equations in Banach spaces, namely the Contraction Mapping Principle and the Fredholm alternative.

5.1. The Contraction Mapping Principle

A mapping T from a normed linear space $\mathscr{V}$ into itself is called a *contraction mapping* if there exists a number $\theta < 1$ such that

$$\|Tx - Ty\| \leqslant \theta \|x - y\| \quad \text{for all } x, y \in \mathscr{V} \tag{5.1}$$

Theorem 5.1. *A contraction mapping T in a Banach space $\mathscr{B}$ has a unique fixed point, that is there exists a unique solution $x \in \mathscr{B}$ of the equation $Tx = x$.*

Proof. (*Method of successive approximations.*) Let $x_0 \in \mathscr{B}$ and define a sequence $\{x_n\} \subset \mathscr{B}$ by $x_n = T^n x_0$, $n = 1, 2, \ldots$ Then if $n \geqslant m$, we have

$$\begin{aligned}
\|x_n - x_m\| &\leqslant \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \quad \text{by the triangle inequality} \\
&= \sum_{j=m+1}^{n} \|T^{j-1}x_1 - T^{j-1}x_0\| \\
&\leqslant \sum_{j=m+1}^{n} \theta^{j-1}\|x_1 - x_0\| \quad \text{by (5.1)} \\
&\leqslant \frac{\|x_1 - x_0\|\theta^m}{1-\theta} \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Consequently $\{x_n\}$ is a Cauchy sequence and, since $\mathscr{B}$ is complete, converges to an element $x \in \mathscr{B}$. Clearly T is also a continuous mapping and hence we have

$$Tx = \lim Tx_n = \lim x_{n+1} = x$$

so that x is a fixed point of T. The uniqueness of x follows immediately from (5.1). □

In the statement of Theorem 5.1, the space $\mathscr{B}$ can obviously be replaced by any closed subset.

5.2. The Method of Continuity

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be normed linear spaces. A linear mapping $T\colon \mathscr{V}_1 \to \mathscr{V}_2$ is *bounded* if the quantity

$$\|T\| = \sup_{x \in \mathscr{V}_1, x \neq 0} \frac{\|Tx\|_{\mathscr{V}_2}}{\|x\|_{\mathscr{V}_1}} \tag{5.2}$$

is finite. It is easy to show that a linear mapping T is bounded if and only if it is continuous. The invertibility of a bounded linear mapping may sometimes be deduced from the invertibility of a similar mapping through the following theorem, which is known in applications as the *method of continuity*.

Theorem 5.2. *Let $\mathscr{B}$ be a Banach space, $\mathscr{V}$ a normed linear space and let L_0, L_1 be bounded linear operators from $\mathscr{B}$ into $\mathscr{V}$. For each $t \in [0, 1]$, set*

$$L_t = (1-t)L_0 + tL_1$$

and suppose that there is a constant C such that

$$(5.3) \qquad \|x\|_{\mathscr{B}} \leqslant C\|L_t x\|_{\mathscr{V}}$$

for $t \in [0, 1]$. Then L_1 maps $\mathscr{B}$ onto $\mathscr{V}$ if and only if L_0 maps $\mathscr{B}$ onto $\mathscr{V}$.

Proof. Suppose that L_s is onto for some $s \in [0, 1]$. By (5.3), L_s is one-to-one and hence the inverse mapping $L_s^{-1}: \mathscr{V} \to \mathscr{B}$ exists. For $t \in [0, 1]$ and $y \in \mathscr{V}$, the equation $L_t x = y$ is equivalent to the equation

$$\begin{aligned} L_s(x) &= y + (L_s - L_t)x \\ &= y + (t-s)L_0 x - (t-s)L_1 x \end{aligned}$$

which in turn, is equivalent to the equation

$$x = L_s^{-1}y + (t-s)L_s^{-1}(L_0 - L_1)x$$

The mapping T from $\mathscr{B}$ into itself given by $Tx = L_s^{-1}y + (t-s)L_s^{-1}(L_0 - L_1)x$ is clearly a contraction mapping if

$$|s-t| < \delta = [C(\|L_0\| + \|L_1\|)]^{-1}$$

and hence the mapping L_t is onto for all $t \in [0, 1]$, satisfying $|s-t| < \delta$. By dividing the interval $[0, 1]$ into subintervals of length less than δ, we see that the mapping L_t is onto for all $t \in [0, 1]$ provided it is onto for any fixed $t \in [0, 1]$, in particular for $t = 0$ or $t = 1$. □

5.3. The Fredholm Alternative

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be normed linear spaces. A mapping $T: \mathscr{V}_1 \to \mathscr{V}_2$ is called *compact* (or *completely continuous*) if T maps bounded sets in $\mathscr{V}_1$ into relatively compact sets in $\mathscr{V}_2$ or equivalently T maps bounded sequences in $\mathscr{V}_1$ into sequences in $\mathscr{V}_2$ which contain convergent subsequences. It follows that a compact linear mapping is also continuous but the converse is not true in general unless $\mathscr{V}_2$ is finite dimensional.

The Fredholm alternative (or Riesz-Schauder theory) concerns compact linear operators from a space $\mathscr{V}$ into itself and is an extension of the theory of linear mappings in finite dimensional spaces.

Theorem 5.3. *Let T be a compact linear mapping of a normed linear space $\mathscr{V}$ into itself. Then either* (i) *the homogeneous equation*

$$x - Tx = 0$$

has a nontrivial solution $x \in \mathscr{V}$ or (ii) *for each $y \in \mathscr{V}$ the equation*

$$x - Tx = y$$

has a uniquely determined solution $x \in \mathscr{V}$. Furthermore, in case (ii), *the operator $(I - T)^{-1}$ whose existence is asserted there is also bounded.*

The proof of Theorem 5.3 depends upon the following simple result of Riesz.

Lemma 5.4. *Let $\mathscr{V}$ be a normed linear space and $\mathscr{M}$ a proper closed subspace of $\mathscr{V}$. Then for any $\theta < 1$, there exists an element $x_\theta \in \mathscr{V}$ satisfying $\|x_\theta\| = 1$ and* dist $(x_\theta, \mathscr{M}) \geq \theta$.

Proof. Let $x \in \mathscr{V} - \mathscr{M}$. Since $\mathscr{M}$ is closed, we have

$$\text{dist}\,(x, \mathscr{M}) = \inf_{y \in M} \|x - y\| = d > 0.$$

Consequently there exists an element $y_\theta \in \mathscr{M}$ such that

$$\|x - y_\theta\| \leq \frac{d}{\theta},$$

so that, defining

$$x_\theta = \frac{x - y_\theta}{\|x - y_\theta\|},$$

we have $\|x_\theta\| = 1$ and for any $y \in \mathscr{M}$,

$$\begin{aligned} \|x_\theta - y\| &= \frac{\|x - y_\theta - \|y_\theta - x\|\, y\|}{\|y_\theta - x\|} \\ &\geq \frac{d}{\|y_\theta - x\|} \geq \theta. \end{aligned}$$

The lemma is thus proved. □

The element y is called the *orthogonal projection* of x on $\mathscr{M}$. Theorem 5.6 also shows that any closed proper subspace of $\mathscr{H}$ is orthogonal to some element of $\mathscr{H}$.

5.7. The Riesz Representation Theorem

The Riesz representation theorem provides an extremely useful characterization of the bounded linear functionals on a Hilbert space as inner products.

Theorem 5.7. *For every bounded linear functional F on a Hilbert space $\mathscr{H}$, there is a uniquely determined element $f \in \mathscr{H}$ such that $F(x)=(x, f)$ for all $x \in \mathscr{H}$ and* $\|F\| = \|f\|$.

Proof. Let $\mathscr{N}=\{x|F(x)=0\}$ be the null space of F. If $\mathscr{N}=\mathscr{H}$, the result is proved by taking $f=0$. Otherwise, since $\mathscr{N}$ is a closed subspace of $\mathscr{H}$, there exists by Theorem 5.6 an element $z\neq 0, \in \mathscr{H}$ such that $(x, z)=0$ for all $x \in \mathscr{N}$. Hence $F(z)\neq 0$ and moreover for any $x \in \mathscr{H}$

$$F\left(x-\frac{F(x)}{F(z)}z\right)=F(x)-\frac{F(x)}{F(z)}F(z)=0$$

so that the element $x-\frac{F(x)}{F(z)}z \in \mathscr{N}$. This means that

$$\left(x-\frac{F(x)}{F(z)}z,\ z\right)=0,$$

that is, that

$$(x, z)=\frac{F(x)}{F(z)}\|z\|^2$$

and hence $F(x)=(f, x)$ where $f=zF(z)/\|z\|^2$. The uniqueness of f is easily proved and is left to the reader. To show that $\|F\| = \|f\|$, we have first, by the Schwarz inequality,

$$\|F\| = \sup_{x\neq 0}\frac{|(x,f)|}{\|x\|}\leqslant \sup_{x\neq 0}\frac{\|x\|\ \|f\|}{\|x\|}=\|f\|;$$

and secondly,

$$\|f\|^2=(f,f)=F(f)\leqslant \|F\|\ \|f\|,$$

so that $\|f\|\leqslant\|F\|$, and hence $\|F\|=\|f\|$. □

Theorem 5.7 shows that the dual space of a Hilbert space may be identified with the space itself and consequently that Hilbert spaces are reflexive.

5.8. The Lax-Milgram Theorem

The Riesz representation theorem suffices for the treatment of linear elliptic equations that are variational, that is, they are the Euler-Lagrange equations of certain multiple integrals. For general divergence structure equations we will require a slight extension of Theorem 5.7 due to Lax and Milgram. A bilinear form **B** on a Hilbert space $\mathscr{H}$ is called *bounded* if there exists a constant K such that

$$|\mathbf{B}(x, y)| \leqslant K\|x\| \, \|y\| \quad \text{for all } x, y \in \mathscr{H} \tag{5.10}$$

and *coercive* if there exists a number $\nu > 0$ such that

$$\mathbf{B}(x, x) \geqslant \nu\|x\|^2 \quad \text{for all } x \in \mathscr{H}. \tag{5.11}$$

A particular example of a bounded, coercive bilinear form is the inner product itself.

Theorem 5.8. *Let* **B** *be a bounded, coercive bilinear form on a Hilbert space* $\mathscr{H}$. *Then for every bounded linear functional* $F \in \mathscr{H}^*$, *there exists a unique element* $f \in \mathscr{H}$ *such that*

$$\mathbf{B}(x, f) = F(x) \quad \textit{for all } x \in \mathscr{H}.$$

Proof. By virtue of Theorem 5.7, there exists a linear mapping $T\colon \mathscr{H} \to \mathscr{H}$ defined by $\mathbf{B}(x, f) = (x, Tf)$ for all $x \in \mathscr{H}$. Furthermore $\|Tf\| \leqslant K\|f\|$ by (5.10) so that T is bounded. By (5.11) we obtain $\nu\|f\|^2 \leqslant \mathbf{B}(f, f) = (f, Tf) \leqslant \|f\| \, \|Tf\|$, so that

$$\nu\|f\| \leqslant \|Tf\| \leqslant K\|f\| \quad \text{for all } f \in \mathscr{H}.$$

This estimate implies that T is one-to-one, has closed range (see Problem 5.3) and that T^{-1} is bounded. Suppose that T is not onto $\mathscr{H}$. Then there exists an element $z \neq 0$ satisfying $(z, Tf) = 0$ for all $f \in \mathscr{H}$. Choosing $f = z$, we obtain $(z, Tz) = \mathbf{B}(z, z) = 0$ implying $z = 0$ by (5.11). Consequently T^{-1} is a bounded linear mapping on $\mathscr{H}$. We then have $F(x) = (x, g) = \mathbf{B}(x, T^{-1}g)$ for all $x \in \mathscr{H}$ and some unique $g \in \mathscr{H}$ and the result is proved with $f = T^{-1}g$. □

5.9. The Fredholm Alternative in Hilbert Spaces

Theorems 5.3 and 5.5 are of course applicable to compact operators in Hilbert spaces. Let us derive now for Hilbert spaces our earlier remarks concerning adjoints in Banach spaces. In light of Theorem 5.7, we define the adjoint slightly differently *have uniquely determined solutions $x \in \mathscr{H}$ for every $y \in \mathscr{H}$, and the inverse mappings $(\lambda I - T)^{-1}, (\lambda I - T^*)^{-1}$ are bounded. If $\lambda \in \Lambda$, the null spaces of the mappings $\lambda I - T$, $\lambda I - T^*$ have positive finite dimension and the equations (5.13) are solvable if and only if y is orthogonal to the null space of $\lambda I - T^*$ in the first case and $\lambda I - T$ in the other.*

5.10. Weak Compactness

Let $\mathscr{V}$ be a normed linear space. A sequence $\{x_m\}$ *converges weakly* to an element $x \in \mathscr{V}$ if $f(x_n) \to f(x)$ for all f in the dual space $\mathscr{V}^*$. By the Riesz representation theorem, Theorem 5.7, a sequence $\{x_n\}$ in a Hilbert space $\mathscr{H}$ will converge weakly to $x \in \mathscr{H}$ if $(x_n, y) \to (x, y)$ for all $y \in \mathscr{H}$. The following result is useful in the Hilbert space approach to differential equations.

Theorem 5.12. *A bounded sequence in a Hilbert space contains a weakly convergent subsequence.*

Proof. Let us assume initially that $\mathscr{H}$ is separable and suppose that the sequence $\{x_n\} \subset \mathscr{H}$ satisfies $\|x_n\| \leqslant M$. Let $\{y_m\}$ be a dense subset of $\mathscr{H}$. By the Cantor diagonal process we obtain a subsequence $\{x_{n_k}\}$ of our original sequence satisfying $(x_{n_k}, y_m) \to \alpha_m \in \mathbb{R}$ as $k \to \infty$. The mapping $f: \{y_m\} \to \mathbb{R}$ defined by $f(y_m) = \alpha_m$ may consequently be extended to a bounded linear functional f on $\mathscr{H}$ and hence by the Riesz representation theorem, there exists an element $x \in \mathscr{H}$ satisfying $(x_{n_k}, y) \to f(y) = (x, y)$ as $k \to \infty$, for all $y \in \mathscr{H}$. Hence the subsequence $\{x_{n_k}\}$ converges weakly to x.

To extend the result to an arbitrary Hilbert space $\mathscr{H}$, we let $\mathscr{H}_0$ be the closure of the linear hull of the sequence $\{x_n\}$. Then by our previous argument there exists a subsequence $\{x_{n_k}\} \subset \mathscr{H}_0$ and an element $x \in \mathscr{H}_0$ satisfying $(x_{n_k}, y) \to (x, y)$ for all $y \in \mathscr{H}_0$. But by Theorem 5.6, we have for arbitrary $y \in \mathscr{H}$, $y = y_0 + y_1$, where $y_0 \in \mathscr{H}_0$, $y_1 \in \mathscr{H}_0^\perp$. Hence $(x_{n_k}, y) = (x_{n_k}, y_0) \to (x, y_0) = (x, y)$ for all $y \in \mathscr{H}$ so that $\{x_{n_k}\}$ converges weakly to x, as required. □

The first part of the proof of Theorem 5.12 extends automatically to reflexive Banach spaces with separable dual spaces (see Problem 5.4). The result is true however for arbitrary reflexive Banach spaces (see [YO]).

Notes

The material in this chapter is standard and can be found in texts on functional analysis such as [DS], [EW] and [YO].

Jan. 3, 2018

① Functional Analysis. • Banach space • Hilbertspace •

Appendix A

Here, we collect without proof a few basic results about Sobolev spaces. A general reference to this topic is Gilbarg-Trudinger [1], or Adams [1].

Sobolev Spaces

Let Ω be a domain in $\mathbb{R}^n$. For $u \in L^1_{loc}(\Omega)$ and any multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, with $|\alpha| = \sum_{j=1}^n \alpha_j$, define the distibutional derivative $D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$ by letting

$$< \varphi, D^\alpha u > = \int_\Omega (-1)^{|\alpha|} u \, D^\alpha \varphi \, dx, \tag{A.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. We say $D^\alpha u \in L^p(\Omega)$, if there is a function $g_\alpha \in L^p(\Omega)$ satisfying

$$< \varphi, D^\alpha u > = < \varphi, g_\alpha > = \int_\Omega \varphi g_\alpha \, dx,$$

for all $\varphi \in C_0^\infty(\Omega)$. In this case we identify $D^\alpha u$ with $g_\alpha \in L^p(\Omega)$.

Given this, for $k \in \mathbb{N}_0$, $1 \le p \le \infty$, define the space

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega); D^\alpha u \in L^p(\Omega) \quad \text{for all} \quad \alpha\colon |\alpha| \le k\},$$

with norm

$$\|u\|^p_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p}, \quad \text{if} \;\; 1 \le p < \infty,$$

respectively, with norm

$$\|u\|_{W^{k,\infty}} = \max_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty}.$$

Note that the distributional derivative (A.1) is continuous with respect to weak convergence in $L^1_{loc}(\Omega)$.

Many properties of $L^p(\Omega)$ carry over to $W^{k,p}(\Omega)$.

A.1 Theorem. *For any $k \in \mathbb{N}_0$, $1 \le p \le \infty$, $W^{k,p}(\Omega)$ is a Banach space. $W^{k,p}(\Omega)$ is reflexive if and only if $1 < p < \infty$. Moreover, $W^{k,2}(\Omega)$ is a Hilbert space with scalar product*

$$(u, v)_{W^{k,2}} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u \, D^\alpha v \, dx \ ,$$

inducing the norm above.

For $1 \le p < \infty$, $W^{k,p}(\Omega)$ also is separable. In fact, we have the following result due to Meyers and Serrin; see Adams [1; Theorem 3.16].

A.2 Theorem. *For any $k \in \mathbb{N}_0, 1 \le p < \infty$, the subspace $W^{k,p} \cap C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

The completion of $W^{k,p} \cap C^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $H^{k,p}(\Omega)$. By Theorem A.2, $W^{k,p}(\Omega) = H^{k,p}(\Omega)$. In particular, if $p = 2$ it is customary to use the latter notation.

Finally, $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$; in particular, $H_0^{k,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^{k,2}(\Omega)$, with dual $H^{-k}(\Omega)$. $\mathcal{D}^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{D^{k,p}}^p = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}^p .$$

Hölder Spaces

A function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous with exponent $\beta > 0$ if

$$[u]^{(\beta)} = \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty.$$

For $m \in \mathbb{N}_0$, $0 < \beta \le 1$, denote

$$C^{m,\beta}(\Omega) = \big\{u \in C^m(\Omega);\ D^\alpha u \text{ is Hölder continuous with exponent } \beta \text{ for all } \alpha\colon |\alpha| = m\big\} .$$

If Ω is relatively compact, $C^{m,\beta}(\bar{\Omega})$ becomes a Banach space in the norm

$$\|u\|_{C^{m,\beta}} = \sum_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty} + \sum_{|\alpha|=m} [D^\alpha u]^{(\beta)}.$$

The space $C^{m,\beta}(\Omega)$ on an open domain $\Omega \subset \mathbb{R}^n$ carries a Fréchet space topology, induced by the $C^{m,\beta}$-norms on compact sets exhausting Ω. Finally, we may set $C^{m,0}(\Omega) := C^m(\Omega)$. Observe that for $0 < \beta \le 1$ smooth functions are *not* dense in $C^{m,\beta}(\bar{\Omega})$.

Imbedding Theorems

Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. X is (continuously) embedded into Y (denoted $X \hookrightarrow Y$) if there exists an injective linear map $i\colon X \to Y$ and a constant C such that

$$\|i(x)\|_Y \le C\|x\|_X, \quad \text{for all } x \in X.$$

In this case we will often simply identify X with the subspace $i(X) \subset Y$.

X is compactly embedded into Y if i maps bounded subsets of X into relatively compact subsets of Y.

For the spaces that we are primarily interested in we have the following results. First, from Hölder's inequality we obtain:

A.3 Theorem. *For $\Omega \subset \mathbb{R}^n$ with Lebesgue measure $\mathcal{L}^n(\Omega) < \infty$, $1 \le p < q \le \infty$, we have $L^q(\Omega) \hookrightarrow L^p(\Omega)$. This ceases to be true if $\mathcal{L}^n(\Omega) = \infty$.*

For Hölder spaces, by the theorem of Arzéla-Ascoli we have the following compactness result; see Adams [1; Theorems 1.30, 1.31].

A.4 Theorem. *Suppose Ω is a relatively compact domain in $\mathbb{R}^n$, and let $m \in \mathbb{N}_0$, $0 \le \alpha < \beta \le 1$. Then $C^{m,\beta}(\bar{\Omega}) \hookrightarrow C^{m,\alpha}(\bar{\Omega})$ compactly.*

Finally, for Sobolev spaces we have (see Adams [1; Theorem 5.4]):

A.5 Theorem *(Sobolev embedding theorem). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$, $1 \le p \le \infty$. Then the following holds:*
(1°) If $kp < n$, we have $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le \frac{np}{n-kp}$; the embedding is compact, if $q < \frac{np}{n-kp}$.
(2°) If $0 \le m < k - \frac{n}{p} < m+1$, we have $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\bar{\Omega})$, for $0 \le \alpha \le k - m - \frac{n}{p}$; the embedding is compact, if $\alpha < k - m - \frac{n}{p}$.

Compactness of the embedding $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q < \frac{np}{n-kp}$ is a consequence of the Rellich-Kondrakov theorem; see Adams [1; Theorem 6.2]. Theorem A.5 is valid for $W_0^{k,p}(\Omega)$-spaces on arbitrary bounded domains Ω.

Density Theorem

By Theorem A.2, Sobolev functions can be approximated by functions enjoying any degree of smoothness in the interior of Ω. Some regularity condition on the boundary $\partial\Omega$ is necessary if smoothness up to the boundary is required:

A.6 Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1, and let $k \in \mathbb{N}$, $1 \le p < \infty$. Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

More generally, it suffices that Ω has the segment property; see for instance Adams [1; Theorem 3.18].

Trace and Extension Theorems

For a domain Ω with C^k-boundary $\partial\Omega = \Gamma$, $k \in \mathbb{N}$, $1 < p < \infty$, denote by $W^{k-\frac{1}{p},p}(\Gamma)$ the space of "traces" $u|_\Gamma$ of functions $u \in W^{k,p}(\Omega)$. We think of $W^{k-\frac{1}{p},p}(\Gamma)$ as the set of equivalence classes $\{\{u\} + W_0^{k,p}(\Omega); u \in W^{k,p}(\Omega)\}$, endowed with the trace norm

$$\|u|_\Gamma\|_{W^{k-\frac{1}{p},p}(\Gamma)} = \inf\{\|v\|_{W^{k,p}(\Omega)}; u - v \in W_0^{k,p}(\Omega)\}.$$

By this definition, $W^{k-\frac{1}{p},p}(\Gamma)$ is a Banach space. Moreover, in case $k = 1$, $p = 2$ the trace operator $u \mapsto u|_{\partial\Omega}$ is a linear isometry of the (closed) orthogonal complement of $H_0^{1,2}(\Omega)$ in $H^{1,2}(\Omega)$ onto $H^{\frac{1}{2},2}(\Gamma)$. By the open mapping theorem this provides a bounded "extension operator" $H^{\frac{1}{2},2}(\Gamma) \to H^{1,2}(\Omega)$. In general, we have:

A.7 Theorem. *For any Ω with C^k-boundary Γ, $k \in \mathbb{N}$, $1 < p < \infty$ there exists a continuous linear extension operator* ext: $W^{k-\frac{1}{p},p}(\Gamma) \to W^{k,p}(\Omega)$ *such that* $(\mathrm{ext}(u))\big|_\Gamma = u$, *for all* $u \in W^{k-\frac{1}{p},p}(\Gamma)$.

See Adams [1; Theorem 7.53 and 7.55].

Covering $\partial\Omega = \Gamma$ by coordinate patches and defining Sobolev spaces $W^{k,p}(\Gamma)$ as before via such charts (see Adams [1; 7.51]), an equivalent norm for $W^{s,p}(\Gamma)$, where $k < s < k+1$, $s = k + \sigma$, is given by

$$\|u\|_{\tilde{W}^{s,p}} = \left\{ \|u\|^p_{W^{k,p}} + \sum_{|\alpha|=k} \int_\Gamma \int_\Gamma \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n-1+\sigma p}}\, dx\, dy \right\}^{1/p} ;$$

see Adams [1; Theorem 7.48].

From this, the following may be deduced:

A.8 Theorem. *Suppose Ω is a bounded domain with C^k-boundary Γ, $k \in \mathbb{N}$, $1 < p < \infty$. Then $W^{k,p}(\Gamma) \hookrightarrow W^{k-\frac{1}{p},p}(\Gamma) \hookrightarrow W^{k-1,p}(\Gamma)$ and both embeddings are compact.*

In particular, we have

$$H^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega) \tag{A.2}$$

compactly, for any bounded domain of class C^1.

Poincaré Inequality

For a bounded domain Ω of diameter d and $u \in H_0^{1,2}(\Omega)$ there holds

$$\int_\Omega |u|^2 dx \le d^2 \int_\Omega |\nabla u|^2\, dx\ . \tag{A.3}$$

This follows immediately from Hölder's inequality and the mean value theorem. (It suffices to consider $\Omega \subset [0,d] \times \mathbb{R}^{n-1} = S$, $u \in C_0^\infty(\Omega) \subset C_0^\infty(S)$.) More generally, we state:

A.9 Theorem. *For any bounded domain Ω of class C^1 there exists a constant $c = c(\Omega)$ such that for any $u \in H^{1,2}(\Omega)$ we have*

$$\int_\Omega |u|^2\, dx \le c \int_\Omega |\nabla u|^2\, dx + c \int_{\partial\Omega} |u|^2\, do\ .$$

Proof. The argument is modelled on Nečas [1; p. 18 f.]. Suppose by contradiction that for a sequence (u_m) in $H^{1,2}(\Omega)$ there holds

$$\|u_m\|^2_{L^2(\Omega)} \ge m \left(\|\nabla u_m\|^2_{L^2(\Omega)} + \|u_m\|^2_{L^2(\partial\Omega)} \right)\ . \tag{A.4}$$

By homogeneity, we may normalize

$$\|u_m\|^2_{L^2(\Omega)} = 1 \ .$$

But then (u_m) is bounded in $H^{1,2}(\Omega)$, and we may assume that $u_m \to u$ weakly. Moreover, by Theorem A.5.(1°), it follows that $u_m \to u$ strongly in $L^2(\Omega)$ and by (A.2) also $u_m|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$.

But (A.4) also implies that $\nabla u_m \to 0$ in $L^2(\Omega)$, and $u_m|_{\partial\Omega} \to 0$ in $L^2(\partial\Omega)$. Hence, $u \in H_0^{1,2}(\Omega)$; moreover, $\nabla u = 0$. By (A.3) therefore, $u \equiv 0$. But $\|u\|_{L^2(\Omega)} = \lim_{m\to\infty} \|u_m\|_{L^2(\Omega)} = 1$. Contradiction. □

In the same spirit the following variant of Poincaré's inequality may be derived.

A.10 Theorem. *Let $A_R = B_{2R}(0) \setminus B_R(0) \subset \mathbb{R}^n$ denote the annulus of size R in $\mathbb{R}^n$. There exists a constant $c = c(n,p)$ such that for any $R > 0$, any $u \in H^{1,p}(A_R)$ there holds*

$$\int_{A_R} |u - \overline{u}_R|^p \, dx \le c\, R^p \int_{A_R} |\nabla u|^p \, dx \ ,$$

where $\overline{u}_R$ denotes the mean of u over the annulus A_R.

Proof. Scaling with R, we may assume that $R = 1$, $A_R = A_1 =: A$. Moreover, it suffices to consider $\overline{u} = 0$. If for a sequence (u_m) in $H^{1,p}(A)$ with $\overline{u}_m = 0$ we have

$$1 = \int_A |u_m|^p \, dx \ge m \int_A |\nabla u_m|^p \, dx \ ,$$

by Theorem A.5 we conclude that $u_m \to u \equiv const. = \overline{u} = 0$ in $L^p(A)$. Contradiction. □

Appendix B

In this appendix we recall some fundamental estimates for elliptic equations. A basic reference is Gilbarg-Trudinger [1].

On a domain $\Omega \subset \mathbb{R}^n$ we consider second order elliptic differential operators of the form

$$Lu = -a_{ij}\frac{\partial^2}{\partial x_j \partial x_j}u + b_i \frac{\partial}{\partial x_i}u + cu, \tag{B.1}$$

or in divergence form

$$Lu = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}u\right) + cu\ , \tag{B.2}$$

with bounded coefficients $a_{ij} = a_{ji}, b_i$, and c satisfying the ellipticity condition

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

with a uniform constant $\lambda > 0$, for all $\xi \in \mathbb{R}^n$. By convention, repeated indeces are summed from 1 to n. The standard example is the operator $L = -\Delta$. If $a_{ij} \in C^1$, then any operator of type (B.2) also falls into category (B.1) with $b_j = -\frac{\partial}{\partial x_i}a_{ij}$.

Schauder Estimates

Let us first consider the (classical) C^α-setting; see Gilbarg-Trudinger [1; Theorems 6.2, 6.6].

B.1 Theorem. *Let L be an elliptic operator of type (B.1), with coefficients of class C^α, and let $u \in C^2(\Omega)$. Suppose $Lu = f \in C^\alpha(\bar{\Omega})$. Then $u \in C^{2,\alpha}(\Omega)$, and for any $\Omega' \subset\subset \Omega$ we have*

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\bar{\Omega})}\right). \tag{B.3}$$

If in addition Ω is of class $C^{2+\alpha}$, and if $u \in C^o(\bar{\Omega})$ coincides with a function $u_o \in C^{2+\alpha}(\bar{\Omega})$ on $\partial\Omega$, then $u \in C^{2,\alpha}(\bar{\Omega})$ and

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\bar{\Omega})} + \|u_o\|_{C^{2,\alpha}}(\bar{\Omega})\right) \tag{B.4}$$

with constants C possibly depending on L, Ω, n, α, and – in the first case – on Ω'.

L^p-Theory

For solutions in Sobolev spaces the Calderón-Zygmund inequality is the counterpart of the Schauder estimates for classical solutions; see Gilbarg-Trudinger [1; Theorems 9.11, 9.13].

B.2 Theorem. *Let L be elliptic of type (B.1) with continuous coefficients a_{ij}. Suppose $u \in H^{2,p}_{loc}(\Omega)$ satisfies $Lu = f$ in Ω with $f \in L^p(\Omega)$, $1 < p < \infty$. Then for any $\Omega' \subset\subset \Omega$ we have*

$$\|u\|_{H^{2,p}(\Omega')} \leq C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right) . \tag{B.5}$$

If in addition Ω is of Class $C^{1,1}$, and if there exists a function $u_o \in H^{2,p}(\Omega)$ such that $u - u_o \in H^{1,p}_o(\Omega)$, then

$$\|u\|_{H^{2,p}(\Omega)} \leq C\left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|u_o\|_{H^{2,p}(\Omega)}\right) \tag{B.6}$$

The constants C may depend on L, Ω, n, p, and – in the first case – on Ω'.

Weak Solutions

Let L be elliptic of divergence type (B.2), $f \in H^{-1}(\Omega)$. A function $u \in H^{1,2}_0(\Omega)$ weakly solves the equation $Lu = f$ if

$$\int_\Omega \left(a_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}\varphi + cu\varphi\right)dx - \int_\Omega f\varphi dx = 0, \text{ for all } \varphi \in C^\infty_o(\Omega).$$

The integral

$$\mathcal{L}(u,\varphi) = \int_\Omega \left(a_{ij}\frac{\partial}{\partial x_i}u\frac{\partial}{\partial x_j}\varphi + cu\varphi\right) dx$$

continuously extends to a symmetric bilinear form $\mathcal{L}$ on $H^{1,2}_0(\Omega)$, the Dirichlet form associated with the operator L.

A Regularity Result

As an application we consider the equation

$$-\Delta u = g(\cdot, u) \text{ in } \Omega, \tag{B.7}$$

on a domain $\Omega \subset \mathbb{R}^n$, with a Caratheodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$; that is, assuming $g(x,u)$ is measurable in $x \in \Omega$ and continuous in $u \in \mathbb{R}$. Moreover, we will assume that g satisfies the growth condition

$$|g(x,u)| \leq C\left(1 + |u|^p\right) \tag{B.8}$$

where $p \leq \frac{n+2}{n-2}$, if $n \geq 3$. By (B.8) and Theorem A.5, for any $u \in H^{1,2}(\Omega)$ the composed function $g(\cdot, u(\cdot)) \in H^{-1}(\Omega)$; see also Theorem C.2. The following estimate is essentially due to Brezis-Kato [1], based on Moser's [1] iteration technique.

B.3 Lemma. *Let Ω be a domain in $\mathbb{R}^n$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \Omega$ there holds*

$$(1^\circ) \qquad |g(x,u)| \le a(x)(1+|u|)$$

with a function $a \in L^{n/2}_{loc}(\Omega)$. Also let $u \in H^{1,2}_{loc}(\Omega)$ be a weak solution of equation (B.7). Then $u \in L^q_{loc}(\Omega)$ for any $q < \infty$. If $u \in H^{1,2}_0(\Omega)$, and $a \in L^{n/2}(\Omega)$, then $u \in L^q(\Omega)$ for any $q < \infty$.

Proof. Choose $\eta \in C_0^\infty(\Omega)$ and for $s \ge 0$, $L \ge 0$ let $\varphi = \varphi_{s,L} = u \min\{|u|^{2s}, L^2\}\eta^2 \in H_0^{1,2}(\Omega)$, with supp $\varphi \subset\subset \Omega$. Testing (B.7) with φ, we obtain

$$\begin{aligned}
\int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx &+ \tfrac{s}{2} \int_{\{x\in\Omega\ ;\ |u(x)|^s \le L\}} \left|\nabla(|u|^2)\right|^2 |u|^{2s-2}\eta^2\, dx \\
&\le -2\int_\Omega \nabla u\, u \min\{|u|^{2s}, L^2\}\nabla\eta\eta\, dx \\
&\quad + \int_\Omega a(1+2|u|^2)\min\{|u|^{2s}, L^2\}\eta^2\, dx \\
&\le \tfrac12 \int_\Omega |\nabla u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx \\
&\quad + c\int_\Omega |u|^2 \min\{|u|^{2s}, L^2\}|\nabla\eta|^2\, dx \\
&\quad + 3\int_\Omega |a|\,|u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx + \int_\Omega |a|\eta^2\, dx\ .
\end{aligned}$$

Suppose $u \in L^{2s+2}_{loc}(\Omega)$. Then we may conclude that with constants depending on the L^{2s+2}-norm of u, restricted to supp(η), there holds

$$\begin{aligned}
\int_\Omega \left|\nabla\big(u \min\{|u|^s, L\}\eta\big)\right|^2 dx &\le \\
&\le c + c\cdot \int_\Omega |a|\,|u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx \\
&\le c + cK\int_\Omega |u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx \\
&\qquad + c\int_{\{x\in\Omega\ ;\ |a(x)|\ge K\}} |a|\,|u|^2 \min\{|u|^{2s}, L^2\}\eta^2\, dx \\
&\le c(1+K) + \left(c\cdot\int_{\{x\in\Omega\ ;\ |a(x)|\ge K\}} |a|^{n/2}\, dx\right)^{2/n} \cdot \\
&\qquad \cdot \left(\int_\Omega \left|u\min\{|u|^s, L\}\eta\right|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}} \\
&\le c(1+K) + \varepsilon(K)\cdot\int_\Omega \left|\nabla\big(u\min\{|u|^s, L\}\eta\big)\right|^2 dx\ ,
\end{aligned}$$

where

$$\varepsilon(K) = \left(\int_{\{x \in \Omega \ ; \ |a(x)| \geq K\}} |a|^{n/2} \, dx \right)^{2/n} \to 0 \qquad (K \to \infty) \ .$$

Fix K such that $\varepsilon(K) = \frac{1}{2}$ and observe that for this choice of K (and s as above) we now may conclude that

$$\int_{\{x \in \Omega \ ; \ |u(x)|^s \leq L\}} \left| \nabla \left(|u|^{s+1} \eta \right) \right|^2 dx \leq c \int_\Omega \left| \nabla \left(u \min\{|u|^s, L\} \eta \right) \right|^2 dx$$
$$\leq c(1 + K)$$

remains uniformly bounded in L. Hence we may let $L \to \infty$ to derive that

$$|u|^{s+1} \eta \in H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \ ;$$

that is, $u \in L_{loc}^{\frac{(2s+2)n}{n-2}}(\Omega)$.

Now iterate, letting $s_0 = 0$, $s_i + 1 = (s_{i-1} + 1)\frac{n}{n-2}$, if $i \geq 1$, to obtain the conclusion of the lemma. If $u \in H_0^{1,2}(\Omega)$, we may let $\eta = 1$ to obtain that $u \in L^q(\Omega)$ for all $q < \infty$. □

To apply Lemma B.3, note that, if $u \in H_{loc}^{1,2}(\Omega)$ weakly solves (B.7) with a Carathéodory function g with polynomial growth

$$g(x, u) \leq C\left(1 + |u|^{p-1}\right) ,$$

and if $p \leq \frac{2n}{n-2}$ for $n > 2$, then u weakly solves the equation

$$-\Delta u = a(x)\left(1 + |u|\right)$$

with

$$a(x) = \frac{g\left(x, u(x)\right)}{1 + |u(x)|} \in L_{loc}^{n/2}(\Omega) \ .$$

By Lemma B.3, therefore, $u \in L_{loc}^q(\Omega)$, for any $q < \infty$. In view of our growth condition for g this implies that $\Delta u = -g(u) \in L_{loc}^q(\Omega)$ for any $q < \infty$. Thus, by the Caldéron-Zygmund inequality, Theorem B.2, $u \in H_{loc}^{2,q}(\Omega)$, for any $q < \infty$, whence also $u \in C_{loc}^{1,\alpha}(\Omega)$ by the Sobolev embedding theorem, Theorem A.5, for any $\alpha < 1$. Moreover, if $u \in H_0^{1,2}(\Omega)$, and if $\partial\Omega \in C^2$, by the same token it follows that $u \in H^{2,q} \cap H_0^{1,2}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$. Now we may proceed using Schauder theory. In particular, if g is Hölder continuous, then $u \in C^2(\Omega)$ and is a non-constant, classical C^2-solution of equation (B.7). Finally, if g and $\partial\Omega$ are smooth, higher regularity (up to the boundary) can be obtained by iterating the Schauder estimates.

Maximum Principle

A basic tool for proving existence of solutions to elliptic boundary value problems in Hölder spaces is the maximum principle.

We state this in a form due to Walter [1; Theorem 2], allowing for more general coefficients c in the operator L than in classical versions.

B.4 Theorem. *Suppose L is elliptic of type (B.1) on a domain Ω and suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$L\,u \geq 0 \text{ in } \Omega, \text{ and } u \geq 0 \text{ on } \partial\Omega.$$

Moreover, suppose there exists $h \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that

$$L\,h \geq 0 \text{ in } \Omega, \text{ and } h > 0 \text{ on } \Omega.$$

Then either $u > 0$ in Ω, or $u = \beta h$ for some $\beta \leq 0$.

In particular, let L be given by (B.2) with coefficients $a_{ij} \in C^{1,\alpha}(\bar{\Omega}), c \in C^{\alpha}(\bar{\Omega})$. Then L is self-adjoint and possesses a complete set of eigenfunctions (φ_j) in $H_0^{1,2}(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ with eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. Moreover, φ_1 has constant sign, say, $\varphi_1 > 0$ in Ω. Suppose that the first Dirichlet eigenvalue

$$\lambda_1 = \inf_{u \neq 0} \frac{(L\,u, u)_{L^2}}{(u, u)_{L^2}} > 0.$$

Then in Theorem B.4 we may choose $h = \varphi_1$, and the theorem implies that any solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of $Lu \geq 0$ in Ω, and such that $u \geq 0$ on Ω either is positive throughout Ω or vanishes identically.

The strong maximum principle is based on the Hopf boundary maximum principle; see Walter [1; p. 294]:

B.5 Theorem. *Let L be elliptic of type (B.1) on the ball $B = B_R(0) \subset \mathbb{R}^n$, with $c \geq 0$. Suppose $u \in C^2(B) \cap C^1(\overline{B})$ satisfies $Lu \geq 0$ in B, $u \geq 0$ on ∂B, and $u \geq \gamma > 0$ in $B_\rho(0)$ for some $\rho < R, \gamma > 0$. Then there exists $\delta = \delta(L, \gamma, \rho, R) > 0$ such that*

$$u(x) \geq \delta\big(R - |x|\big) \qquad \text{in } B\ .$$

In particular, if $u(x_0) = 0$ for some $x_0 \in \partial B$, then the interior normal derivative of u at the point x_0 is strictly positive.

Weak Maximum Principle

For weak solutions of elliptic equations we have the following analogue of Theorem B.4.

B.6 Theorem. *Suppose L is elliptic of type (B.2) and suppose the Dirichlet form of $\mathcal{L}$ is positive definite on $H_0^{1,2}(\Omega)$ in the sense that*

$$\mathcal{L}(u, u) > 0 \text{ for all } u \in H_0^{1,2}(\Omega),\ u \neq 0.$$

Then, if $u \in H^{1,2}(\Omega)$ weakly satisfies $L\,u \geq 0$ in the sense that

$$\mathcal{L}(u, \varphi) \geq 0 \quad \text{for all non-negative } \varphi \in H_0^{1,2}(\Omega),$$

and $u \geq 0$ *on* $\partial\Omega$, *it follows that* $u \geq 0$ *in* Ω.

Proof. Choose $\varphi = u_- = \max\{-u, 0\} \in H_0^{1,2}(\Omega)$. Then

$$0 \leq \mathcal{L}(u, u_-) = -\mathcal{L}(u_-, u_-) \leq 0$$

with equality if and only if $u_- \equiv 0$; that is $u \geq 0$. □

Theorem B.6 can be used to strengthen the boundary maximum principle Theorem B.5:

B.7 Theorem. *Let* L *satisfy the hypotheses of Theorem B.6 in* $\Omega = B_R(0) = B \subset \mathbb{R}^n$ *with coefficients* $a_{ij} \in C^1$. *Suppose* $u \in C^2(B) \cap C^1(\overline{B})$ *satisfies* $Lu \geq 0$ *in* B, $u \geq 0$ *on* ∂B *and* $u \geq \gamma > 0$ *in* $B_\rho(0)$ *for some* $\rho < R, \gamma > 0$. *Then there exists* $\delta = \delta(L, \gamma, \rho, R) > 0$ *such that* $u(x) \geq \delta\big(R - |x|\big)$ *in* B.

Proof. We adapt the proof of Walter [1; p. 294]. For large $C > 0$ the function $v = \exp\big(C(R^2 - |x|^2)\big) - 1$ satisfies $Lv \leq 0$ in $B \setminus B_\rho(0)$. Moreover, for small $\varepsilon > 0$ the function $w = \varepsilon v$ satisfies $w \leq u$ for $|x| \leq \rho$ and $|x| = R$. Hence, Theorem B.6 – applied to $u - w$ on $B \setminus B_\rho(0)$ – shows that $u \geq w$ in $B \setminus B_\rho(0)$. □

Application

As an application, consider the operator $L = -\Delta - \delta$, where $\delta < \lambda_1$, the first Dirichlet eigenvalue of $-\Delta$ on Ω. Let $u \in H_0^{1,2}(\Omega)$ or $u \in C^2(\Omega) \cap C^o(\bar{\Omega})$ weakly satisfy $Lu \leq C_o$ in $\Omega, u \leq 0$ on $\partial\Omega$, and choose $v(x) = C\big(C - |x - x_0|^2\big)$ with $x_0 \in \Omega$ and C sufficiently large to achieve that $v > 0$ on $\bar{\Omega}$ and $Lv \geq C_0$. Then $w = v - u$ satisfies

$$Lw \geq 0 \text{ in } \Omega, \ w > 0 \text{ on } \partial\Omega ,$$

and hence w is non-negative throughout Ω. Thus

$$u \leq v \qquad \text{in } \Omega .$$

More generally, results like Theorem B.4 or B.5 can be used to obtain L^∞- or even Lipschitz a priori bounds of solution to elliptic boundary value problems by comparing with suitably constructed "barriers".

CHAPTER 4

Weak Solutions, Part II

4.1. Guide

This chapter covers the well-known theory of De Giorgi-Nash-Moser. We present both the approach of De Giorgi and of Moser so students can make comparisons and can see that the ideas involved are essentially the same. The classical paper [**12**] is certainly very nice material for further reading. One may also wish to compare the results in [**12**] and [**7**].

4.2. Local Boundedness

In the following three sections we will discuss the De Giorgi-Nash-Moser theory for linear elliptic equations. In this section we will prove the local boundedness of solutions. In the next section we will prove Hölder continuity. Then in Section 4.4 we will discuss the Harnack inequality. For all results in these three sections there is no regularity assumption of coefficients.

The main theorem of this section is the following boundedness result.

THEOREM 4.1. *Suppose* $a_{ij} \in L^\infty(B_1)$ *and* $c \in L^q(B_1)$ *for some* $q > n/2$ *satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in B_1,\ \xi \in \mathbb{R}^n \quad \text{and} \quad |a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

for some positive constants λ *and* Λ. *Suppose that* $u \in H^1(B_1)$ *is a subsolution in the following sense*

$$(*) \qquad \int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi \leq \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1 .$$

If $f \in L^q(B_1)$, *then* $u^+ \in L^\infty_{\text{loc}}(B_1)$. *Moreover, there holds for any* $\theta \in (0,1)$ *and any* $p > 0$

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

where $C = C(n, \lambda, \Lambda, p, q)$ *is a positive constant.*

In the following we use two approaches to prove this theorem, the one by De Giorgi and the other by Moser.

PROOF. We first prove for $\theta = 1/2$ and $p = 2$.

METHOD 1. Approach by De Giorgi.

Consider $v = (u-k)^+$ for $k \geq 0$ and $\zeta \in C_0^1(B_1)$. Set $\varphi = v\zeta^2$ as the test function. Note $v = u - k$, $Dv = Du$ a.e. in $\{u > k\}$ and $v = 0$, $Dv = 0$ a.e. in $\{u \leq k\}$. Hence if we substitute such defined φ in $(*)$, we integrate in the set $\{u > k\}$.

By Hölder inequality we have

$$\begin{aligned}\int a_{ij} D_i u D_j \varphi &= \int a_{ij} D_i u D_j v \zeta^2 + 2a_{ij} D_i u D_j \zeta\, v\zeta \\ &\geq \lambda \int |Dv|^2\zeta^2 - 2\Lambda \int |Dv||D\zeta| v\zeta \\ &\geq \frac{\lambda}{2}\int |Dv|^2\zeta^2 - \frac{2\Lambda^2}{\lambda}\int |D\zeta|^2 v^2 .\end{aligned}$$

Hence we obtain

$$\int |Dv|^2\zeta^2 \leq C\left\{\int v^2|D\zeta|^2 + \int |c|v^2\zeta^2 + k^2\int |c|\zeta^2 + \int |f| v\zeta^2\right\}$$

from which the estimate

$$\int |D(v\zeta)|^2 \leq C\left\{\int v^2|D\zeta|^2 + \int |c|v^2\zeta^2 + k^2\int |c|\zeta^2 + \int |f| v\zeta^2\right\}$$

follows.

Recall the Sobolev inequality for $v\zeta \in H_0^1(B_1)$

$$\left(\int_{B_1} (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} \leq c(n)\int_{B_1} |D(v\zeta)|^2$$

where $2^* = 2n/(n-2)$ for $n > 2$ and $2^* > 2$ is arbitrary if $n = 2$. Hölder inequality implies that with $\delta > 0$ small and $\zeta \leq 1$

$$\begin{aligned}\int |f| v\zeta^2 &\leq \left(\int |f|^q\right)^{\frac{1}{q}} \left(\int |v\zeta|^{2^*}\right)^{\frac{1}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}} \\ &\leq c(n)\|f\|_{L^q}\left(\int |D(v\zeta)|^2\right)^{\frac{1}{2}} |\{v\zeta \neq 0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}} \\ &\leq \delta\int |D(v\zeta)|^2 + c(n,\delta)\|f\|_{L^q}^2 |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{q}} .\end{aligned}$$

Note $1 + \frac{2}{n} - \frac{2}{q} > 1 - \frac{1}{q}$ if $q > n/2$. Therefore we have the following estimate:

$$\int |D(v\zeta)|^2 \leq C\left\{\int v^2|D\zeta|^2 + \int |c|v^2\zeta^2 + k^2\int |c|\zeta^2 + F^2|\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\}$$

where $F = \|f\|_{L^q(B_1)}$.

We claim that there holds

$$\text{(4.1)} \qquad \int |D(v\zeta)|^2 \leq C\left\{\int v^2|D\zeta|^2 + (k^2 + F^2)|\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\}$$

if $|\{v\zeta \neq 0\}|$ is small $|\{v\zeta \neq 0\}|$ is small

It is obvious if $c \equiv 0$. In fact, in this special case there is no restriction on the set $\{v\zeta \neq 0\}$. In general, Hölder inequality implies that

$$\int |c|v^2\zeta^2 \leq \left(\int |c|^q\right)^{\frac{1}{q}} \left(\int (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}-\frac{1}{q}}$$
$$\leq c(n) \int |D(v\zeta)|^2 \left(\int |c|^q\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}},$$

and

$$\int |c|\zeta^2 \leq \left(\int |c|^q\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}.$$

Therefore we have

$$\int |D(v\zeta)|^2 \leq$$
$$C\left\{\int v^2|D\zeta|^2 + \int |D(v\zeta)|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}} + (k^2+F^2)|\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\}.$$

This implies (4.1) if $|\{v\zeta \neq 0\}|$ is small.

To continue we obtain by Sobolev inequality

$$\int (v\zeta)^2 \leq \left(\int (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \leq c(n) \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}}.$$

Therefore we have

$$\int (v\zeta)^2 \leq C\left\{\int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{1}{q}}\right\}$$

if $|\{v\zeta \neq 0\}|$ is small. Hence there exists an $\varepsilon > 0$ such that

$$\int (v\zeta)^2 \leq C\left\{\int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\varepsilon} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\varepsilon}\right\}$$

if $|\{v\zeta \neq 0\}|$ is small. Choose the cut-off function in the following way. For any fixed $0 < r < R \leq 1$ choose $\zeta \in C_0^\infty(B_R)$ such that $\zeta \equiv 1$ in B_r and $0 \leq \zeta \leq 1$ and $|D\zeta| \leq 2(R-r)^{-1}$ in B_1. Set

$$A(k,r) = \{x \in B_r; u \geq k\}.$$

We conclude that for any $0 < r < R \leq 1$ and $k > 0$

$$(4.2) \qquad \int_{A(k,r)} (u-k)^2 \leq$$
$$C\left\{\frac{1}{(R-r)^2}|A(k,R)|^{\varepsilon} \int_{A(k,R)} (u-k)^2 + (k+F)^2|A(k,R)|^{1+\varepsilon}\right\}$$

if $|A(k,R)|$ is small. Note

$$|A(k,R)| \leq \frac{1}{k} \int_{A(k,R)} u^+ \leq \frac{1}{k}\|u^+\|_{L^2}.$$

Hence (4.2) holds if $k \geq k_0 = C\|u^+\|_{L^2}$ for some large C depending only on λ and Λ.

Next we would show that there exists some $k = C(k_0 + F)$ such that

$$\int_{A(k,1/2)} (u-k)^2 = 0\,.$$

To continue we take any $h > k \geq k_0$ and any $0 < r < 1$. It is obvious that $A(k,r) \supset A(h,r)$. Hence we have

$$\int_{A(h,r)} (u-h)^2 \leq \int_{A(k,r)} (u-k)^2$$

and

$$|A(h,r)| = |B_r \cap \{u - k > h - k\}| \leq \frac{1}{(h-k)^2} \int_{A(k,r)} (u-k)^2\,.$$

Therefore by (4.2) we have for any $h > k \geq k_0$ and $1/2 \leq r < R \leq 1$

$$\int_{A(h,r)} (u-h)^2 \leq C\left\{\frac{1}{(R-r)^2}\int_{A(h,R)} (u-h)^2 + (h+F)^2|A(h,R)|\right\}|A(h,R)|^{\varepsilon}$$

$$\leq C\left\{\frac{1}{(R-r)^2} + \frac{(h+F)^2}{(h-k)^2}\right\}\frac{1}{(h-k)^{2\varepsilon}}\left(\int_{A(k,R)} (u-k)^2\right)^{1+\varepsilon}$$

or

$$\|(u-h)^+\|_{L^2(B_r)} \leq C\left\{\frac{1}{R-r} + \frac{h+F}{h-k}\right\}\frac{1}{(h-k)^{\varepsilon}}\|(u-k)^+\|^{1-\varepsilon}_{L^2(B_R)}\,. \tag{4.3}$$

Now we carry out the iteration. Set $\varphi(k,r) = \|(u-k)^+\|_{L^2(B_r)}$. For $\tau = 1/2$ and some $k > 0$ to be determined. Define for $\ell = 0, 1, 2, \ldots$,

R=1

$$k_\ell = k_0 + k(1 - \tfrac{1}{2^\ell}) \quad (\leq k_0 + k)$$
$$r_\ell = \tau + \tfrac{1}{2^\ell}(1-\tau)\,.$$

Obviously we have

$$k_\ell - k_{\ell-1} = \tfrac{k}{2^\ell}\,, \qquad r_{\ell-1} - r_\ell = \tfrac{1}{2^\ell}(1-\tau)\,.$$

Therefore we have for $\ell = 0, 1, 2, \ldots$

$$\varphi(k_\ell, r_\ell) \leq C\left\{\frac{2^\ell}{1-\tau} + \frac{2^\ell(k_0 + F + k)}{k}\right\}\frac{2^{\varepsilon\ell}}{k^{\varepsilon}}[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}$$

$$\leq \frac{C}{1-\tau}\cdot\frac{k_0 + F + k}{k^{1+\varepsilon}}\cdot 2^{(1+\varepsilon)\ell}\cdot[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}\,.$$

Next we prove inductively for any $\ell = 0, 1, \ldots$,

$$\varphi(k_\ell, r_\ell) \leq \frac{\varphi(k_0, r_0)}{\gamma^\ell} \quad \text{for some } \gamma > 1 \tag{4.4}$$

if k is sufficiently large. Obviously it is true for $\ell = 0$. Suppose it is true for $\ell - 1$. We write

$$[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \le \left\{ \frac{\varphi(k_0, r_0)}{\gamma^{\ell-1}} \right\}^{1+\varepsilon} = \frac{\varphi(k_0, r_0)^\varepsilon}{\gamma^{\ell\varepsilon-(1+\varepsilon)}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Then we obtain

$$\varphi(k_\ell, r_\ell) \le \frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \frac{k_0 + F + k}{k^{1+\varepsilon}} \cdot [\varphi(k_0, r_0)]^\varepsilon \cdot \frac{2^{\ell(1+\varepsilon)}}{\gamma^{\ell\varepsilon}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Choose γ first such that $\gamma^\varepsilon = 2^{1+\varepsilon}$. Note $\gamma > 1$. Next, we need

$$\frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \left(\frac{\varphi(k_0, r_0)}{k} \right)^\varepsilon \cdot \frac{k_0 + F + k}{k} \le 1.$$

Therefore we choose

$$k = C_*\{k_0 + F + \varphi(k_0, r_0)\}$$

for C_* large. Let $\ell \to +\infty$ in (4.4). We conclude

$$\varphi(k_0 + k, \tau) = 0.$$

Hence we have

$$\sup_{B_{1/2}} u^+ \le (C_* + 1)\{k_0 + F + \varphi(k_0, r_0)\}.$$

Recall $k_0 = C\|u^+\|_{L^2(B_1)}$ and $\varphi(k_0, r_0) \le \|u^+\|_{L^2(B_1)}$. This finishes the proof.

Next we give the second proof of Theorem 4.1.

METHOD 2. Approach by Moser.

First we explain the idea. By choosing the test function appropriately, we will estimate the L^{p_1} norm of u in a smaller ball by the L^{p_2} norm of u for $p_1 > p_2$ in a larger ball, that is,

$$\|u\|_{L^{p_1}(B_{r_1})} \le C\|u\|_{L^{p_2}(B_{r_2})}$$

for $p_1 > p_2$ and $r_1 < r_2$. This is a reversed Hölder inequality. As a sacrifice C behaves like $\frac{1}{r_2 - r_1}$. By iteration and careful choice of $\{r_i\}$ and $\{p_i\}$, we will obtain the result.

For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \ge m. \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \le \bar{u}$. Set the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1)$$

for some $\beta \ge 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. Direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}). \end{aligned}$$