shall see some examples and some more delicate ways of handling the possible loss of compactness. See Section 4; see also Chapter III.

In applications, the conditions of the following special case of Theorem 1.1 can often be checked more easily.

1.2 Theorem. *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fullfilled:*
(1°) $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.
(2°) For any $u \in M$, any sequence (u_m) in M such that $u_m \to u$ weakly in V there holds:

$$E(u) \leq \liminf_{m\to\infty} E(u_m) \ .$$

Then E is bounded from below on M and attains its infimum in M.

The concept of minimizing sequences offers a direct and (apparently) constructive proof.

Proof. Let $\alpha_0 = \inf_M E$ and let (u_m) be a minimizing sequence in M, that is, satisfying $E(u_m) \to \alpha_0$. By coerciveness, (u_m) is bounded in V. Since V is reflexive, by the Eberlein-Šmulian theorem (see Dunford-Schwartz [1; p. 430]) we may assume that $u_m \to u$ weakly for some $u \in V$. But M is weakly closed, therefore $u \in M$, and by weak lower semi-continuity

$$E(u) \leq \liminf_{m\to\infty} E(u_m) = \alpha_0 \ . \qquad \square$$

Examples. An important example of a sequentially weakly lower semi-continous functional is the norm in a Banach space V. Closed and convex subsets of Banach spaces are important examples of weakly closed sets. If V is the dual of a separable normed vector space, Theorem 1.2 and its proof remain valid if we replace weak by weak*-convergence.

We present some simple applications.

Degenerate Elliptic Equations.

1.3 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, $p \in [2, \infty[$ with conjugate exponent q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in H^{-1,q}(\Omega)$, the dual of $H_0^{1,p}(\Omega)$, be given. Then there exists a weak solution $u \in H_0^{1,p}(\Omega)$ of the boundary value problem*

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega \tag{1.2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

in the sense that u satisfies the equation

$$(1.4) \qquad \int_\Omega (\nabla u |\nabla u|^{p-2} \nabla\varphi - f\varphi)dx = 0 \ , \qquad \forall \varphi \in C_0^\infty(\Omega) \ .$$

Proof. Remark that the left part of (1.4) is the directional derivative of the p-functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \ - \ \int_\Omega f u \, dx$$

on the Banach space $V = H_0^{1,p}(\Omega)$ in direction φ; that is, problem (1.2), (1.3) is of variational form.

Note that $H_0^{1,p}(\Omega)$ is reflexive. Moreover, E is coercive. In fact, we have

$$\begin{aligned} E(u) &\geq \frac{1}{p}\|u\|^p_{H_0^{1,p}} \ - \ \|f\|_{H^{-1,q}} \, \|u\|_{H_0^{1,p}} \geq \frac{1}{p}\left(\|u\|^p_{H_0^{1,p}} - c\|u\|_{H_0^{1,p}}\right) \\ &\geq c^{-1}\|u\|^p_{H_0^{1,p}} \ - \ C. \end{aligned}$$

Finally, E is (sequentially) weakly lower semi-continuous: It suffices to show that for $u_m \rightharpoonup u$ weakly in $H_0^{1,p}(\Omega)$ we have

$$\int_\Omega f \, u_m \, dx \quad \to \quad \int_\Omega f \, u \, dx \ .$$

Since $f \in H^{-1,q}(\Omega)$, however, this follows from the very definition of weak convergence. Hence Theorem 1.2 is applicable and there exists a minimizer $u \in H_0^{1,p}(\Omega)$ of E, solving (1.4). □

Remark that for $p \geq 2$ the p-Laplacian is strongly monotone in the sense that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot (\nabla u - \nabla v) \, dx \geq c\|u - v\|^p_{H_0^{1,p}} \ .$$

In particular, the solution u to (1.4) is unique.

If f is more regular, say $f \in C^{m,\alpha}(\overline{\Omega})$, we would expect the solution u of (1.4) to be more regular as well. This is true if $p = 2$, see Appendix B, but in the degenerate case $p > 2$, where the uniform ellipticity of the p-Laplace operator is lost at zeros of $|\nabla u|$, the best that one can hope for is $u \in C^{1,\alpha}(\overline{\Omega})$; see Uhlenbeck [1], Tolksdorf [2; p.128], Di Benedetto [1].

In Theorem 1.3 we have applied Theorem 1.2 to a functional on a reflexive space. An example in a non-reflexive setting is given next.

Since the sphere of radius 1 around p lies in the convex hull of finitely many vectors $q_0, q_1, \ldots, q_{nN}$, by continuity of F in u and convexity in p the right hand side of this inequality remains uniformly bounded in a neighborhood of (u,p). Hence $F(\cdot,\cdot)$ is locally Lipschitz continous in p, locally uniformly in $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Therefore, if $u_m \to u$, $p_m \to p$ we have

$$\begin{aligned} |F(u_m, p_m) - F(u,p)| &\le |F(u_m, p_m) - F(u_m, p)| + |F(u_m, p) - F(u,p)| \\ &\le c|p_m - p| + o(1) \to 0 \qquad \text{as } m \to \infty, \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$, as desired.

4. In the scalar case (N=1), if F is C^2 for example, the existence of a minimizer u for E implies that the *Legendre condition*

$$\sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(x,u,p)\, \xi_\alpha \xi_\beta \ge 0, \qquad \text{for all } \xi \in \mathbb{R}^n$$

holds at all points $(x, u = u(x), p = \nabla u(x))$, see for instance Giaquinta [1; p.11 f]. This condition in turn implies the convexity of F in p.

The situation is quite different in the vector-valued case $N > 1$. In this case, in general only the *Legendre-Hadamard condition*

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1} F_{p^i_\alpha p^j_\beta}(x,u,p) \xi_\alpha \xi_\beta \eta^i \eta^j \ge 0 \ , \qquad \text{for all } \xi \in \mathbb{R}^n, \ \eta \in \mathbb{R}^N$$

will hold at a minimizer, which is much weaker then convexity. (Giaquinta [1; p.12]).

In fact, in Section 3 below we shall see how, under certain additional structure conditions on F, the convexity assumption in Theorem 1.6 can be weakened in the vector-valued case.

2. Constraints

Applying the direct methods often involves a delicate interplay between the functional E, the space of admissible functions M, and the topology on M. In this section we will see how, by means of imposing constraints on admissible functions and/or by a suitable modification of the variational problem, the direct methods can be successfully employed also in situations where their use seems highly unlikely at first.

Note that we will not consider constraints that are dictated by the problems themselves, such as physical restrictions on the response of a mechanical system. Constraints of this type in general lead to variational inequalities, and we refer to Kinderlehrer-Stampacchia [1] for a comprehensive introduction to this field. Instead, we will show how certain variational problems can be solved by adding virtual – that is, purely technical – constraints to the conditions defining the admissible set, thus singling out distinguished solutions.

Semi-Linear Elliptic Boundary Value Problems

We start by deriving the existence of positive solutions to non-coercive, semi-linear elliptic boundary value problems by a constrained minimization method. Such problems are motivated by studies of flame propagation (see for example Gel'fand [1; (15.5), p.357]) or arise in the context of the Yamabe problem (see Chapter III.4).

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let $p > 2$. If $n \geq 3$ we also assume that p satisfies the condition $p < 2^* = \frac{2n}{n-2}$. For $\lambda \in \mathbb{R}$ consider the problem

$$-\Delta u + \lambda u = u|u|^{p-2} \quad \text{in } \Omega \,, \tag{2.1}$$

$$u > 0 \quad \text{in } \Omega \,, \tag{2.2}$$

$$u = 0 \quad \text{on } \partial\Omega \,. \tag{2.3}$$

Also let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ denote the eigenvalues of the operator $-\Delta$ on $H_0^{1,2}(\Omega)$. Then we have the following result:

2.1 Theorem. *For any $\lambda > -\lambda_1$ there exists a positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to problem (2.1)–(2.3).*

Proof. Observe that Equation (2.1) is the Euler-Lagrange equation of the functional

$$\tilde{E}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda |u|^2\right) dx - \frac{1}{p}\int_\Omega |u|^p \, dx$$

on $H_0^{1,2}(\Omega)$ which is neither bounded from above nor from below on this space. However, using the homogeneity of (2.1) a solution of problem (2.1)–(2.3) can also be obtained by solving a constrained minimization problem for the functional

$$E(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda |u|^2\right) dx$$

on the Hilbert space $H_0^{1,2}(\Omega)$, restricted to the set

$$M = \{u \in H_0^{1,2}(\Omega)\,;\ \int_\Omega |u|^p \, dx = 1\} \,.$$

We verify that $E : M \to \mathbb{R}$ satisfies the hypotheses of Theorem 1.2. By the Rellich-Kondrakov theorem the injection $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ is completely continuous for $p < 2^*$, if $n \geq 3$, respectively for any $p < \infty$, if $n = 1, 2$; see Theorem A.5 of the appendix. Hence M is weakly closed in $H_0^{1,2}(\Omega)$.

Recall the Rayleigh-Ritz characterization

$$\lambda_1 = \inf_{\substack{u \in H_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} \tag{2.4}$$

of the smallest Dirichlet eigenvalue. This gives the estimate

$$E(u) \geq \frac{1}{2} \min\{1, (1 + \frac{\lambda}{\lambda_1})\} \|u\|^2_{H_0^{1,2}} . \tag{2.5}$$

From this, coerciveness of E for $\lambda > -\lambda_1$ is immediate.

Weak lower semi-continuity of E follows from weak lower semi-continuity of the norm in $H_0^{1,2}(\Omega)$ and the Rellich-Kondrakov theorem. By Theorem 1.2 therefore E attains its infimum at a point $\underline{u}$ in M. Remark that since $E(u) = E(|u|)$ we may assume that $\underline{u} \geq 0$.

To derive the variational equation for E first note that E is continuously Fréchet-differentiable in $H_0^{1,2}(\Omega)$ with

$$\langle v, DE(u) \rangle = \int_\Omega \big(\nabla u \nabla v + \lambda u v\big) \, dx \ .$$

Moreover, letting

$$G(u) = \int_\Omega |u|^p \, dx - 1 \ ,$$

$G : H_0^{1,2}(\Omega) \to \mathbb{R}$ also is continuously Fréchet-differentiable with

$$\langle v, DG(u) \rangle = p \int_\Omega u|u|^{p-2} v \, dx \ .$$

In particular, at any point $u \in M$

$$\langle u, DG(u) \rangle = p \int_\Omega |u|^p \, dx = p \neq 0 \ ,$$

and by the implicit function theorem the set $M = G^{-1}(0)$ is a C^1-submanifold of $H_0^{1,2}(\Omega)$.

Now, by the Lagrange multiplier rule, there exists a parameter $\mu \in \mathbb{R}$ such that

$$\begin{aligned} \langle v, (DE(\underline{u}) - \mu DG(\underline{u})) \rangle &= \int_\Omega \big(\nabla \underline{u} \nabla v + \lambda \underline{u} v - \mu \underline{u} |\underline{u}|^{p-2} v\big) \, dx \\ &= 0, \qquad \text{for all } v \in H_0^{1,2}(\Omega) \ . \end{aligned}$$

Inserting $v = \underline{u}$ into this equation yields that

$$2E(\underline{u}) = \int_\Omega \big(|\nabla \underline{u}|^2 + \lambda |\underline{u}|^2\big) \, dx = \mu \int_\Omega |\underline{u}|^p \, dx = \mu \ .$$

Since $\underline{u} \in M$ cannot vanish identically, from (2.5) we infer that $\mu > 0$. Scaling with a suitable power of μ, we obtain a weak solution $u = \mu^{\frac{1}{p-2}} \cdot \underline{u} \in H_0^{1,2}(\Omega)$ of (2.1), (2.3) in the sense that

$$\int_\Omega (\nabla u \nabla v + \lambda uv - u|u|^{p-2}v)\, dx = 0 \;, \qquad \text{for all } v \in H_0^{1,2}(\Omega) \;. \tag{2.6}$$

Moreover, (2.2) holds in the weak sense $u \geq 0$, $u \not\equiv 0$. To finish the proof we use the regularity result Lemma B.3 of the appendix and the observations following it to obtain that $u \in C^2(\Omega)$. Finally, by the strong maximum principle $u > 0$ in Ω ; see Theorem B.4.

□

Observe that, at least for the kind of nonlinear problems considered here, by Lemma B.3 of the appendix the regularity theory is taken care of and in the following we may concentrate on proving existence of (weak) solutions. However, additional structure conditions may imply further useful properties of suitable solutions. An example is symmetry.

2.2 Symmetry. By a result of Gidas-Ni-Nirenberg [1; Theorem 2.1, p. 216, and Theorem 1, p.209], if Ω is symmetric with respect to a hyperplane, say $x_1 = 0$, any positive solution u of (2.1), (2.3) is even in x_1, that is, $u(x_1, x') = u(-x_1, x')$ for all $x = (x_1, x') \in \Omega$, and $\frac{\partial u}{\partial x_1} < 0$ at any point $x = (x_1, x') \in \Omega$ with $x_1 > 0$. In particular, if Ω is a ball, any positive solution u is radially symmetric. The proof of this result uses a variant of the Alexandrov-Hopf reflection principle and the maximum principle. This method lends itself to numerous applications in many different contexts; in Chapter III.4 below we shall see that it is even possible to derive a-priori bounds from this method in the setting of a parabolic equation on the sphere.

Perron's Method in a Variational Guise

In the previous example the constraint built into the definition of M had the effect of making the restricted functional $E = \tilde{E}|_M$ coercive. Moreover, this constraint only changed the Euler-Lagrange equations by a factor which could be scaled away using the homogeneity of the right hand side of (2.1).

In the next application we will see that sometimes also inequality constraints can be imposed without changing the Euler-Lagrange equations at a minimizer.

2.3 Weak sub- and super-solutions. Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Let $u_0 \in H_0^{1,2}(\Omega)$ be given. Consider the equation

$$-\Delta u = g(\cdot, u) \qquad \text{in } \Omega \;, \tag{2.7}$$

$$u = u_0 \qquad \text{on } \partial\Omega \;. \tag{2.8}$$

By definition $u \in H^{1,2}(\Omega)$ is a (weak) *sub-solution* to (2.7–8) if $u \le u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi \, dx - \int_\Omega g(\,\cdot\,,u)\varphi \, dx \le 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\ ,\ \varphi \ge 0\ .$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) *super-solution* to (2.7–8) if in the above the reverse inequalities hold.

2.4 Theorem. *Suppose $\underline{u} \in H^{1,2}(\Omega)$ is a sub-solution while $\overline{u} \in H^{1,2}(\Omega)$ is a super-solution to problem (2.7–8) and assume that with constants $\underline{c}, \overline{c} \in \mathbb{R}$ there holds $-\infty < \underline{c} \le \underline{u} \le \overline{u} \le \overline{c} < \infty$, almost everywhere in Ω. Then there exists a weak solution $u \in H^{1,2}(\Omega)$ of (2.7–8), satisfying the condition $\underline{u} \le u \le \overline{u}$ almost everywhere in Ω.*

Proof. With no loss of generality we may assume $u_0 = 0$. Let $G(x,u) = \int_0^u g(x,v)\, dv$ denote a primitive of g. Note that (2.7–8) formally are the Euler-Lagrange equations of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u)\, dx\ .$$

However, our assumptions do not allow the conclusion that E is finite or even differentiable on $V := H_0^{1,2}(\Omega)$ – the smallest space where we have any chance of verifying coerciveness. Instead we restrict E to

$$M = \left\{u \in H_0^{1,2}(\Omega)\ ;\ \underline{u} \le u \le \overline{u} \text{ almost everywhere}\right\}\ .$$

Since $\underline{u}, \overline{u} \in L^\infty$ by assumption, also $M \subset L^\infty$ and $G(x,u(x)) \le c$ for all $u \in M$ and almost every $x \in \Omega$.

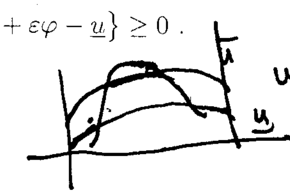
Now we can verify the hypotheses of Theorem 1.2: Clearly, $V = H_0^{1,2}(\Omega)$ is reflexive. Moreover, M is closed and convex, hence weakly closed. Since M is essentially bounded, our functional $E(u) \ge \frac{1}{2}\|u\|^2_{H_0^{1,2}(\Omega)} - c$ is coercive on M. Finally, to see that E is weakly lower semi-continuous on M, it suffices to show that

$$\int_\Omega G(x,u_m)\, dx \to \int_\Omega G(x,u)\, dx$$

if $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$, where $u_m, u \in M$. But – passing to a subsequence, if necessary – we may assume that $u_m \to u$ pointwise almost everywhere; moreover, $|G(x,u_m(x))| \le c$ uniformly. Hence we may appeal to Lebesgue's theorem on dominated convergence.

From Theorem 1.2 we infer the existence of a relative minimizer $u \in M$. To see that u weakly solves (2.7), for $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ let $v_\varepsilon = \min\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M$ with

$$\begin{aligned} \varphi^\varepsilon &= \ \ \max\{0, u + \varepsilon\varphi - \overline{u}\} \ge 0\ ,\\ \varphi_\varepsilon &= -\min\{0, u + \varepsilon\varphi - \underline{u}\} \ge 0\ . \end{aligned}$$



Note that $\varphi_\varepsilon, \varphi^\varepsilon \in H_0^{1,2} \cap L^\infty(\Omega)$.

E is differentiable in direction $v_\varepsilon - u$. Since u minimizes E in M we have

$$0 \leq \langle (v_\varepsilon - u), DE(u) \rangle = \varepsilon \langle \varphi, DE(u) \rangle - \langle \varphi^\varepsilon, DE(u) \rangle + \langle \varphi_\varepsilon, DE(u) \rangle \ ,$$

so that

$$\langle \varphi, DE(u) \rangle \geq \frac{1}{\varepsilon} \left[\langle \varphi^\varepsilon, DE(u) \rangle - \langle \varphi_\varepsilon, DE(u) \rangle \right] .$$

Now, since $\overline{u}$ is a supersolution to (2.7), we have

$$\begin{aligned}
\langle \varphi^\varepsilon, DE(u) \rangle &= \langle \varphi^\varepsilon, DE(\overline{u}) \rangle + \langle \varphi^\varepsilon, DE(u) - DE(\overline{u}) \rangle \\
&\geq \langle \varphi^\varepsilon, DE(u) - DE(\overline{u}) \rangle \\
&= \int_{\Omega_\varepsilon} \big\{ \nabla(u - \overline{u}) \nabla(u + \varepsilon\varphi - \overline{u}) - \\
&\qquad - \big(g(x,u) - g(x,\overline{u}) \big) (u + \varepsilon\varphi - \overline{u}) \big\} \, dx \\
&\geq \varepsilon \int_{\Omega_\varepsilon} \nabla(u - \overline{u}) \nabla\varphi \, dx - \varepsilon \int_{\Omega_\varepsilon} \big| g(x,u) - g(x,\overline{u}) \big| \, |\varphi| \, dx \ ,
\end{aligned}$$

where $\Omega^\varepsilon = \big\{ x \in \Omega \ ; \ u(x) + \varepsilon\varphi(x) \geq \overline{u}(x) > u(x) \big\}$. Note that $\mathcal{L}^n(\Omega^\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that

$$\langle \varphi^\varepsilon, DE(u) \rangle \geq o(\varepsilon)$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that

$$\langle \varphi_\varepsilon, DE(u) \rangle \leq o(\varepsilon)$$

whence

$$\langle \varphi, DE(u) \rangle \geq 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Reversing the sign of φ and since $C_0^\infty(\Omega)$ is dense in $H_0^{1,2}(\Omega)$ we finally see that $DE(u) = 0$, as claimed. □

2.5 A special case. Let Ω be a smooth bounded domain in $\mathbb{R}^n, n \geq 3$, and let

$$g(x,u) = k(x)u - u|u|^{p-2} \tag{2.9}$$

where $p = \frac{2n}{n-2}$, and where k is a continuous function such that

$$1 \leq k(x) \leq K < \infty$$

uniformly in Ω. Suppose $u_0 \in C^1(\overline{\Omega})$ satisfies $u_0 \geq 1$ on $\partial\Omega$.
Then $\underline{u} \equiv 1$ is a sub-solution while $\overline{u} \equiv c$ for large $c > 1$ is a super-solution to equation (2.7–8). Consequently, (2.7–8) admits a solution $u \geq 1$.

2.6 Remark. The sub-super-solution method can also be applied to equations on manifolds. In the context of the Yamabe problem it has been used by Loewner-Nirenberg [1] and Kazdan-Warner [1]; see Chapter III.4. The nonlinear term in this case is precisely (2.9).

$$
\text{(I.36)} \qquad \begin{cases} \Delta u + u^p + u^q = 0 & \text{in} \quad B_R, \\ u > 0 & \text{in} \quad B_R, \\ u = 0 & \text{on} \quad \partial B_R, \end{cases}
$$

possesses at least two solutions for every R large. (See [LN] for the details.) This also raises the following question: is it true that the equation (I.35) with $\frac{n}{n-2} < p < n^* < q$ always has a positive radial entire solution? If the answer is affirmative, then (I.36) with the same range of p and q will also have at least two solutions for every R large. The condition $\frac{n}{n-2} < p < n^* < q$ would be almost optimal in the following sense: if $p \leq \frac{n}{n-2}$ or $p < q < n^*$, then (I.35) has no positive radial entire solutions (see e.g. [NS2]). And, if $n^* \leq p \leq q$, then positive radial entire solutions of (I.35) do exist. We shall return to such questions in Part III.

§7. Monotone iteration scheme

The main idea here is to reduce solving a differential equation to solving two differential inequalities with an ordered relationship between their solutions. This method is, in some sense, constructive and fairly general.

Let

$$Lu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i},$$

where $a_{ij}(x) = a_{ji}(x)$ is smooth and there exists a constant $\mu > 0$ such that $a_{ij}(x)\xi^i\xi^j \geq \mu|\xi|^2$ for all $\xi \in \mathbb{R}^n$, and $x \in \Omega$. Set the boundary operator to be

$$Bu = u$$

on Γ_1 and

$$Bu = \frac{\partial u}{\partial \nu} + bu$$

on Γ_2 where $b(x) \geq 0$ on Γ_2, ν is the unit outer normal to Γ_2, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 are disjoint and are both closed (one of them may be empty). Note that this includes Dirichlet, Neumann ($b \equiv 0$) and regular oblique derivative boundary conditions.

Consider the problem

(I.37) $$\begin{cases} Lu + f(x,u) = 0 & \text{in } \Omega, \\ Bu = g & \text{on } \partial\Omega, \end{cases}$$

where f is, say, C^1 in u, (locally) Holder continuous in $(x,u) \in \Omega \times \mathbb{R}$. We say that φ is a ***super-solution*** (or an upper-solution) of (I.37) if

(I.38) $$\begin{cases} L\varphi + f(x,\varphi) \leq 0 & \text{in } \Omega, \\ B\varphi \geq g & \text{on } \partial\Omega; \end{cases}$$

and ψ is a ***sub-solution*** (or a lower-solution) of (I.37) if

$$(\mathrm{I}.39) \qquad \begin{cases} L\psi + f(x,\psi) \geq 0 & \text{in } \Omega, \\ B\psi \leq g & \text{on } \partial\Omega. \end{cases}$$

The main existence result is the following

THEOREM. *Let* φ *be a super-solution of* (I.37), ψ *be a sub-solution of* (I.37) *and* $\varphi \geq \psi$ *in* Ω. *Then there exists a solution* u *of* (I.37) *with* $\psi \leq u \leq \varphi$.

REMARKS. (i) This is sometimes called the barrier method (φ and ψ are the barriers). This is a classical method, it goes back at least to Bieberbach. We shall follow Sattinger's proof [S].

(ii) The proof of the theorem is referred to as the "monotone iteration scheme", which is more useful and more important than the statement of the theorem itself.

Proof. Choose a number $\Lambda > 0$ so large that $\frac{\partial f}{\partial u} + \Lambda > 0$ for all $(x,u) \in \bar{\Omega} \times [\inf \psi, \sup \varphi]$, and the spectrum of $(L-\Lambda, B)$ is strictly contained in the left-half plane. In particular, this implies that the (nonlinear) operator T defined by $w = Tu$ where w is the solution of

$$(\mathrm{I}.40) \qquad \begin{cases} (L-\Lambda)w = -[f(x,u) + \Lambda u] & \text{in } \Omega, \\ Bw = g & \text{on } \partial\Omega, \end{cases}$$

is well-defined, say, in $C^2(\Omega)$.

The crucial property of T that we need is the *monotonicity*; i.e. if $u \le v$ in Ω, then $Tu \le Tv$ in Ω. Setting $w = Tv - Tu$, we have

$$\begin{cases} (L-\Lambda)w = f(x,u) + \Lambda u - f(x,v) - \Lambda v & \text{in} \quad \Omega, \\ Bw = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If we write

$$f(x,u)+\Lambda u-f(x,v)-\Lambda v = \begin{cases} \left[\frac{f(x,u) - f(x,v)}{u - v} + \Lambda\right](u-v) & \text{if } u \neq v, \\ 0 & \text{if } u = v, \end{cases}$$

then we see that by the choice of Λ, the following holds

$$\text{(I.41)} \qquad \begin{cases} (L - \Lambda)w \le 0 & \text{in} \quad \Omega, \\ Bw \ge 0 & \text{on} \quad \partial\Omega, \end{cases}$$

since $u-v \le 0$ in Ω. Now suppose that $\min_{\bar{\Omega}} w < 0$. Let $w(P) = \min_{\bar{\Omega}} w$, then $w(P) < 0$, $P \in \bar{\Omega}$ and we have two cases:

(i) $P \in \Omega$. Then

$$0 \ge (L-\Lambda)w(P) = \sum_{i,j=1}^{n} a_{ij}(P)\frac{\partial^2 w}{\partial x_i \partial x_j}(P) + 0 - \Lambda w(P)$$

$$\ge 0 - \Lambda w(P) > 0,$$

a contradiction.

(ii) $P \in \partial\Omega$. Clearly $P \notin \Gamma_1$ since on Γ_1 we have $0 \le Bw = w$. So $P \in \Gamma_2$, and $\frac{\partial w}{\partial \nu}(P) + b(P)w(P) \ge 0$, therefore $\frac{\partial w}{\partial \nu}(P) \ge 0$. On the other hand, since $(L-\Lambda)(-w) \ge 0$, $\min_{\bar{\Omega}}(-w) = (-w)(P) > 0$ and $-\Lambda < 0$, the Hopf boundary point lemma implies that $\frac{\partial(-w)}{\partial\nu} > 0$ at P, a contradiction. Thus $w \geq 0$ in Ω and the monotonicity of T is established.

We can now start iteration. Let $u_1 = T\varphi$. Then, $u_1 \leq \varphi$. For,

$$(L-\Lambda)(\varphi - u_1) = (L-\Lambda)\varphi + [f(x,\varphi) + \Lambda\varphi]$$
$$= L\varphi + f(x,\varphi) \leq 0,$$

thus if we set $w = \varphi - u_1$, then w satisfies (I.41), i.e.

$$\begin{cases} (L - \Lambda)w \leq 0 & \text{in} \quad \Omega, \\ Bw \geq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Therefore $w \geq 0$, i.e. $u_1 \leq \varphi$ in Ω. Next, setting $u_2 = Tu_1$, $u_3 = Tu_2, \cdots, u_n = Tu_{n-1}, \cdots$, we see from the monotonicity property of T that $\varphi \geq u_1 \geq u_2 \geq \cdots \geq u_k \geq \cdots$. Similarly, starting from ψ, we set $v_1 = T\psi$, $v_2 = Tv_1, \cdots$ and we obtain $\psi \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq \cdots$. Moreover, $\psi \leq \varphi$ which implies $T\psi \leq T\varphi$, i.e. $v_1 \leq u_1$. By induction, $v_k \leq u_k$ for all k. Thus

$$\psi \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq \cdots \leq u_k \leq \cdots \leq u_2 \leq u_1 \leq \varphi.$$

Setting $\bar{u} = \lim u_k$ and $\bar{v} = \lim v_k$, we have $\bar{v} \leq \bar{u}$.

We claim that $\bar{v} = T\bar{v}$ and $\bar{u} = T\bar{u}$. From (I.40), since $-[f(x,u_k) + \Lambda u_k]$ is uniformly bounded in L^∞ (independent of k), by elliptic L^p estimates we see that $\{Tu_{k+1}\}$ is uniformly bounded in $W^{2,p}$ for any p large. Thus $\{Tu_k\}$ is uniformly bounded in $C^{1,\alpha}$ (independent of k) which, in turn, implies that the right-hand side of (I.40),

$-[f(x,u_k) + \Lambda u_k]$ is also uniformly bounded in $C^{1,\alpha}$. Now Schauder estimates assure us that $\{Tu_k\}$ is uniformly bounded in $C^{2,\alpha}$. Since the embedding $C^{2,\alpha} \longrightarrow C^2$ is compact and u_k converges to $\bar{u}$ pointwise already, we conclude that u_k converges to $\bar{u}$ in $C^2(\bar{\Omega})$. Now, letting k go to ∞ in the following

$$\begin{cases} (L-\Lambda)u_{k+1} = -[f(x,u_k) + \Lambda u_k], \\ Bu_{k+1} = g, \end{cases}$$

we have

$$\begin{cases} (L-\Lambda)\bar{u} = -[f(x,\bar{u}) + \Lambda\bar{u}], \\ B\bar{u} = g \end{cases}$$

i.e. $\bar{u} = T\bar{u}$ and $\bar{u}$ is a solution of (I.37). Similarly, $\bar{v}$ is also a solution and $\psi \leq \bar{v} \leq \bar{u} \leq \varphi$.

<u>REMARKS</u>. (i) The above argument also proves that all possible solutions u with $\psi \leq u \leq \varphi$ must also satisfy $\bar{v} \leq u \leq \bar{u}$.

(ii) The solutions $\bar{v}$, $\bar{u}$ are generally "stable" ($\bar{u}$, $\bar{v}$ may coincide).

(iii) This theorem has had lots of applications, see, e.g. [KW].

(iv) We now take up the existence question for the problem (13) in the Introduction, i.e. the problem

$$
\text{(I.42)} \qquad \begin{cases} \Delta u + u^\gamma = 0 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}
$$

where $0 < \gamma < 1$ and Ω is an arbitrary bounded smooth domain in $\mathbb{R}^n$. First, we set $\psi = \epsilon\varphi_1$ where $\varphi_1 > 0$ is the first eigenfunction of Δ on Ω with zero Dirichlet boundary data, and $\epsilon > 0$ is sufficiently small. We compute, for $\epsilon > 0$ small enough, we have

$$
\Delta\psi + \psi^\gamma = \epsilon\varphi_1\left(\frac{1}{\epsilon^{1-\gamma}\varphi_1^{1-\gamma}} - \lambda_1\right) \geq 0
$$

since $1-\gamma > 0$ and $1/\varphi_1^{1-\gamma}$ is bounded away from 0 (due to the boundary condition $\varphi_1 = 0$ on $\partial\Omega$). Thus ψ is a sub-solution of (I.42). To construct a super-solution, we let ρ be the solution of

$$
\begin{cases} \Delta\rho + 1 = 0 & \text{in} \quad \Omega, \\ \rho = 0 & \text{on} \quad \partial\Omega. \end{cases}
$$

By the strong maximum principle, we have $\rho > 0$ in Ω. Now, set $\varphi = \alpha\rho$, where $\alpha > 0$ is so large that $\varphi > \psi$ (this is possible since $\frac{\partial\rho}{\partial\nu} < 0$ on $\partial\Omega$ by the Hopf boundary point lemma) and

$$
\Delta\varphi + \varphi^\gamma = \alpha(\alpha^{\gamma-1}\varphi^\gamma - 1) \leq 0.
$$

(Note that $\alpha >> 1$ implies $\alpha^{\gamma-1} << 1$ since $\gamma < 1$.) Thus φ is a super-solution of (I.42) with $\varphi \geq \psi$ in Ω and it follows that (I.42) has a solution. In fact, this solution is also unique. (See the last theorem in this section.)

(v) Let us now try to apply this method to the Lane-Emden equation (I.1) (or (2)). Suppose that, somehow, we have succeeded in doing this. Then, we would have a solution u of (I.1) and a super-solution $\varphi \geq u$ in Ω of (I.1). Since $u = 0$ on $\partial\Omega$ and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, we have

$$0 \geq \int_{\partial\Omega} (\varphi \frac{\partial u}{\partial \nu} - u \frac{\partial \varphi}{\partial \nu}) ds = \int_{\Omega} (\varphi \Delta u - u \Delta \varphi)$$

$$\geq \int_{\Omega} (u\varphi^p - \varphi u^p) = \int_{\Omega} \varphi u (\varphi^{p-1} - u^{p-1}).$$

This implies that $\varphi \equiv u$ in Ω; i.e. φ was already a solution of (I.1). Or, equivalently, this says that, in order to find a pair of super- and sub-solutions with the correct order, one must first find a solution. This indicates that this barrier method does not seem to be useful in treating "super-linear" Dirichlet problems.

The following result gives the existence of multiple solutions which seems due to H. Amann [Am].

<u>THEOREM</u>. *Suppose that* $v_1 \geq v_2 \geq v_3 \geq v_4$ *where* v_1, v_3 *are strict super-solutions to* (I.37) *and* v_2, v_4 *are strict sub-solutions to* (I.34). *Then there exist three solutions to* (I.37).

Heuristically, it is easy to "prove" this theorem. From our previous theorem, there exist two solutions $\underline{u}$, $\overline{u}$ of

(I.37) with $v_1 \geq \bar{u} \geq v_2$, $v_3 \geq \underline{u} \geq v_4$. Since both solutions are "stable" and $\bar{u} \geq \underline{u}$, there must be an "unstable" solution in between. (This may be made precise by considering the corresponding parabolic initial value problem.) Indeed, the first proof which was due to Amann did use parabolic approach.

REMARK. Both theorems above may be extended to more general nonlinearities involving ∇u. For instance, they generalize to the problem

$$\begin{cases} Lu + f(x,u,Du) = 0 & \text{in} \quad \Omega, \\ Bu = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where L, B are as before and $f : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function such that $\frac{\partial f}{\partial \xi}$, $\frac{\partial f}{\partial \eta}$ exist and are continuous ((x,ξ,η) denotes a generic point in $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$), and f satisfies the "natural" growth condition: there exists an increasing function $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|f(x,\xi,\eta)| \leq c(|\xi|)(1 + |\eta|^2)$$

for $(x,\xi,\eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$.

For the proofs, we refer the readers to [AC].

We shall end our discussion on Dirichlet problems with an easy uniqueness result.

Proposition. *Suppose that*

(I.43) $\frac{f(t)}{t}$ *is decreasing in* $t > 0$ *and* $\lim_{t \downarrow 0} \frac{f(t)}{t} > \lambda_1$,

then (I.4) *has at most one positive solution.*

Proof. Suppose that (I.4) has two solutions u and v. Observe that by (I.43) we can choose $\epsilon > 0$ so small that $\epsilon\varphi_1$ is a sub-solution of (I.4), and, $\epsilon\varphi_1 < u$ and $\epsilon\varphi_1 < v$ (since $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$ never vanish on $\partial\Omega$ by the Hopf boundary point lemma), where $\varphi_1 > 0$ is the first eigenfunction of Δ on Ω with $\varphi_1 = 0$ on $\partial\Omega$. Now we start our monotone iteration at $\epsilon\varphi_1$ and go up. Eventually we reach a solution w. Since u, v are also supersolutions to (I.4), we have $u \geq w$ and $v \geq w$. Since $u \not\equiv v$, we may assume that $u > w$ in Ω. Applying Green's identity, we have

$$0 = \int_\Omega (u\Delta w - w\Delta u) = \int_\Omega [wf(u) - uf(w)]$$

$$= \int_\Omega uw\left[\frac{f(u)}{u} - \frac{f(w)}{w}\right] < 0,$$

a contradiction.

REMARKS. (i) If $\frac{f(t)}{t}$ is decreasing in $\mathbb{R}_+$ and $\lim_{t \downarrow 0} \frac{f(t)}{t} \leq \lambda_1$, then $f(t) < \lambda_1 t$ in $\mathbb{R}_+$. Then

$$0 = \int_\Omega (\varphi_1 \Delta u - u\Delta\varphi_1) = \int_\Omega [\lambda_1 u\varphi_1 - \varphi_1 f(u)]$$

$$= \int_\Omega u\varphi_1(\lambda_1 - \frac{f(u)}{u}) > 0,$$

which implies that (I.4) has no solution.

(ii) It is clear that $f(u) = u^\gamma$, $0 < \gamma < 1$, satisfies (I.43). Thus the uniqueness of (I.42) (or (13)) follows.

§8. Miscellaneous remarks

(1) For the stability of solutions to the Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

we refer the interested readers to the end of §1 in Part II below for some results and a conjecture.

(2) For the related inhomogeneous equation

$$\text{(I.44)} \qquad \begin{cases} \Delta u + |u|^{p-1}u = g(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $g \in L^2$ and $u \in H_0^1(\Omega)$, there has been a lot of work. In general, (I.44) may not have any positive solutions, thus one is interested in finding just a solution of (I.44). The most recent progress was due to Bahri and Lions [BL]. They showed that if $p < \frac{n}{n-2}$, then there exist infinitely many solutions. (See [BL] for references to other earlier works.)