4. The Concentration-Compactness Principle

As we have seen in our analysis of the Plateau problem, Section 2.7, a very serious complication for the direct methods to be applicable arises in the presence of non-compact group actions.

If, in a terminology borrowed from physics which we will try to make more precise later, the action is a "manifest" symmetry – as in the case of the conformal group of the disc acting on Dirichlet's integral for minimal surfaces – we may be able to eliminate the action by a suitable normalization. This was the purpose for introducing the three-point-condition on admissible functions for the Plateau problem in the proof of Theorem 2.8. However, if the action is "hidden" such a normalization is not possible and there is no hope that *all* minimizing sequences converge to a minimizer. Even worse, the variational problem may not have a solution. For such problems, P.-L. Lions developed his concentration-compactness principle. On the basis of this principle, for many constrained minimization problems it is possible to state necessary and sufficient conditions for the convergence of all minimizing sequences satisfying the given constraint. These conditions involve a delicate comparison of the given functional in variation and a (family of) functionals "at infinity" (on which the group action is manifest).

Rather than dwell on abstract notions we prefer to give an example – a variant of problem (2.1), (2.3) – which will bring out the main ideas immediately.

4.1 Example. Let $a\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function $a > 0$ and suppose that

$$a(x) \to a_\infty > 0 \qquad (|x| \to \infty) .$$

We look for positive solutions u of the equation

$$-\Delta u + a(x)u = u|u|^{p-2} \qquad \text{in } \mathbb{R}^n , \tag{4.1}$$

decaying at infinity, that is

$$u(x) \to 0 \qquad \text{as } |x| \to \infty . \tag{4.2}$$

Here $p > 2$ may be an arbitrary number, if $n = 1, 2$. If $n \geq 3$ we suppose that $p < \frac{2n}{n-2}$. This guarantees that the imbedding

$$H^{1,2}(\Omega) \to L^p(\Omega)$$

is compact for any $\Omega \subset\subset \mathbb{R}^n$.

Note that (4.1) is the Euler-Lagrange equation of the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(x)|u|^2\right) dx$$

on $H^{1,2}(\mathbb{R}^n)$, restricted to the unit sphere

$$M = \{u \in H^{1,2}(\mathbb{R}^n) \; ; \; \int_{\mathbb{R}^n} |u|^p \, dx = 1\}$$

in $L^p(\mathbb{R}^n)$. Moreover, if $a(x) \equiv a_\infty$, E is invariant under translations

$$u \mapsto u_{x_0}(x) = u(x - x_0) \; .$$

In general, for any $u \in H^{1,2}(\mathbb{R}^n)$, after a substitution of variables

$$E(u_{x_0}) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a(\,\cdot\, + x_0)|u|^2\right) dx \to \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a_\infty |u|^2\right) dx$$

as $|x_o| \to \infty$, whence it may seem appropriate to call

$$E^\infty(u) := \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + a_\infty |u|^2\right) dx$$

the *functional at infinity* associated with E. The following result is due to Lions [2; Theorem I.2].

4.2 Theorem. *Suppose*

$$I := \inf_M E < \inf_M E^\infty =: I^\infty \; , \tag{4.3}$$

then there exists a positive solution $u \in H^{1,2}(\mathbb{R}^n)$ of equation (4.1). Moreover, condition (4.3) is necessary and sufficient for the relative compactness of all minimizing sequences for E in M.

Proof. Clearly, (4.3) is necessary for the convergence of all minimizing sequences in M. Indeed, suppose $I^\infty \le I$ and let (u_m) be a minimizing sequence for E^∞. Then also $(\tilde{u}_m)$, given by $\tilde{u}_m = u_m(\cdot + x_m)$, is a minimizing sequence for E^∞, for any sequence (x_m) in $\mathbb{R}^n$. Choosing $|x_m|$ large enough such that

$$\left|E(\tilde{u}_m) - E^\infty(\tilde{u}_m)\right| \le \frac{1}{m} \; ,$$

moreover, $(\tilde{u}_m)$ is a minimizing sequence for E. In addition, we can achieve that

$$\tilde{u}_m \to 0 \qquad \text{locally in } L^2 \; ,$$

whence $(\tilde{u}_m)$ cannot be relatively compact.

Note that this argument also proves that the weak inequality $I \le I^\infty$ always holds true, regardless of the particular choice of the function a.

We now show that condition (4.3) is also sufficient. The existence of a positive solution to (4.1) then follows as in the proof of Theorem 2.1.

Let (u_m) be a minimizing sequence for E in M such that

$$E(u_m) \to I \; .$$

We may assume that $u_m \to u$ weakly in $L^p(\mathbb{R}^n)$. By continuity, a is uniformly positive on $\mathbb{R}^n$. Hence we also have

$$\|u_m\|^2_{H^{1,2}} \le c\, E(u_m) \le C < \infty \ ,$$

and in addition we may assume that $u_m \rightharpoonup u$ weakly in $H^{1,2}(\mathbb{R}^n)$ and pointwise almost everywhere. Denote $u_m = v_m + u$. Observe that by Vitali's theorem

$$\begin{aligned}
\int |u_m|^p\, dx - \int |u_m - u|^p\, dx &= -\iint_0^1 \frac{d}{d\vartheta}|u_m - \vartheta u|^p\, d\vartheta\, dx \\
&= p\iint_0^1 u(u_m - \vartheta u)|u_m - \vartheta u|^{p-2}\, d\vartheta\, dx \qquad (4.4)\\
&\to p\iint_0^1 u(u - \vartheta u)|u - \vartheta u|^{p-2}\, d\vartheta\, dx = \int |u|^p\, dx \ ,
\end{aligned}$$

where $\int \dots dx$ denotes integration over $\mathbb{R}^n$; that is

$$\int_{\mathbb{R}^n} |u|^p\, dx + \int_{\mathbb{R}^n} |v_m|^p\, dx \to 1 \ .$$

Similarly

$$\begin{aligned}
E(u_m) &= E(v_m + u) = \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \big\{ \big(|\nabla u|^2 + 2\nabla u \nabla v_m + |\nabla v_m|^2\big) \\
&\quad + a(x)\big(|u|^2 + 2uv_m + |v_m|^2\big)\big\}\, dx \qquad (4.5)\\
&= E(u) + E(v_m) + \int_{\mathbb{R}^n} \big(\nabla u \nabla v_m + a(x) u v_m\big)\, dx
\end{aligned}$$

and the last term converges to zero by weak convergence $v_m = u_m - u \rightharpoonup 0$ in $H^{1,2}(\mathbb{R}^n)$.

Moreover, letting

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n \ ; \ |a(x) - a_\infty| \ge \varepsilon\} \subset\subset \mathbb{R}^n,$$

since $v_m \to 0$ locally in L^2, the integral

$$\begin{aligned}
\int_{\mathbb{R}^n} (a(x) - a_\infty)|v_m|^2\, dx &\le \\
&\le \varepsilon \int_{\mathbb{R}^n} |v_m|^2 + \sup_{\mathbb{R}^n} |a(x)| \int_{\Omega_\varepsilon} |v_m|^2\, dx \\
&\le c\varepsilon + o(1) \ .
\end{aligned}$$

Here and in the following, $o(1)$ denotes error terms such that $o(1) \to 0$ as $m \to \infty$. Hence this integral can be made arbitrarily small if we first choose $\varepsilon > 0$ and then let $m \ge m_0(\varepsilon)$ be sufficiently large. That is, we have

$$E(u_m) = E(u) + E^\infty(v_m) + o(1) \ .$$

By homogeneity, if we denote $\lambda = \int_{\mathbb{R}^n} |u|^p \, dx$,

$$E(u) = \lambda^{2/p} E\left(\lambda^{-1/p} u\right) \geq \lambda^{2/p} I, \quad \text{if } \lambda > 0 \, ,$$
$$E^\infty(v_m) = (1-\lambda)^{2/p} E^\infty\left((1-\lambda)^{-1/p} v_m\right) \geq (1-\lambda)^{2/p} I^\infty + o(1), \text{ if } \lambda < 1 \, .$$

Hence we obtain the estimate for $\lambda \in [0,1]$

$$\begin{aligned} I &= E(u_m) + o(1) = E(u) + E^\infty(v_m) + o(1) \\ &\geq \lambda^{2/p} I + (1-\lambda)^{2/p} I^\infty + o(1) \\ &\geq \left(\lambda^{2/p} + (1-\lambda)^{2/p}\right) I + o(1) \, . \end{aligned}$$

Since $p > 2$ this implies that $\lambda \in \{0,1\}$. But if $\lambda = 0$, we obtain that

$$I \geq I^\infty + o(1) > I$$

for large m; a contradiction.

Therefore $\lambda = 1$; that is, $u_m \to u$ in L^p, and $u \in M$. By convexity of E, moreover,

$$E(u) \leq \liminf_{m \to \infty} E(u_m) = I \, ,$$

and u minimizes E in M. Hence also $E(u_m) \to E(u)$. Finally, by (4.5)

$$\begin{aligned} \|u_m - u\|^2_{H^{1,2}} &\leq cE(u_m - u) \\ &= c\left(E(u_m) - E(u)\right) + o(1) \to 0 \, , \end{aligned}$$

and $u_m \to u$ strongly in $H^{1,2}(\mathbb{R}^n)$. The proof is complete. □

Regarding $|u_m|^p \, dx$ as a measure on $\mathbb{R}^n$, a systematic approach to such problems is possible via the following lemma (P.L. Lions [1; p. 115 ff.]).

4.3 Concentration-Compactness Lemma I. *Suppose μ_m is a sequence of probability measures on $\mathbb{R}^n$: $\mu_m \geq 0$, $\int_{\mathbb{R}^n} d\mu_m = 1$. There is a subsequence (μ_m) such that one of the following three conditions holds:*

1°) (Compactness) There exists a sequence $x_m \subset \mathbb{R}^n$ such that for any $\varepsilon > 0$ there is a radius $R > 0$ with the property that

$$\int_{B_R(x_m)} d\mu_m \geq 1 - \varepsilon$$

for all m.

2°) (Vanishing) For all $R > 0$ there holds

$$\lim_{m \to \infty} \left(\sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_m \right) = 0 \, .$$

(3°) (Dichotomy) There exists a number λ, $0 < \lambda < 1$, such that for any $\varepsilon > 0$ there is a number $R > 0$ and a sequence (x_m) with the following property: Given $R' > R$ there are non-negative measures μ_m^1, μ_m^2 such that

$$0 \le \mu_m^1 + \mu_m^2 \le \mu_m \ ,$$
$$\operatorname{supp}(\mu_m^1) \subset B_R(x_m),\ \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R'}(x_m) \ ,$$
$$\limsup_{m\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_m^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_m^2 \right| \right) \le \varepsilon \ .$$

Proof. The proof is based on the notion of concentration function

$$Q(r) = \sup_{x\in\mathbb{R}^n} \left(\int_{B_r(x)} d\mu \right)$$

of a non-negative measure, introduced by P. Lévy [1].

Let Q_m be the concentration functions associated with μ_m. Note that (Q_m) is a sequence of non-decreasing, non-negative bounded functions on $[0,\infty[$ with $lim_{R\to\infty} Q_m(R) = 1$. Hence, (Q_m) is locally bounded in BV on $[0,\infty[$ and there exists a subsequence (μ_m) and a bounded, non-negative, non-decreasing function Q such that

$$Q_m(R) \to Q(R) \qquad (m\to\infty) \ ,$$

for almost every $R > 0$. Let

$$\lambda = \lim_{R\to\infty} Q(R) \ .$$

Clearly $0 \le \lambda \le 1$. If $\lambda = 0$, we have "vanishing", case (2°). Suppose $\lambda = 1$. Then for some $R_0 > 0$ we have $Q(R_0) > \frac{1}{2}$. For any $m \in \mathbb{N}$ let x_m satisfy

$$Q_m(R_0) \le \int_{B_{R_0}(x_m)} d\mu_m + \frac{1}{m} \ .$$

Now for $0 < \varepsilon < \frac{1}{2}$ fix R such that $Q(R) > 1 - \varepsilon > \frac{1}{2}$ and let y_m satisfy

$$Q_m(R) \le \int_{B_R(y_m)} d\mu_m + \frac{1}{m} \ .$$

Then for large m we have

$$\int_{B_R(y_m)} d\mu_m + \int_{B_{R_0}(x_m)} d\mu_m > 1 = \int_{\mathbb{R}^n} d\mu_m \ .$$

It follows that for such m

$$B_R(y_m) \cap B_{R_0}(x_m) \ne \emptyset \ .$$

That is, $B_R(y_m) \subset B_{2R+R_0}(x_m)$ and hence

$$1 - \varepsilon \le \int_{B_{2R+R_0}(x_m)} d\mu_m$$

for large m. Choosing R even larger, if necessary, we can achieve that (1°) holds for all m.

If $0 < \lambda < 1$, given $\varepsilon > 0$ choose R and a sequence (x_m) – depending on ε and R – such that

$$\int_{B_R(x_m)} d\mu_m > \lambda - \varepsilon \ ,$$

for $m \ge m_0(\varepsilon)$. Enlarging $m_0(\varepsilon)$, if necessary, we can also find a sequence $R_m \to \infty$ such that

$$Q_m(R) \le Q_m(R_m) < \lambda + \varepsilon \ ,$$

for $m \ge m_0(\varepsilon)$. Now let $\mu_m^1 = \mu_m \llcorner B_R(x_m)$, the restriction of μ_m to $B_R(x_m)$. Similarly, define $\mu_m^2 = \mu_m \llcorner (\mathbb{R}^n \setminus B_{R_m}(x_m))$. Obviously

$$0 \le \mu_m^1 + \mu_m^2 \le \mu_m \ ,$$

and, given $R' > R$, for large m we also have

$$\operatorname{supp}(\mu_m^1) \subset B_R(x_m), \ \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R_m}(x_m) \subset \mathbb{R}^n \setminus B_{R'}(x_m) \ .$$

Finally, for $m \ge m_0(\varepsilon)$ we can achieve

$$\begin{aligned} \left|\lambda - \int_{\mathbb{R}^n} d\mu_m^1\right| + \left|1 - \lambda - \int_{\mathbb{R}^n} d\mu_m^2\right| &= \\ = \left|\lambda - \int_{B_R(x_m)} d\mu_m\right| + \left|\int_{B_{R_m}(x_m)} d\mu_m - \lambda\right| &< 2\varepsilon \ , \end{aligned}$$

which concludes the proof. □

In the context of Theorem 4.2 Lemma 4.3 may be applied to $\mu_m = |u_m|^p \, dx$, $m \in \mathbb{N}$. Dichotomy in this case is made explicit in (4.4). In view of the compactness of the embedding $H^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ on bounded domains Ω for all $p < \frac{2n}{n-2}$ the situation dealt with in Example 4.1 is referred to as the *locally compact case*.

If a problem is conformally invariant, in particular invariant under the non-compact group of dilatations of $\mathbb{R}^n$ acting via

$$u \to u_R(x) = u(x/R), \ R > 0 \ ,$$

not even local compactness can hold.