MATH 516-101-2022 Homework One
Due Date: September 21st, 2022

1. (10pts) Find the explicit solution to

$$u_t + 2u_x + 3u_y + 4u = e^x$$
$$u(x, y, 0) = x^2$$

2. (20pts) Let $f \in C_c^{\alpha}(R^2)$, where $\alpha \in (0, 1)$. Here $f \in C^{\alpha}$ means $f \in C^0$ and $|f(x) - f(y)| \leq C|x - y|^{\alpha}$. Show that the following function

$$v(x) = \frac{1}{2\pi} \int_{R^2} (\log |x - y|) f(y) dy$$

satisfies

(a) $v \in C^2$

(b) $\Delta v(x) = f(x), x \in R^2$.

3. (20pts) This problem concerns the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} \frac{1}{(1 + |y|)^l} dy \tag{1}$$

where $n \geq 3$.

a) Show that for $l \in (2, n)$, $|u(x)| \leq \frac{C}{|x|^{l-2}}$ for $|x| > 1$

b) Show that for $l = n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}} \log(|x| + 2)$ for $|x| > 1$

c) Show that for $l > n$, then $|u(x)| \leq \frac{C}{|x|^{n-2}}$ for $|x| > 1$

Hint: For $|x| = R >> 1$, divide the integral into three parts

$$\int_{R^n} (...) dy = \int_{|y-x|<\frac{|x|}{2}} (...) + \int_{\frac{|x|}{2}<|y-x|<2|x|} (...) + \int_{|y-x|>2|x|} (...)$$

and estimate each parts. For example in the region $|y - x| < \frac{|x|}{2}$ we have $|y| > |x| - |x - y| > \frac{|x|}{2}$ and

$$\int_{|y-x|<\frac{|x|}{2}} \frac{1}{|x - y|^{n-2}} |f(y)| dy \leq \int_0^{\frac{|x|}{2}} \frac{r^{n-1}}{r^{n-2}} dr \frac{C}{|x|^l} \leq \frac{C}{|x|^{l-2}}$$

4. (10pts) For $n > 2$, the Kelvin transform is defined by

$$v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$$

Suppose u satisfies $-\Delta u(x) = u^p$. Find out the new equation satisfied by v. Find out for which exponent p the equation is invariant.

5. (10pts) Let u be a harmonic function in $B_1(0) \backslash \{0\} = \{0 < |x| < 1\}$ be such that $\lim_{x \to 0} |x|^{n-2} u(x) = 0$. Show that $u \in C^2(B_1(0))$. Here $n \geq 3$.

6. (10pts) Let u be harmonic in $\Omega \subset \mathbb{R}^n$. Show that for all $x_0 \in \Omega$

$$|\nabla u(x_0)| \leq \frac{n}{d_0} [\sup_{\Omega} u - u(x_0)], d_0 = d(x, \partial\Omega).$$

Hint: Do gradient estimate for $\sup_{\Omega} u - u(x)$. Note that this function is nonnegative.

7. (20pts) Let $G(x, y) = \Gamma(|x - y|) - H(x, y)$ be the Green's function in Ω and define:

$$v(x) = \int_{\Omega} G(x, y) f(y) dy$$

Suppose that f is bounded and integrable in Ω. Show that $\lim_{x \to x_0, x \in \Omega} v(x) = 0$ for any $x_0 \in \partial\Omega$.