$$|u(x)-u(x_0)| = \left|\int_{\partial B} K(x,y)(\varphi(y)-\varphi(x_0))\,ds_y\right|$$

$$= \int_{|y-x_0|\le\delta} K(x,y)\,|\varphi(y)-\varphi(x_0)|\,ds$$

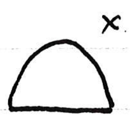
$$+ \int_{|y-x_0|>\delta} K(x,y)\,|\varphi(y)-\varphi(x_0)|\,ds_y$$

$$\le \varepsilon + 2M \frac{(R^2-|x|^2)R^{n-2}}{(\delta/2)^n}.$$

#

Other Green's Functions.

Ex. Half-Space 

Ex. Half-ball 

Ex. Quarter ball 

Finally, we are ready to discuss Perron's Method. The main goal of Perron's Method is the following theorem

Theorem I.13. Under some smoothness condition on Ω (e.g. $\partial\Omega \in C^2$), then $\forall g \in C^0(\partial\Omega)$, $\exists 1$ a sol'n to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

We first define: $u \in C^2(\Omega)$ is called subharmonic (superharmonic) in a domain Ω if $\Delta u \geq 0$ (or $\Delta u \leq 0$) in Ω

<u>Def:</u> A fcn $u \in C^0(\Omega)$ is called subharmonic in Ω if $\forall B \subset\subset \Omega$ and $\forall$ harmonic h in B, i.e. $h \in C^2(B) \cap C(\bar{B})$ and $\Delta h = 0$ in B, satisfying $u \leq h$ on ∂B we have $u \leq h$ in B

Corollary: A harmonic function in Ω is both superharmonic and a sub-harmonic fcn.

Lemma 1. (Strong M.P.). Assume Ω is connected. If a subharmonic fcn u attains its supremum in Ω, then $u \equiv$ Constant in Ω.

Lemma 2. u Subharmonic, v superharmonic, $u - v \leq 0$ on $\partial\Omega$. Then either $v > u$ in Ω or $v \equiv u$.

Proof: $\{x \mid \sup(u-v) = M\} = \Omega_1$

open: $x_1 \in \Omega_1$, $B_{\rho_0}(x_1) \subset\subset \Omega$. Then $B_{\rho_0}(x_1) \in \Omega_1$. If not, $\exists$ a ball $B_\rho(x^2)$, $0 < \rho < \rho_0$, $x^2 \in \partial B$, $(u-v)(x^2) < M$. Let h_1, h_2 be harmonic in B $h_1 = u$ on ∂B, $h_2 = v$ on ∂B. Then, if $x \in B$

$$M \geq \max_{\partial B} (u-v) = \max_{\partial B} (h_1 - h_2)$$

$$\geq h_1(x) - h_2(x) \geq u(x) - v(x)$$

Set $x = x^1 \Rightarrow u(x^1) - v(x^1) = M \Rightarrow h_1 - h_2 \equiv$ Constant #

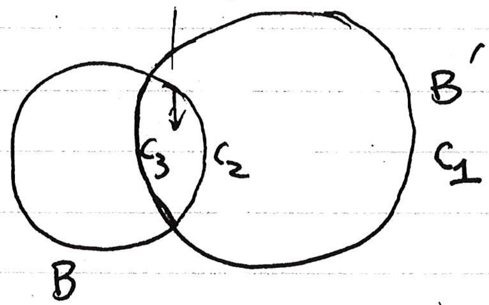
Harmonic Lifting. Let u be subharmonic in Ω, $B \subset\subset \Omega$ a ball, $\bar{u}$ harmonic in B s.t. $\bar{u} = u$ on ∂B. Then

$$U(x) = \begin{cases} \bar{u} : & x \in B \\ u(x) : & x \in \Omega \setminus \bar{B} \end{cases}$$

is called a harmonic lifting of u in B

Lemma3. U is subharmonic in Ω

Proof: Let $B' \subset\subset \Omega$ and h harmonic in B', $h \geq U$ on $\partial B'$.
we have to show that $h \geq U$ in B'.



$$\begin{cases} \Delta h = \Delta \bar{u} = 0 \\ h \geq u = \bar{u} \text{ on } C_2 \\ h \geq U = \bar{u} \text{ on } C_3 \end{cases}$$

$\Rightarrow h \geq \bar{u} = U$ on $B' \cap B$

on $C_1 = \partial B' \setminus B$ we have $h \geq U \equiv u$

$\underline{B' \setminus B}$ $\begin{cases} \Delta h = 0 \text{ in } B' \\ h \geq U = u \text{ on } C_1 \\ h \geq U = \bar{u} \geq u \text{ on } C_1 \end{cases}$

$h \geq u$.

on $C_3 = \partial B' \cap B$, according to the definition of $\bar{u}$, $h \geq U = \bar{u} \geq u$

on $C_2 = \partial B \cap B$, $\bar{u} = u = U$

Combining these inequalities, $\Rightarrow h \geq u$ on $\partial B'$, hence $h \geq u$ in B'
Then $U \leq h$ in $B' \setminus B$

Since $U \equiv u$ in $B' \setminus B$. It remains to show that $U \leq h$ in $B' \cap B$.
on $\partial(B' \cap B)$ we have $h \geq U$, and assumption $h \geq U$ in B'. since $U \equiv \bar{u}$ in $B \cap B'$, h is harmonic $\Rightarrow h \geq U$ in $B \cap B'$

Lemma4. Let $u_1, \cdots, u_N$ be subharmonic in Ω. Then
$u = \max\{u_1, \cdots, u_N\}$ is also subharmonic

Def: u is subharmonic, $u \leq \phi$ on $\partial\Omega \Leftrightarrow u$ is a sub-soln

Lemma 5. $\underline{u}$ is a subfcn, $\overline{u}$ is a superfcn with respect to φ. Then $\underline{u} \le \overline{u}$ in Ω

Set $S_\varphi = \{ v \in C(\Omega)$ subharmonic in Ω, $v \le \varphi$ on $\partial\Omega \}$.

First S_φ is not empty, $v = \inf_{\partial\Omega} \varphi$

Theorem I.14. (Perron). The function

$$u(x) := \sup_{v \in S_\varphi} v(x)$$

is harmonic in Ω

Proof: (i) We have in Ω

$$\inf_{\partial\Omega} \varphi \le u(x) \le \sup_{\partial\Omega} \varphi$$

(ii) Let $y \in \Omega$ be fixed. Then $\exists$ a sequence $v_n \in S_\varphi$ with $\lim_{n\to+\infty} v_n(y) = u(y)$.

Let $B = B_R(y) \subset\subset \Omega$, R sufficiently small, and let V_n be the harmonic Lifting of v_n in B. Then $V_n \in S_\varphi$

$$\lim V_n(y) = u(y) \qquad -(2.5)$$

Proof of (2.5): $v_n(y) \le V_n(y)$ since $v_n = V_n$ on ∂B, V_n is harmonic in B and v_n is subharmonic.

$$u(y) \le \lim_{n\to+\infty} V_n(y)$$

On the other hand, since $V_n \in S_\varphi$, we have

$$V_n(y) \le \sup_{v \in S_\varphi} v(y) = u(y)$$

and $u(y) \le \lim V_n \le u(y)$.

(iii) $\forall h$ harmonic in B, we have

$$\sup_{B_\rho(y)} |D^\alpha h| \le C \sup_{B_R} |h|$$

• $\exists$ a subsequence $V_{n_k} \to$ uniformly in $B_\rho(y)$ to a harmonic fcn v. and

$$v(x) \le u(x), \quad x \in B_R(y)$$

Since $V_n(x) \le u(x)$ on $B_R(y)$.

At the center y it is,

$$v(y) = u(y). \qquad (9)$$

(iv). Claim $v(x) = u(x)$, $x \in B$.

Proof: If not, $\exists\, z \in B$, $v(z) < u(z)$. Then $\exists\, u_0 \in S_\varphi$, $v(z) < u_0(z)$.

Set

$$\omega_k(x) = \max(u_0(x), V_{n_k}(x)).$$

Let W_k be the harmonic lifting of ω_k in B.

A subsequence of W_k converges uniformly on cpt of B to w harmonic in B s.t.

$$v(x) \le w(x) \le u(x), \qquad x \in B = B_R(y). \qquad (10)$$

By (9)–(10), $v(y) = w(y) = u(y)$.

Since $\Delta v = \Delta w = 0$.

Strong M. P. $\quad v(x) = w(x)$, $x \in B$.

$w(z) = v(z) < u_0(z)$.

• By the definition of $\omega_n(x)$ and W_n

$$u_0(x) \le \omega_{n_k}(x) \le W_{n_k}(x)$$

$$u_0(x) \le w(x), \quad x \in B \qquad \Rightarrow v(z) < u_0(z) \le w(z), \text{ Contradiction}\#$$

Boundary Behavior

Def: A $C(\bar{\Omega})$-fcn $w = w_\xi$ is called a barrier at $\xi \in \partial\Omega$ relative to Ω if

(i) w is superharmonic in Ω

(ii) $w > 0$ in $\bar{\Omega} \setminus \{\xi\}$ and $w(\xi) = 0$

An important fature of the barrier concept is that it is a local property of the boundary $\partial\Omega$. Namely, let us define w to be a local barrier at $\xi \in \partial\Omega$ if $\exists$ a nbhd N of ξ such that w satisfies the definition in $\Omega \cap N$.

Let B be a ball satisfying $\xi \in B \subset\subset N$ and $m = \inf_{N-B} w > 0$.

$$\bar{w}(x) = \begin{cases} \min(m, w(x)), & x \in \bar{\Omega} \cap B \\ m, & x \in \bar{\Omega} \setminus B \end{cases}$$

is then a barrier at ξ relative to Ω.

ξ is called a regular point if $\exists$ a local barrier

Theorem I.14. Let u be a harmonic fcn defined in Ω by the Perron method with boundary data φ. If ξ is a regular point of $\partial\Omega$ and if φ is continuous at ξ, then

$$\lim_{x \to \xi, x \in \Omega} u(x) = \phi(\xi)$$

Proof: Fix $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|\phi(x) - \phi(\xi)| < \varepsilon$, $\forall x \in \partial\Omega$, $|x - \xi| < \delta$.
Let $M = \sup_{\partial\Omega} |\varphi|$. Let w be a barrier at ξ. Then $\exists$ a $k = k(\varepsilon)$ s.t

$$k w(x) > 2M \quad \text{if } |x - \xi| \geq \delta$$

Then $\phi(\zeta)+\varepsilon+kw(x)$ is a super-sol'n relative to φ.

$\phi(\zeta)-\varepsilon-kw(x)$ is a sub-sol'n

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

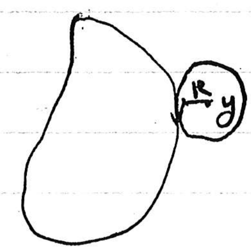
$$\phi(\zeta)-\varepsilon-kw(x) \le u(x) \le \phi(\zeta)-\varepsilon-kw(x)$$

$$|u(x)-\phi(\zeta)| \le \varepsilon + kw(x).$$

#

Examples of Local barrier.

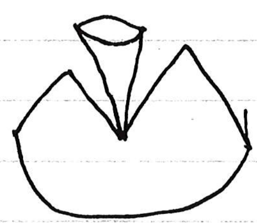
1). Exterior sphere.



$B_R \cap \bar{\Omega} = \{\zeta\}$

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n}, & n \ge 3 \\ \log\left(\frac{|x-y|}{R}\right) \end{cases}$$

is a local barrier



2). Exterior Cone.

$r^\lambda f(\theta)$.

$r^\lambda \cos(\mu\theta)$. $\quad \Delta w = r^{\lambda-2}(\lambda^2-\mu^2)\cos\mu\theta, \quad |\mu\theta| \le \frac{\pi}{2}$.

Perron's Method not easy to execute numerically. Another method is <u>Dirichlet Energy Method</u>

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad (1)$$

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u$$

$$\Lambda_0 = \{ \underbrace{u = g \text{ on } \partial\Omega}_{u \in C^2(\Omega) \cap C(\bar\Omega)} \}$$

<u>Theorem</u>: 1) If $u \in \Lambda_0$ is a sol'n to (1), then

$E[u] \le E[v], \quad \forall v \in \Lambda_0$

2) If $u \in \Lambda_0$ such that

$E[u] \le E[v], \quad \forall v \in \Lambda_0$

then u satisfies (1).

Proof: $v = u + w, \qquad w = 0$ on $\partial\Omega$

$$\int_\Omega |\nabla u + w|^2 = \int_\Omega 2\nabla u \cdot \nabla w + |\nabla w|^2$$

$$= 2\int_\Omega \nabla(w\nabla u) - w\Delta u + |\nabla w|^2$$

$$= +\int_\Omega w f + |\nabla w|^2 + \int_\Omega w(-\Delta u)$$

$$E[u+w] = E[u] + \frac{1}{2}\int_\Omega |\nabla w|^2 \quad \to \text{true } \forall\, \underset{w=0}{w} \in C^2(\Omega) \cap C(\bar\Omega)$$

So if we consider

$$\inf_{u \in \Lambda_0} E[u] = C_0$$

if C_0 is attained by some u and $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then we ar done.

Problem 1: Is C_0 attained?

Ex. 1. (Hadamard 1906): Let $D = \{x \in \mathbb{R}^2 \mid |x|^2 < 1\}$

Let $u: D \to \mathbb{R}$ be given by

$$u(r,\theta) = \sum_{n=1}^{+\infty} n^{-2} r^{n!} \sin(n\theta)$$

It is easy to check $\Delta u = 0$

series converges absolutely uniformly in $\bar{D}$. Hence u is harmonic in D and continuous in $\bar{D}$.

On the other hand

$$E(u) = \int_D |\nabla u|^2 \geq \int_0^{2\pi} \int_0^{\rho} |\partial_r u|^2 r\,dr\,d\theta$$

$$= \sum_{n=1}^{+\infty} \frac{\pi n!}{2n^4} \rho^{2n!} \geq \sum_{n=1}^{m} \frac{\pi n!}{2n^4} \rho^{2n!}$$

$\forall \rho < 1$, $\forall$ integer m. Hence $E(u) = \infty$.

To conclude, there exists a Dirichlet datum $g \in C(\partial D)$ for which the Dirichlet problem is perfectly solvable, but the sol'n cannot be obtained by minimizing the Dirichlet energy.

Ex. 2. Consider $\begin{cases} \Delta u = 0 & \text{in } D = B_1 \\ u = 0 & \text{on } \partial B_1 \end{cases}$

$$k \in \mathbb{N}, \quad u_k(r,\theta) = \begin{cases} k a_k, & r < e^{-2k} \\ -a_k(k + \log r), & e^{-2k} < r < e^{-k} \\ 0, & e^{-k} < r < 1 \end{cases}$$

$\varepsilon^{-\frac{n-2}{2}} \eta\left(\frac{x}{\varepsilon}\right)$ $n \geq 3$

$\frac{1}{\sqrt{\log \frac{1}{\varepsilon}}} \left[\log(\varepsilon^2 + r^2) - l\right.$

u_k is continuous, piecewise smooth

$$E(u_k) = 2\pi \int_0^1 |\partial_r u_k|^2 r\,dr = 2\pi\, a_k^2 \log r \Big|_{e^{-2k}}^{e^{-k}} = 2\pi k a_k^2$$

Choosing $a_k = k^{-2/3}$. Then $E(u_k) \to 0$

However $u_k(0) = k a_k = k^{1/3}$ diverges as $k \to +\infty$.

Remedy:

- First, we show that E has a minimizer in a class that contains Λ_0 as a subset (Sobolev Space)
- Show that minimizer is in fact in Λ_0 (Regularity Theory)

Part I.3 Heat Equation

$$\begin{cases} u_t = \Delta u + f, & x \in R^n,\ t > 0 \\ u(x,0) = g(x), & x \in R^n,\ t = 0 \end{cases}$$

$u \in C^2$, $x \in R^n$, $t > 0$.

A formal solution is obtained by Fourier transformation. We first consider $f = 0$

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-ix\cdot\xi} u(x)\,dx$$

$$u(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \hat{u}(\xi)\,d\xi$$

$$\begin{cases} \hat{u}_t + |\xi|^2 \hat{u} = 0 \\ \hat{u}(0) = \hat{g} \end{cases}$$

$$u(x,t) = (2\pi)^{-n/2} \int e^{ix\cdot\xi - |\xi|^2 t} \hat{g}(\xi)\, d\xi \quad = \int K(x,y,t) f(y)\, dy$$

$$K(x,y,t) = (2\pi)^{-n} \int e^{i(x-\xi)\cdot\xi - |\xi|^2 t} d\xi$$

$$\xi = \frac{i(x-y)}{2t} + \frac{1}{\sqrt{t}} \eta$$

$$\int e^{-|\eta|^2} d\eta = \left(\int_{-\infty}^{\infty} e^{-s^2} ds \right)^n = \pi^{n/2}$$

$$K(x,y,t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

we obtain Poisson's formula

<u>Theorem 1</u>. Let f be continuous and bounded for $x \in \mathbb{R}^n$. Then

$$u(x,t) = \int K(x,y,t) f(y)\, dy$$

$u \in C^\infty$, $u_t = \Delta u$

$\lim_{x \to 0} u(x,t) = f(x)$.

The proof follows from basic properties of K

(a) $K \in C^\infty$

(b) $\left(\frac{\partial}{\partial t} - \Delta_x\right) K(x,y,t) = 0$, $t > 0$

(c) $K(x,y,t) > 0$, $t > 0$

(d) $\int K(x,y,t)\, dy = 1$, $\forall x \in \mathbb{R}^n$, $t > 0$

(e) $\forall \delta > 0$, $\lim_{t \to 0} \int_{|y-x| > \delta} K(x,y,t)\, dy = 0$, uniformly $\forall x \in \mathbb{R}^n$

Then

$$|u(x,t)-f(\xi)| = \left| \int K(x,y,t)\,(f(y)-f(\xi))\,dy \right|$$

$$\leq \int_{|y-x|<\delta} + \int_{|x-y|>\delta}$$

By the same type of argument one proves that if f is measura[ble] and

$$|f(x)| \leq M e^{a|x|^2}, \quad a, M > 0$$

then $u(x,t)$ satisfies $u_t = \Delta u$, $0<t<\frac{1}{4a}$

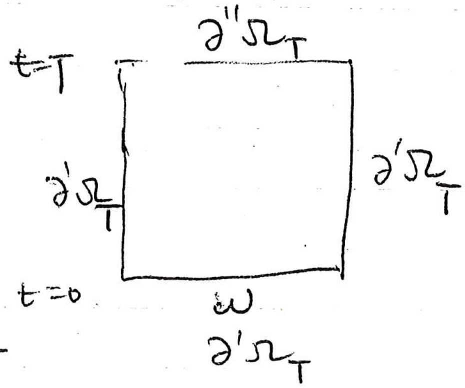
infinite-speed

Maximum Principle, uniqueness, and regularity

$$\Omega_T = \{(x,t) \mid x \in \Omega,\ 0<t<T\}$$

$$\partial'\Omega_T = \{(x,t) \mid x \in \partial\Omega,\ 0 \leq t \leq T \text{ or } x \in \Omega,\ t=0\}$$

$$\partial''\Omega_T = \{(x,t) \mid x \in \Omega,\ t=T\}$$



Theorem. $u_t - \Delta u \leq 0$. Then

$$\max_{\bar{\Omega}_T} u = \max_{\partial'\Omega_T} u$$

Maximum Principle $\Rightarrow$ Uniqueness in Ω_T

Proof: $v = u - kt$

$v_t - \Delta v < 0$ #

We can extend the maximum principle and uniqueness theorem to the case where Ω is the "slab"

$$\Omega = \{(x,t) \mid x \in \mathbb{R}^n,\ 0<t<T\}$$

Theorem. $u_t - \Delta u \leq 0, \quad 0<t<T, \quad x \in R^n$

$u(x,t) \leq M e^{ax^2} \quad 0<t<T, \quad x \in R^n$

$u(x,0) = f(x) \quad x \in R^n$

Then $u(x,t) \leq \sup_z f(z) \quad 0 \leq t \leq T, \; x \in R^n.$

Corollary: $\begin{cases} u_t - \Delta u = 0 & 0<t<T \\ u(x,0) = f(x) \end{cases}$

is unique if $|u(x,t)| \leq M e^{a|x|^2}, \quad 0<t<T$

Proof: It is sufficient to show that under $4aT<1$ since we can divide $0<t<T$ into small intervals

$u(x,t) \leq \sup_y u(y, k\tau) \leq \sup_y u(y,0) \quad k\tau \leq t \leq (k+1)\tau.$

Let $4a(T+\varepsilon) < 1$.

Given a fixed y, $\mu > 0$

$$V_\mu(x,t) = u(x,t) - \mu \left(4\pi(T+\varepsilon-t)\right)^{-n/2} \exp\left[|x-y|^2/4(T+\varepsilon-t)\right]$$

$0 \leq t \leq T$

$\partial_t V_\mu - \Delta V_\mu = u_t - \Delta u \leq 0$

Consider

$\Omega_T = \{(x,t) \mid |x-y|<\rho, \; 0<t<T\}$

$V_\mu(y,t) \leq \max_{\partial'\Omega} V_\mu$

on the plane part $\partial'\Omega$, $V_\mu(x,0) \leq u(x,0) \leq \sup_z f(z)$

on the curved part $|x-y| = \rho, \quad 0 \leq t < T$

$$V_\mu(x,t) \leq M e^{a|x|^2} - \mu \left(4\pi(T+\varepsilon-t)\right)^{-n/2} \exp\left[e^2/4(T+\varepsilon-t)\right]$$

$$\leq M e^{a(|y|+\rho)^2} - \mu \left(4\pi(T+\varepsilon)\right)^{-\frac{n}{2}} e^{\rho^2/4(T+\varepsilon)}$$

$\leq \sup f(z)$

if ρ is large enough, $\rho = \rho(\mu, \varepsilon, T)$.

$$\max_{\partial'\Omega} V_\mu (\cancel{y,t) = u(y,t)}) \le \sup f(z)$$

$$V_\mu(y,t) = u(y,t) - \mu\,(4(T+\varepsilon-t))^{-n/2} \le \sup f(z)$$

Letting $\mu \to 0$. #

The growth condition is necessary.

$$\begin{cases} u_t = u_{xx}, & t > 0 \\ u(x,0) = \cancel{g(t)}, 0 \end{cases}$$

Solve
$$\begin{cases} u_t = u_{xx} \\ u(0,t) = g(t), \\ u_x(0,t) = 0 \end{cases}$$

$$u = \sum_{j=0}^{+\infty} g_j(t)\, x^j$$

$$g_0 = g,\quad g_1 = 0,\quad g_j'(t) = (j+2)(j+1)\, g_{j+2} \Rightarrow g_{j+2} = \frac{1}{(j+1)(j+2)} g_j'(t)$$

$$u(x,t) = \sum_{k=0}^{+\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}$$

Now we choose
$$g(t) = \begin{cases} \exp[-t^{-\alpha}], & t > 0 \\ 0, & t \le 0, \end{cases} \qquad \alpha > 1$$

Then $\exists\ \theta = \theta(\alpha) > 0$ s.t.
$$|g^{(k)}(t)| \le \frac{k!}{(\theta t)^k} \exp\left[-\frac{1}{2} t^{-\alpha}\right]$$

Since $k!/(2k)! < 1/k!$

$$\sum_{k=0}^{+\infty} \left| \frac{g^{(k)}(t)}{(2k)!} x^{2k} \right| \le \sum_{k=0}^{+\infty} \frac{|x|^{2k}}{k!\,(\theta t)^k} \exp\left[-\tfrac{1}{2} t^{-\alpha}\right]$$

$$= \exp\left[\frac{1}{t}\left(\frac{|x|^2}{\theta} - \frac{1}{2} t^{1-\alpha}\right)\right]$$

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = 0 \end{cases} \qquad u \in C^\infty(\mathbb{R}^{n+1})$$

u is not bounded by $e^{a|x|^2}$ (for t small)

Inhomogeneous heat equation

$$\begin{cases} u_t - \Delta u = f \\ u = 0 \end{cases}$$

Duhamel's Principle: We solve

$$\begin{cases} u_t(\cdot\,; s) - \Delta u(\cdot, s) = 0, & t > s \\ u(\cdot, s) = f(\cdot\,; s) & t = s \end{cases}$$

Then

$$u(x,t) = \int_0^t u(x,t;s)\,ds$$

In fact, formally, $u_t = u(x,t;t) + \int_0^t \Delta u$
$= f(x,t) - \Delta u$

$$u(x,t) = \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy\,ds$$

Theorem: $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support

Then

$$\begin{cases} u_t - \Delta u = f \\ u(x,0) = 0 \end{cases}$$

Proof. $u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy\, ds$

$$u_t = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

$$\Delta u = \int_0^t \int \Phi(y,s) \Delta_x f(x-y, t-s)\, dy\, ds$$

$$u_t - \Delta u = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) [\partial_t - \Delta_x] f(x-y, t-s)\, dy\, ds + \int \Phi(y,t) f(x-y,0)\, dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) [(-\partial_s - \Delta_y)] f(x-y,t-s)]\, dy\, ds + \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) [-\partial_s - \Delta_y] f(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

$$= I_\varepsilon + J_\varepsilon + K$$

$$|J_\varepsilon| \le (\|f_t\|_{L^\infty} + \|D^2 f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\, dy\, ds \le \varepsilon C$$

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} [(\partial_s - \Delta_y)\Phi(y,s)] f(x-y, t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy$$

$$u_t - \Delta u = \lim_{\varepsilon \to 0} \int \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy$$
$$= f(x,t). \quad (\text{as } \varepsilon \to 0)$$
#

Summary

1) $f \in C^2_1$, compact support $\Rightarrow$
$g \in C^0$

$$u(x,t) = \int \Phi(x-y,t) g(y)\, dy + \int_0^t \int \Phi(x-y, t-s) f(y,s)\, dy\, ds$$

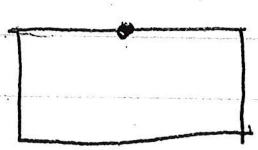
Solves
$$\begin{cases} u_t - \Delta u = f \\ u = g \end{cases}$$

2) This sol'n is unique if
$$|u(x,t)| \le e^{a|x|^2}$$

2) Maximum Principle works only for parabolic boundary

Finally, we discuss the local smoothness of heat equation.

Fix (x_0, t_0).
parabolic cylinder



$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0).$$
$$Q_r \equiv Q_r(0,0).$$

Thm. Suppose $u \in C^2_1(Q_r)$, $u_t - \Delta u = 0$ in Q_r

Then $u \in C^\infty(Q_r)$ and

$$\max_{Q_{\frac{r}{2}}} |D^k_x D^l_t u| \leq \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(Q_r)}$$

Observe dimension balance

Proof: 1) First assume $u \in C^\infty(Q_r)$.

Let $\zeta(x,t)$ be a cut-off fcn

$$\zeta \in C^\infty_{x,t}, \quad \zeta\Big|_{Q_{3/4}} = 1, \quad \zeta\Big|_{Q^c_{\frac{7}{8}}} = 0$$

Let $v = \zeta u$

$$v_t - \Delta u = (\zeta_t - \Delta\zeta) u - 2\nabla_x u \nabla_x \zeta = f$$

$$v = \tilde{v}(x,t) = \int_{-1}^t \int_{R^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds$$

$=$ integrate by parts

$u = v$ in $Q_{3/4}$

In $Q_{1/2}$ $D^k_x D^l_t u = D^k_x D^l_t \tilde{v} = \int_{-1}^t \int D^k_x D^l_t \underbrace{[\Phi(\zeta_t - \Delta\zeta) + 2\nabla(\Phi \nabla\zeta)]}_{\text{smooth}}$

$$|D^k_x D^l_t u(x,t)| \leq \int_{-1}^t \int_{B_1} |D^k_x D^l_t (\Phi(x-y, t-s))(\zeta_t - \Delta\zeta)|\, |u(y,s)|\, dy\, ds$$

$$\leq \int_{-1}^t \int_{B_1} C_{k,l} |u(y,s)|\, dy\, ds = C_{k,l} \|u\|_{L^1(Q_1)}$$

2) The general case

$u^\varepsilon = u * \eta$, η_ε mollifier in x,t

$$u^\varepsilon(x,t) = \iint_{Q_1} K(x,t; y,s)\, u^\varepsilon(y,s)\, dy\, ds \qquad \varepsilon \to 0 :$$