MATH 516-101

overage: Introduction of PDEs

Part I. Classical Solutions to PDEs

Part II. Soboler Spaces

Part III. Weak Solutions to PDEs

Part IV. Regularity of Weak Solutions: From weak to strong

Examples of PDE

0. Transport Equation $u_t + b\cdot\nabla u = 0$
1. Laplace Equation $\Delta u = 0$
2. heat Equation $u_t = \Delta u$
3. Wave Equation $u_{tt} = \Delta u$

linear

Nonlinear PDEs

$|\nabla u| = 1$ (Eikonal Equation)

$\Delta u + u^2 = 0$ (Emden-Fowler)

$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, (p-laplacian).

$u_t + u u_x = 0$ (Burgers)

$u_{tt} - \Delta u + \sin u = 0$ (Klein-Gordon).

$i u_t - \Delta u - u^3 = 0$ (Nonlinear Schrödinger Equation)

$\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ (Minimal Surface Equation)

$\begin{cases} u_t - \Delta u + u\cdot\nabla u + \nabla p = 0 \\ \mathrm{div}\, u = 0 \end{cases}$ (Navier-Stokes Equation)

Questions :

- Existence & Uniqueness
  what do we mean by a sol'n ?
- Stability of IVP & BVP.
- Other Properties.

Classical Solutions.

all derivatives of the sol'n in the equation are continuous and the equation holds pointwisely.

Weak sol'ns :

the equation holds in a weak sense → Prerequistes → L^p-space

Part I. Classical solutions to linear PDE

§.1 Transport Equation

$u_t + b\cdot Du = f(x,t)$, $u(0,x) = g(x)$

$u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$

Parametrize initial data $(x_0(\xi), t_0(\xi))$. $\frac{dt}{ds} = 1$

Method of Characteristics (MATH 400).

$$\frac{dt}{1} = \frac{dx_1}{b_1} =$$

$$\begin{cases} \frac{dt}{ds} = 1, & t(0) = 0 \\ \frac{dx_j}{ds} = b_j, & x_j(0) = \xi \\ \frac{du}{ds} = f(x,t), & u(0) = g(\xi) \end{cases}$$

$t = s$

$x_j = b_j s + \xi$

$\Rightarrow \xi = x - bt.$

$$u = g(x) = g(\xi) = g(x - bt)$$

$$u = g(\xi) + \int_0^s f(\underbrace{x}_{b\tau + x - bt}, \tau)\, d\tau$$

$$= g(x-bt) + \int_0^t f(b\tau + x - bt, \tau)$$

$$= g(x-bt) + \int_0^t f(x + (s-t)b, s$$

Existence & Uniqueness: provided $b \in C^1$

2.2. Laplace equation & Poisson Equation

$-\Delta u = 0$ Laplace equation

$-\Delta u = f$ Poisson equation

Classical sol'n: $u \in C^2$, $f \in C^0$, equation holds pointwisely

Aim: Solution formula in a bounded domain Ω with suitable boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Existence & Uniqueness in the class of C^0 functions

$\Omega = R^n$, Laplace equation

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \underbrace{\Delta_{S^{n-1}}}_{\text{Laplace-Beltrami operator on } S^{n-1}} u$$

radial sol'ns. $u_{rr} + \frac{n-1}{r} u_r = 0$

$$u = \begin{cases} c_1 r^{2-n} + c_2, & n > 2 \\ c_1 \log r + c_2, & n = 2 \end{cases}$$

Newtonian Potential

$$V(x) = \int_{R^n} f(y) \frac{1}{|x-y|^{n-2}} dy, \quad n > 2$$

$$= \int_{R^n} f(y) \ln \frac{1}{|x-y|} dy, \quad n = 2$$

<u>Theorem</u>: $f \in C_c^1(R^n)$. Then

$V \in C^2(R^n)$ and

$$\Delta V = -(n-2)\omega_n f(x), \quad x \in R^n, \quad n \geq 3$$

$$\Delta V = -2\pi f(x), \quad x \in R^n, \quad n=2$$

Consider $n=3$ only

<u>Proof</u> $V_{x_i} = \int_{R^n} f(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) dy$

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(\frac{1}{|x-y|}\right)$$

$$f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) = -\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right) + f_{y_i}(y)\frac{1}{|x-y|}$$

$$V_{x_i}(x) = \int_{R^n} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)dy$$

$$= \int_{R^n \setminus B_\rho(x)} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)dy + \int_{B_\rho(x)} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)dy$$

$$= \int_{R^n \setminus B_\rho(x)} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)dy$$

$$+ \int_{B_\rho(x)} \left(-\frac{\partial}{\partial y_i}\left(f(y)\frac{1}{|x-y|}\right) + f_{y_i}(y)\frac{1}{|x-y|}\right)dy$$

$$= \int_{R^n \setminus B_\rho(x)} f(y)\frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)dy$$

$$+ \int_{B_\rho(x)} f_{y_i}(y)\frac{1}{|x-y|} - \int_{\partial B_\rho} f(y)\frac{1}{|x-y|}n_i \, d\sigma_y$$

The first and third integral is C^1
The second integral is C^1 since $f \in C^1$

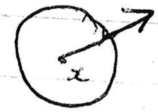
Hence

$$\Delta V = \int_{R^n \setminus B_\rho(x)} f(y)\, \Delta_x \frac{1}{|x-y|}\, dy$$

$$+ \int_{B_\rho} \sum_i f_{y_i}(y) \cdot \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) - \int_{\partial B_\rho} f(y) \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{1}{|x-y|}\, n_i\, d\sigma$$

$$= \int_{B_\rho(x)} \sum_i f_{y_i}(y) \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right)$$

$$- \int_{\partial B_\rho(x)} f(y) \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-y|}\right) n_i\, dS$$



$$= I_1 + I_2$$

where $$|I_1| \leq \int_{B_\rho(x)} \sum |f_{y_i}(y)| \left| \frac{x_i - y_i}{|x-y|^3} \right|$$

$$\leq c \int_{B_\rho(x)} \frac{1}{|x-y|^2}\, dy \leq C\rho \to 0 \text{ as } \rho \to 0$$

$$I_2 = -\int_{\partial B_\rho(x)} f(y) \frac{1}{\rho^2}\, d\sigma \qquad \text{since } n \cdot (y-x) = \rho \text{ if } y \in \partial B_\rho(x)$$

$$= -\frac{1}{\rho^2} f(\bar{y}) \int_{\partial B_\rho(x)} d\sigma$$

$$= -\omega_n f(\bar{y}) \quad , \quad f(\bar{y}) = \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} f\, d\sigma$$

$$\lim_{\rho \to 0} I_2 = -\omega_n f(x)$$

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Other dimensions — Exercise

<u>Remark</u>: 1) Result holds if $f \in C_c^{\alpha}(\mathbb{R}^n)$

2) Result holds if f has sufficient decay

$|f(y)| \leq C(1+|y|)^{-l}$, $l > 2$

Exercise. $$\int \frac{1}{|x-y|^{n-2}} \frac{1}{(1+|y|)^{l}} dy \lesssim \begin{cases} \dfrac{1}{(1+|x|)^{l-2}}, & 2 < l < n \\ \dfrac{1}{(1+|x|)^{n-2}} \log |x|, & l = n \\ \dfrac{1}{(1+|x|)^{n-2}}, & l > n. \end{cases}$$