \dots, f^n$ in $L^2(\Omega)$ s.t.

$$\langle f, v \rangle = \int_\Omega f^0 v + \sum_{i=1}^n f^i v_{x_i}, \quad v \in H_0^1(\Omega) \qquad (1)$$

(ii) $\|f\|_{H^{-1}(\Omega)} = \inf \{ (\int_\Omega \sum_{i=0}^n |f^i|^2)^{1/2} \mid f$ satisfies (1) for $f^0, \dots, f^n \in L^2(\Omega)$

(iii) $(v, u)_{L^2(\Omega)} = \langle v, u \rangle$, $u \in H_0^1(\Omega)$, $v \in L^2(\Omega) \subset H^{-1}(\Omega)$

Proof: 1. Given $u, v \in H_0^1(\Omega)$, inner product $(u, v) := \int_\Omega uv + Du \cdot Dv$

Let $f \in H^{-1}(\Omega)$. By Riesz Represantation theorem

$\exists!\, u \in H_0^1(\Omega)$ satisfying

$(u, v) = \langle f, v \rangle$, $\forall\, v \in H_0^1(\Omega)$. $\quad$ ($\langle f, v \rangle$, dual)

$\int_\Omega Du \cdot Dv + uv \, dx = \langle f, v \rangle$

So $\langle f, v \rangle = \int_\Omega f^0 v + \sum_{i=1}^n f^i v_{x_i} \, dx$, $\quad f^0 = u$, $f^i = (Du)_i$

$\Rightarrow \|f\|_{H^{-1}} \geq \|u\|_{H_0^1(\Omega)}$

2. Assume now $f \in H^{-1}(\Omega)$

$$\langle f, v \rangle = \int_\Omega g^0 v + \sum_{i=1}^n g^i v_{x_i} \, dx$$

$$g^0, \cdots, g^n \in L^2(\Omega)$$

$\cdot\, v = u \Rightarrow$

$$\int_\Omega |Du|^2 + |u|^2 \leq \int_\Omega \sum |g^i|^2 \, dx$$

$$|\langle f, v \rangle| \leq \left(\int_\Omega |f^i|^2 \right)^{1/2}$$

#

Thus the distribution $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is bounded with respect to the $H_0^1(\Omega)$-norm on the dense subset $\mathcal{D}(\Omega)$. It therefore extends in a unique way to a bounded linear functional on $H_0^1(\Omega)$, which we still denote by f. Moreover,

$$\|f\|_{H^{-1}} \leq \left(\sum_{i=0}^{n} \int_\Omega f_i^2 \, dx \right)^{1/2},$$

which proves inequality in the other direction of (4.9). □

The dual space of $H^1(\Omega)$ cannot be identified with a space of distributions on Ω because $\mathcal{D}(\Omega)$ is not a dense subspace. Any linear functional $f \in H^1(\Omega)^*$ defines a distribution by restriction to $\mathcal{D}(\Omega)$, but the same distribution arises from different linear functionals. Conversely, any distribution $T \in \mathcal{D}'(\Omega)$ that is bounded with respect to the H^1-norm extends uniquely to a bounded linear functional on H_0^1, but the extension of the functional to the orthogonal complement $(H_0^1)^\perp$ in H^1 is arbitrary (subject to maintaining its boundedness). Roughly speaking, distributions are defined on functions whose boundary values or trace is zero, but general linear functionals on H^1 depend on the trace of the function on the boundary $\partial\Omega$.

EXAMPLE 4.8. The one-dimensional Sobolev space $H^1(0,1)$ is embedded in the space $C([0,1])$ of continuous functions, since $p > n$ for $p = 2$ and $n = 1$. In fact, according to the Sobolev embedding theorem $H^1(0,1) \hookrightarrow C^{0,1/2}([0,1])$, as can be seen directly from the Cauchy-Schwartz inequality:

$$\begin{aligned}
|f(x) - f(y)| &\leq \int_y^x |f'(t)| \, dt \\
&\leq \left(\int_y^x 1 \, dt \right)^{1/2} \left(\int_y^x |f'(t)|^2 \, dt \right)^{1/2} \\
&\leq \left(\int_0^1 |f'(t)|^2 \, dt \right)^{1/2} |x - y|^{1/2}.
\end{aligned}$$

As usual, we identify an element of $H^1(0,1)$ with its continuous representative in $C([0,1])$. By the trace theorem,

$$H_0^1(0,1) = \{ u \in H^1(0,1) : u(0) = 0, \ u(1) = 0 \}.$$

The orthogonal complement is

$$H_0^1(0,1)^\perp = \{ u \in H^1(0,1) : \text{such that } (u,v)_{H^1} = 0 \text{ for every } v \in H_0^1(0,1) \}.$$

This condition implies that $u \in H_0^1(0,1)^\perp$ if and only if

$$\int_0^1 (uv + u'v') \, dx = 0 \qquad \text{for all } v \in H_0^1(0,1),$$

which means that u is a weak solution of the ODE

$$-u'' + u = 0.$$

It follows that $u(x) = c_1 e^x + c_2 e^{-x}$, so

$$H^1(0,1) = H_0^1(0,1) \oplus E$$

where E is the two dimensional subspace of $H^1(0,1)$ spanned by the orthogonal vectors $\{e^x, e^{-x}\}$. Thus,

$$H^1(0,1)^* = H^{-1}(0,1) \oplus E^*.$$

If $f \in H^1(0,1)^*$ and $u = u_0 + c_1 e^x + c_2 e^{-x}$ where $u_0 \in H_0^1(0,1)$, then

$$\langle f, u\rangle = \langle f_0, u_0\rangle + a_1 c_1 + a_2 c_2$$

where $f_0 \in H^{-1}(0,1)$ is the restriction of f to $H_0^1(0,1)$ and

$$a_1 = \langle f, e^x\rangle, \qquad a_2 = \langle f, e^{-x}\rangle.$$

The constants a_1, a_2 determine how the functional $f \in H^1(0,1)^*$ acts on the boundary values $u(0)$, $u(1)$ of a function $u \in H^1(0,1)$.

4.4. The Poincaré inequality for $H_0^1(\Omega)$

We cannot, in general, estimate a norm of a function in terms of a norm of its derivative since constant functions have zero derivative. Such estimates are possible if we add an additional condition that eliminates non-zero constant functions. For example, we can require that the function vanishes on the boundary of a domain, or that it has zero mean. We typically also need some sort of boundedness condition on the domain of the function, since even if a function vanishes at some point we cannot expect to estimate the size of a function over arbitrarily large distances by the size of its derivative. The resulting inequalities are called Poincaré inequalities.

The inequality we prove here is a basic example of a Poincaré inequality. We say that an open set Ω in $\mathbb{R}^n$ is bounded in some direction if there is a unit vector $e \in \mathbb{R}^n$ and constants a, b such that $a < x \cdot e < b$ for all $x \in \Omega$.

THEOREM 4.9. *Suppose that Ω is an open set in $\mathbb{R}^n$ that is bounded is some direction. Then there is a constant C such that*

$$\int_\Omega u^2\, dx \le C \int_\Omega |Du|^2\, dx \qquad \text{for all } u \in H_0^1(\Omega). \tag{4.11}$$

PROOF. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is sufficient to prove the inequality for $u \in C_c^\infty(\Omega)$. The inequality is invariant under rotations and translations, so we can assume without loss of generality that the domain is bounded in the x_n-direction and lies between $0 < x_n < a$.

Writing $x = (x', x_n)$ where $x' = (x_1, \dots, , x_{n-1})$, we have

$$|u(x', x_n)| = \left|\int_0^{x_n} \partial_n u(x', t)\, dt\right| \le \int_0^a |\partial_n u(x', t)|\, dt.$$

The Cauchy-Schwartz inequality implies that

$$\int_0^a |\partial_n u(x', t)|\, dt = \int_0^a 1 \cdot |\partial_n u(x', t)|\, dt \le a^{1/2}\left(\int_0^a |\partial_n u(x', t)|^2\, dt\right)^{1/2}.$$

Hence,

$$|u(x', x_n)|^2 \le a \int_0^a |\partial_n u(x', t)|^2\, dt.$$

Integrating this inequality with respect to x_n, we get

$$\int_0^a |u(x', x_n)|^2\, dx_n \le a^2 \int_0^a |\partial_n u(x', t)|^2\, dt.$$

A further integration with respect to x' gives

$$\int_\Omega |u(x)|^2\, dx \le a^2 \int_\Omega |\partial_n u(x)|^2\, dx.$$

Since $|\partial_n u| \le |Du|$, the result follows with $C = a^2$. □

Part IV Weak Solutions of 2nd Order Elliptic Equations (L^2- theory)

In this part, we consider

$$\begin{cases} Lu = f(x) & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}, \quad \Omega \text{ bounded domain}$$

$Lu = -\partial_i (a^{ij} \partial_j u) + b^i \partial_i u + cu$ (divergence form)

$Lu = -a^{ij} \partial_{ij} u + b^i \partial_i u + cu$ (non-divergence form

$a_{ij}(x)$ a matrix of measurable function, uniformly ellip

$\lambda |\xi|^2 \le a^{ij}(x) \xi_i \xi_j \le \Lambda |\xi|^2$, $\lambda \le 1 \le \Lambda$

If coefficients $\in C^1$ and $u \in C^2$, div-form = non-diverg

In general, no

"Thm" Assume "nice" coefficients, nice Ω GT, Chapter

(1) $\forall f \in C^\alpha(\bar\Omega)$, $\exists ! \; u \in C^{2,\alpha}(\bar\Omega)$ (Schauder Theory

(2) $\forall f \in L^p(\Omega)$, $\exists ! \; u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ (L^p-theory) GT Chapter 9

(3) $\forall f \in H^{-1}(\Omega)$, $\exists ! \; u \in H^1_0(\Omega)$ for divergence-form GT chapt 8

Evans Chapter 6

Weak sol'ns $a^{ij} \in L^\infty$, $b^i, c \in L^\infty$, $f \in L^2(\Omega)$

Definition: (1) The bilinear form

$$B(u,v) := \int_\Omega \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \, dx$$

$\forall u, v \in H_0^1(\Omega)$

(2) We say that $u \in H_0^1(\Omega)$ is a weak sol'n if

$$B[u,v] = (f,v), \quad \forall v \in H_0^1(\Omega).$$

More generally, we can consider

$$Lu = f^0 - \sum_{i=1}^n \partial_{x_i} f^i \in H^{-1}(\Omega).$$

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Find $w \in H^1(\Omega)$. $\quad \begin{cases} L\tilde{u} = f - Lw & \text{on } \Omega \\ \tilde{u} = 0 & \text{on } \partial\Omega \end{cases}$

$Lw \in H^{-1}(\Omega)$.

<u>Remark</u>: $B(u,v) \neq B(v,u)$ non-symmetric

Lax-Milgram Thm: Let H be a Hilbert space, Let $B: H \times H \to$ be such that

① bilinear

② bounded $|B[u,v]| \leq \alpha \|u\|_H \|v\|_H$

③ positive define, $B[u,u] \geq \beta \|u\|_H^2$

Then $\forall f \in H^*$, a bounded linear fcnl on H

$\exists! \, u \in H$ s.t. $B[u,v] = \langle f, v \rangle, \ \forall v \in H$.

Proof: 1. $\forall$ fixed $u \in H$, $v \to B[u,v]$ is a bdd linear fcnl on H. By Riesz Representation Thm
$\exists! \, A(u) \in H$ s.t.
$B[u,v] = (Au, v)$ (by positive definite)

2. $A : H \to H$ is linear

3. A is bdd from above and below
$(Au, Au) = B[u, Au] \le \alpha \|u\| \, \|Au\|$
below $\beta \|u\|^2 \le B[u,u] = (Au, u) \le \|Au\| \, \|u\|$

4. A is 1-1 and onto
1-1: If $Au = A\tilde{u}$, $0 = \|Au - A\tilde{u}\| \ge \beta \|u - \tilde{u}\|$
Onto: Range $R(A)$ is closed, If $Au_k \to w$, u_k Cauchy
$u_k \to \bar{u}$, $A\bar{u} = w$
$R(A) = H$. If not, $\exists \, w_1 \in H \setminus R(A)$, choose $w_2 \in R(A)$
minimizing $\{ \text{dist}(w_1, w_2) \mid w_2 \in R(A) \}$
$w = w_1 - w_2 \perp R(A)$, $w \ne 0$

but $\beta \|w\|^2 \le B[w,w] = (Aw, w) = 0$.

5. (Existence) For a given f, by Riesz Representation thm again, $\exists! \, w_f$,
$\langle f, v \rangle = (w_f, v)$, $\forall v \in H$
Let $u = A^{-1} w$, then $\forall v \in H$
$B[u,v] = (w, v) = \langle f, v \rangle$

6. Uniqueness. If not, $u, \tilde{u}$ are sol'n, then

Remark. ① we did not assume B is Symmetric

② Since $\beta \|u\|^2 \le B[u,u] = \langle f, u\rangle \le \|f\|_{H^{-1}} \|u\|$

The map $f \to u$ is bdd from H^* to H

To apply Lax-Milgram thm to our weak formulation.

check

1. bilinear, easy
2. bounded, OK if coeff nice
3. positive definite, need modification

Ex. $\Omega = (0,\pi)$, $Lu = -\frac{d^2}{dx^2}u + cu$, Constant

$u_k(x) = \sin kx$

$Lu_k = (k^2 + c)\, u_k$

If $c = -k^2$ for some k

③ $B[u,v] = \int \nabla u \cdot \nabla v + cu \cdot v$, B is not positive define

$B[u_k, v] = 0, \ \forall v \in H_0^1$

① The problem has not uniqueness

② The problem has no existence for some f

If $\exists u$, $Lu = u_k$ then

$\|u_k\|^2 = (Lu, u_k) = (u, Lu_k) = 0$

Lemma: If Ω is bdd, Lipschitz, $a^{ij}, b^i, c \in L^\infty$, (a^{ij}) unif. elliptic

$\exists \alpha, \beta > 0$, $\exists \gamma \ge 0$ s.t.

(i) $|B[u,v]| \le \alpha \|u\| \|v\|$

Pf: (i) Take $\alpha = C\left(\|a^{ij}\|_{L^\infty} + \|b^i\|_{L^\infty} + \|c\|_{\infty}\right)$

(ii) $\lambda \int_\Omega |Du|^2 \le \int a^{ij} \partial_i u \partial_j u = B[u,u] - \text{l.o.T.}$

$$|\text{l.o.T.}| \le C_1 \int |Du||u| + |u|^2$$

$$\le C_1 \varepsilon \int |Du|^2 + \left(\frac{C}{4\varepsilon} + C_1\right) \int |u|^2$$

$$\varepsilon_1 = \frac{\lambda}{2C_1}, \quad \gamma = \frac{C_1}{4\varepsilon} + C_1 + \frac{\lambda}{2}$$

$$\frac{\lambda}{2} \int_\Omega |Du|^2 + |u|^2 \le B[u,u] + \gamma \|u\|_{L^2}^2$$

Rmk: If b. c, vanish, or small, c has good sign can take $\gamma = 0$.

<u>First existence theorem</u>: $\exists\, \gamma \ge 0$ s.t., $\forall\, \mu \ge \gamma$, $\forall\, f \in H^{-1}(\Omega)$,
$\exists !$ weak sol'n $u \in H_0^1(\Omega)$ s.t.

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Pf: $B_\mu[u,v] = B[u,v] + \mu(u,v)$
check the conditions of Lax-Milgram Thm #

Rmk: This defines a linear operator

$$(L+\mu)^{-1}: H^{-1}(\Omega) \to H_0^1(\Omega)$$
$$f \to u$$

It is a bounded linear map

$$\beta \|u\|^2 \le B_\mu[u,u] = \langle f, u \rangle \le \|f\|_{H^{-1}} \|u\|$$

$$\|u\| \le \frac{1}{\beta} \|f\|_{H^{-1}(\Omega)}$$

Q: what happens without adding γu?
We will discuss it using Fredholm Alternative.

Review the previous example

$Lu = -u'' + u = f$ has no sol'n if

$\|f\|^2 = (Lu, f) = (u, L^* f) = 0$

L^* - formal adjoint operator of L

$Lu = -\partial_i(a^{ij}\partial_j u) + b^i u_i + cu$

$$(Lu, v) = \int a^{ij}\partial_j u \partial_i v + b^i \partial_i u v + \int cuv$$

$$= \int a^{ij}\partial_j u \partial_i v - \int u \partial_i(b^i v) + \int cuv$$

$$= (u, L^* v)$$

$$L^* = -\partial_i(a^{ij}\partial_j v + b^i v) + cu.$$

Compact operator: Let X, Y be Banach spaces. A bounded operator $K: X \to Y$ is compact if any bdd sequence $\{x_k\} \subset X$ has a subsequence $\{x_{k_j}\}$ s.t. Kx_{k_j} converges in Y

Ex: ① $id: W^{1,p} \to L^q$ $\quad q < \frac{np}{n-p}$ $\quad$ Compact

② $Y = \mathbb{R}^n$

③ $X, Y = \ell^2$, $K(a_1, \dots,) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \dots)$

④ $X = H_0^1(\Omega)$, $Y = L^2(\Omega)$.

For compact operators, its range is almost finite-dimensional

adjoint operator $K: H \to H$, $(Kx, y) = (x, K^* y)$
Hilbert space

Fredholm Alternative:

Let H be a Hilbert space and $K: H \to H$ a compact linear operator

Then

(i) $\dim \underbrace{N}_{\text{kernel}}(I-K) = \dim (N(I-K^*)) < +\infty$

(ii) $R(I-K)$ is closed and

$$R(I-K) = N(I-K^*)^\perp$$

In particular, $R(I-K) = H$ iff $N(I-K^*) = 0$.

Either $u - Ku = v$, has a unique sol'n to $\forall v \in H$, and $(I-K)^{-1}$ $\exists$ bdd

or $u-Ku=0$ has a sol'n and $u - Ku = v$ has a sol'n iff $(v, w) = 0$, $\forall$ $w - K^*w = 0$

Second existence theorem (Fredholm Alternative)

Assume Ω is bdd Lipschitz domain in R^n, $a^{ij}, b^i, b_*^i, c \in L^\infty(\Omega)$. uniform elliptic. Exactly one of the following two cases holds.

Either $\begin{cases} Lu = f \\ u = 0 \end{cases}$ has a unique weak sol'n $u \in H_0^1(\Omega)$ $\forall f \in L^2$

and $\|L^{-1}\|$ is bdd

or $\begin{cases} Lu = 0 \\ u = 0 \end{cases}$ has a nonzero weak sol'n and

$\begin{cases} Lu = f \\ u = 0 \end{cases}$ has a sol'n iff $(f, v) = 0$, $\forall v \in H_0^1(\Omega)$, $L^* v = 0$.

Proof: Let γ be from previous Thm, we know

$$(L+\gamma)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$$

is a bdd linear operator. Rewrite $Lu=f$ as

$$(L+\gamma)u = \gamma u + f$$

apply $(L+\gamma)^{-1}$:

$$u = (L+\gamma)^{-1}(\gamma u + f)$$

$$u - \gamma (L+\gamma)^{-1} u = (L+\gamma)^{-1} f \qquad (*)$$

Let $K = \gamma (L+\gamma)^{-1}$, $\tilde{f} = (L+\gamma)^{-1} f$

Take $H = \cancel{H_0^1}\, L^2$ Then

$$K : \cancel{H_0^1(\Omega)} \to L^2(\Omega) \to \cancel{H_0^1(\Omega)}\ L^2(\Omega)$$

Claim: K is compact $\Rightarrow \|K u_m\|_{H_0^1(\Omega)} \le C$

Let $\|u_m\|_{\cancel{H_0^1}\,L^2(\Omega)} \le C$. Then $\exists\, u_{m_j} \to u$ in $L^2(\Omega)$

$(L+\gamma)^{-1} u_{m_j}$ is cts $\Rightarrow (L+\gamma)^{-1} u_{m_j} \to (L+\gamma)^{-1} u$

By Fredholm Alternative

either $(*)$ has a sol'n $u \in H$, $\forall \tilde{f} \in H$

or $(*)$ has a nonzero sol'n $u \in H$, for $\tilde{f} = 0$

In the second case, $(*)$ has a sol'n iff $(\tilde{f}, v) = 0$, $\forall v \in N(I-K$

$$v = K^* v \iff L^* v = 0.$$

Rmk: 1. $u \in L^2 \Rightarrow u \in H_0^1$ by equation

2. Unclear if one can allow $f \in H^{-1}$. Make sense Kf.