MATH 516-101 2021-2022 Homework THREE
Due Date: by 6pm, October 20, 2021

10 points each, unless otherwise stated.

1. (20points) Let u solve

$$u_{tt} = u_{xx}, x \in R, t > 0$$

$$u(x,0) = g(x), u_t(x,0) = h(x)$$

Suppose that g and h have compact support. Let $p(t) = \frac{1}{2}\int u_x^2(x,t)$ be the potential energy, and $k(t) = \frac{1}{2}\int u_t^2(x,t)$. Prove
(a) $p(t) + k(t) = p(0) + k(0)$
(b) $p(t) = k(t)$ for all large enough t.

2. Derive a solution formula for the three-dimensional wave equation with radial source

$$u_{tt} = \Delta u + f(r,t), t > 0, r > 0$$

$$u(r,0) = 0, u_t(r,0) = 0$$

3. Let $n = 5$ and $U(r,t)$ satisfy

$$U_{tt} - (U_{rr} + \frac{4}{r}U_r) = 0$$

Show that $\tilde{U} = \frac{1}{r}\partial_r(r^3 U)$ satisfies $\tilde{U}_{tt} - \tilde{U}_{rr} = 0$.

4. Find the first order and second order weak derivatives for the following function $u : R \to R$, if exists:

$$(a) \quad u(x) = \begin{cases} 1 - |x|, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases} ; (b) \quad u(x) = |\cos x|.$$

5. Suppose $u : (a,b) \to R$ and the weak derivative exists and satisfies

$$Du = 0 \ a.e. \text{ in } (a,b)$$

Prove that u is constant a.e. in (a,b).

6. Let $1 < p < +\infty$. Show that if $u, v \in W^{1,p}(\Omega)$ then $\max(u,v), \min(u,v) \in W^{1,p}(\Omega)$. Show that this is not true for $W^{2,p}(\Omega)$.

7. Consider the following function

$$u(x) = \frac{1}{|x|^\gamma}$$

in $\Omega = B_1(0)$. Show that if $\gamma + 1 < n$, the weak derivatives are given by

$$\partial_j u = -\gamma \frac{x_j}{|x|^{\gamma+2}}$$

i.e., you need to show rigorously that

$$\int u\partial_j\phi = \int \phi\gamma\frac{x_j}{|x|^{\gamma+2}}$$

For the condition on γ such that $u \in W^{1,p}$ or $u \in W^{2,p}$.

8. Let $\eta(t) = 1$ for $t \leq 0$ and $\eta(t) = 0$ for $t > 1$. Let $f \in W^{k,p}(R^n)$ and $f_k = f\eta(|x| - k)$. Show that $\|f_k - f\|_{W^{k,p}} \to 0$ as $k \to +\infty$. As a consequence show that $W^{k,p}(R^n) = W_0^{k,p}(R^n)$.

9. Let $R^+ = \{x \in R; x > 0\}$ and assume that $u \in W^{2,p}(R^+)$. Define the symmetric extension of u by setting $Eu(x) = u(|x|)$. Prove that $Eu \in W^{1,p}(R)$ but $Eu \notin W^{2,p}(R)$, in general.