Third Existence Theorem.

Recall: <u>Thm.</u> Let $K: H \to H$ be compact, $\dim K = +\infty$.

Then $Ku = \sigma u$, $u \neq 0$, σ = spectrum of u

① $0 \in \sigma(K)$

② $\sigma(K) - \{0\}$ is finite or else

$\sigma(K) - \{0\}$ is a sequence tending to 0

Thm: (i) $\exists$ at most countable set $\Sigma \subset \mathbb{R}$ s.t.

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak sol'n $\forall f \in L^2(\Omega)$ iff $\lambda \notin \Sigma$

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{+\infty}$

the values of a nondecreasing sequence with $\lambda_k \to +\infty$

Proof: $Lu = \lambda u \iff Lu + \gamma u = (\gamma + \lambda) u$

$$\iff u = \frac{\gamma + \lambda}{\gamma} K u$$

Thm: If $\lambda \notin \Sigma$, then

$$\|u\|_{L^2} \leq C \|f\|_{L^2(\Omega)}$$

$Lu = \lambda u + f$

$u = 0$

Proof: By contradiction #.

Regularity of Weak Sol'ns I.

We will discuss two results on regularity of weak sol'ns, H^2-regularity & Local boundedness (L^∞)

I. H^2-regularity

Let u be a weak sol'n to

$$-\Delta u = f \quad \text{in} \quad R^n$$

Formally
$$\int f^2 = \int (-\Delta u)^2 = \int \sum u_{ii} u_{jj}$$
$$= -\sum \int u_i u_{ijj}$$
$$= \sum \int u_{ij} u_{ij}$$
if $u \in H^2$ already (then we can use $C_0^\infty(R^n)$ to approximate)

Fact: If $u \in H^2$, then $\sum \| u_{ij} \|_{L^2} \le \| f \|_{L^2}$

The problem is to prove that $u \in H^2$.

Idea: discretization (Nirenberg's method)

$$Lu = -\partial_j (a^{ij} u_i) + b^i u_i + c(x) u$$

Theorem 1 (Interior regularity) Assume

$a^{ij} \in C^1(\Omega)$, b^i, $c \in L^\infty(\Omega)$.

$f \in L^2(\Omega)$

Let u be a weak solution to $Lu = f$ in Ω

Then $u \in H^2_{loc}(\Omega)$ and $\forall V \subset\subset \Omega$

Remark: As a consequence, we obtain

$$Lu = f \quad \text{a.e.} \quad \text{in } \Omega$$

To see this

$$B[u,v] = (f,v), \quad \forall v \in C_0^\infty(\Omega)$$

Since $u \in H^2_{loc}(\Omega)$, we can integrate by parts

$$B[u,v] = (Lu, v)$$

Hence $Lu = f$ a.e.

Proof: $\eta = 1$ on V, $\quad \eta \equiv 0$ in W^c

$B(u,v) = (f,v)$

$$\Rightarrow \int a^{ij} u_i v_j = \int \tilde{f} v = \int \left[f - \sum b^i u_i - cu \right] v$$

$$\tilde{f} \in L^2 \text{ by } u \in H^1(\Omega)$$

Let $|h| > 0$ be small, $k \in \{1, \cdots, n\}$

$$v := -D_k^{-h}(\eta^2 D_k^h u) \quad \in H_0^1(\Omega)$$

where $\quad D_k^h u = \dfrac{u(x+he_k) - u(x)}{h}$

So
$$\int a^{ij} u_i v_j = \int a^{ij} u_i \left[D_k^{-h}(\eta^2 D_k^h u) \right]_{x_j}$$

$$= \int a^{ij} u_i \left[D_k^{-h}(\eta^2 D_k^h u)_{x_j} \right]$$

$$= -\int a^{ij} D_k^h u_i \, (\eta^2 D_k^h u)_{x_j}$$

$$\quad - \int D_k^h(a^{ij}) \, u_i \, (\eta^2 D_k^h u)_{x_j}$$

$$= I_1 + I_2$$

$$\int_\Omega v D_k^{-h} w = -\int w D_k^h v \quad , \quad \cancel{D_k^h v = v(\;)}$$

$$-(I_2+I_1) = +\int a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \eta^2 \, dx$$

$$+ \int_\Omega \Big[a^{ij,h} D_k^h u_{x_i} D_k^h u \, 2\eta \eta_{x_j} + (D_k^h a^{ij}) u_{x_i} D_k^h u_{x_j} \eta^2 + (D_k^h a^{ij}) u_{x_j} D_k^h u \, 2\eta\eta_{x_j} \Big]$$

The first term gives

$$\geq \theta \int \eta^2 |D_k^h Du|^2 \, dx \qquad \text{(Main term)}$$

The rest of term

$$\leq \int_\Omega \Big[|D_k^h u_{x_i}| \, |D_k^h u| \, \eta |\nabla\eta| + |\nabla u| \, |D_k^h u_{x_j}| \eta^2 + |\nabla u| \, |D_k^h u| \, \eta |\nabla \eta| \Big]$$

$$\leq \varepsilon \int_\Omega |D_k^h u_{x_i}|^2 \eta^2 + \frac{C}{\varepsilon} \int_W \left(|D_k^h u|^2 + |Du|^2 \right) dx$$

Claim: $\displaystyle \int_W |D_k^h u|^2 \leq C \int_\Omega |\nabla u|^2$

$$u(x+the_k) - u(x) = h \int_0^1 u_{x_k}(x+the_k)\, dt$$

$$|u(x+he_k) - u(x)| \leq |h| \int_0^1 |Du(x+the_k)| \, dt$$

$$\int_W |D_k^h u|^2 \leq \int_W \int_0^1 |Du(x+the_k)|^2 \, dt \, dx$$

$$= \int_0^1 \int_W | \qquad |^2 \, dx \, dt$$

$$\leq C \int_\Omega |Du|^2$$

Thus

$$|A| \geq \frac{\theta}{2}\int_\Omega \eta^2 |D_k^h Du|^2 - C\int_\Omega |Du|^2$$

The rest of the terms can be estimated similarly

$$\int_\Omega |v|^2 \leq C\int_\Omega |D(\eta^2 D_k^h u)|^2 dx$$

$$\leq C\int_W |D_k^h u|^2 + \varepsilon\eta^2 |D_k^h Du|^2 dx$$

$$\leq C\int_\Omega |Du|^2 + \varepsilon\eta^2 |D_k^h Du|^2 dx$$

Finally we have

$$\int_V |D_k^h Du|^2 dx \leq \int_\Omega \eta^2 |D_k^h Du|^2 \leq C\int_\Omega (f^2 + u^2 + |Du|^2)\,dx$$

Hence $\sup_h \|D_k^h u\|_{L^2(V)} < +\infty$

$\exists\, v_i \in L^2(V)$, $h_k \to 0$, $D_k^{h_k} u \to v_i$ weakly in $L^2(V)$

$$\int_V u\phi_{x_i} = \int_\Omega u\,\phi_{x_i} = \lim_{h_k\to 0}\int_\Omega u\, D_i^{-h_k}\phi\, dx$$

$$= -\lim_{h_k\to 0}\int_\Omega D_i^{h_k} u\,\phi$$

$$= -\int_V v_i\phi = -\int_\Omega v_i\phi$$

Thus $v_i = u_{x_i}$ in the weak sense so $Du \in L^p(V)$

and $$\int_V |Du|^{2} \leq \lim_{h\to 0}\int_V |D_k^h Du|^2 \leq C\left(\int_\Omega (f^2 + u^2 + |Du|^2)\, d\right.$$

Thm2 (Higher interior regularity):

$a^{ij}, b^i, c \in C^{m+1}(\Omega)$

$f \in H^m(\Omega)$

Then $u \in H^{m+2}_{loc}(\Omega) = W^{m+2,2}_{loc}(\Omega)$

$$\|u\|_{H^{m+2}(V)} \le C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

Boundary Regularity.

Thm3. $a^{ij} \in C^1(\bar{\Omega})$, $b^i, c \in L^\infty$, $f \in L^2(\Omega)$

$\partial\Omega \in C^2$

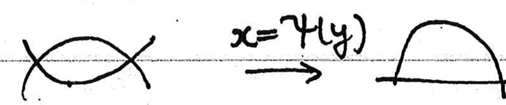
Then

$$\|u\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

Proof: Idea

1) Use boundary strengthen to the standard case

$x = \varphi(y)$, $y = \Phi(x)$



$$\partial_{x_j}(a^{ij}\partial_{x_i}u) = \partial_{y_k}\left(a^{ij}\partial_{y_l}\frac{\partial y_l}{\partial x_i}\right)\frac{\partial y_k}{\partial x_j}, \quad \tilde{u}(y) = u(x).$$

New coefficient

$$\tilde{a}^{kl} = \left(a^{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j}\right) \in C^1 \text{ since } \Phi \in C^2$$

$$Lu = -\partial_{y_k}(\tilde{a}^{kl}\tilde{u}_{y_l}) + \tilde{b}^k\partial_{y_k} + \tilde{c}\tilde{u}$$

$$\tilde{a}^{kl}\xi_k\xi_l = a^{ij}\frac{\partial y_l}{\partial x_i}\frac{\partial y_k}{\partial x_j}\xi_k\xi_l \ge \left|\frac{\partial y_k}{\partial x_i}\xi_k\right|^2$$

2). Model Case $\Omega = B_1^\circ(0) \cap \mathbb{R}^n_+$

$V = B^\circ_{\frac{1}{2}}(0) \cap \mathbb{R}^n_+$, $\quad \eta \equiv 1$ on $B_{\frac{1}{2}}(0)$, $\quad \eta = 0$ on $\mathbb{R}^n \setminus B_1(0)$

Let $h>0$ be small, choose $k \in \{1, \cdots, n-1\}$, and write

$$v(x) = -D_k^{-h}(\eta^2 D_k^h u)$$

$$= -\frac{1}{h} D_k^{-h}\left[\eta^2(x)\left[u(x+he_k) - u(x)\right]\right]$$

$$= \frac{1}{h^2}\Big(\eta^2(x-he_k)\left[u(x) - u(x-he_k)\right]$$
$$- \eta^2(x)\left[u(x+he_k) - u(x)\right]\Big)$$

$u = 0$ on $\{x_n = 0\}$

$v \in W_0^{1,2}(\Omega)$

Same computation

$$\int_V |D_k^h Du|^2 \le C \int_\Omega [f^2 + u^2 + |Du|^2]\, dx.$$

$k = 1, \cdots, n-1$

Similar as before

$$Du \in H^1(V)$$

$$\sum_{\substack{k,l=1 \\ k+l<2n}}^n \|u_{x_k x_l}\|_{L^2} \le C(\|f\|_{L^2} + \|u\|_{H^1})$$

3). For $k = l = n$, we use the equation $\quad Lu = f$ a.e. in Ω

$$a^{nn} u_{x_n x_n} = -\sum_{i+j<2n} a^{ij} u_{x_i x_j} + b^i u_{x_i} + cu - f$$

Thm (Higher boundary regularity). $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$

$f \in H^m(\Omega)$, $\partial\Omega \in C^{m+2}$

$$\|u\|_{H^{m+2}(\Omega)} \le C\left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

Corollary: $-\Delta u = \mu u$, $u = 0$ on $\partial\Omega$

If $\partial\Omega \in C^\infty \Rightarrow u \in C^\infty$