Chapter III

Limit Cases of the Palais-Smale Condition

Condition (P.-S.) may seem rather restrictive. Actually, as Hildebrandt [4; p. 324] records, for quite a while many mathematicians felt convinced that inspite of its success in dealing with one-dimensional variational problems like geodesics (see Birkhoff's Theorem I.4.4 for example, or Palais' [3] work on closed geodesics), the Palais-Smale condition could never play a role in the solution of "interesting" variational problems in higher dimensions.

Recent advances in the Calculus of Variations have changed this view and it has become apparent that the methods of Palais and Smale apply to many problems of physical and/or geometric interest and – in particular – that the Palais-Smale condition will in general hold true for such problems in a broad range of energies. Moreover, the failure of (P.-S.) at certain levels reflects highly interesting phenomena related to internal symmetries of the systems under study, which geometrically can be described as "separation of spheres", or mathematically as "singularities", respectively as "change in topology". Again speaking in physical terms, we might observe "phase transitions" or "particle creation" at the energy levels where (P.-S.) fails.

Such phenomena seem to have first been observed by Sacks-Uhlenbeck [1] and – independently – by Wente [5] in the context of harmonic maps of surfaces, respectively in the context of surfaces of prescribed constant mean curvature. (See Sections 4 and 5 below.) In these cases the term "separation of spheres" has a clear geometric meaning. More recently, Sedlacek [1] has uncovered similar results also for Yang-Mills connections. If interpreted appropriately, very early indications of such phenomena already may be found in the work of Douglas [2], Morse-Tompkins [2] and Shiffman [2] on minimal surfaces of higher genus and/or connectivity. In this case, a "change in topology" in fact sometimes may be observed even physically as one tries to realize a multiply connected or higher genus minimal surface in a soap film experiment. See Jost-Struwe [1] for a modern approach to these results.

Mathematically, it seems that non-compact group actions give rise to these effects. In physics and geometry, of course, such group actions arise naturally as "symmetries" from the requirements of scale or gauge invariance; in particular, in the examples of Sacks-Uhlenbeck and Wente cited above, from conformal invariance.

Moreover, a symmetry may be "manifest" or "broken", that is, perturbed by interaction terms; surprisingly, existence results for problems with non-compact internal symmetries seem to depend on the extent to which the symmetry is broken or perturbed. As in the case of non-compact minimization problems studied in Section I.4, sometimes the perturbation from symmetry can be measured by comparing with a suitable (family of) *limiting problem(s)* where the symmetry is acting. Existence results – for example in the line of Theorem II.6.1 – therefore will strongly depend on energy estimates for critical values.

We start with a simple example.

1. Pohožaev's Non-Existence Result

Let Ω be a domain in $\mathbb{R}^n$, $n > 2$. Consider the limit case $p = 2^* = \frac{2n}{n-2}$ in Theorem I.2.1. Given $\lambda \in \mathbb{R}$ we would like to solve the problem

$$(1.1) \qquad -\Delta u = \lambda u + u|u|^{2^*-2} \qquad \text{in } \Omega ,$$

$$(1.2) \qquad u > 0 \qquad \text{in } \Omega ,$$

$$(1.3) \qquad u = 0 \qquad \text{on } \partial\Omega .$$

(Note that in order to be consistent with the literature, in this section we reverse the sign of λ as compared with Section I.2.1 or Section II.5.8.) As in Theorem I.2.1 we can approach this problem by a direct method and attempt to obtain non-trivial solutions of (1.1), (1.3) as relative minima of the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda|u|^2\right)\,dx\ ,$$

on the unit sphere in $L^{2^*}(\Omega)$,

$$M = \{u \in H_0^{1,2}(\Omega)\ ;\ \|u\|_{L^{2^*}}^{2^*} = 1\}\ .$$

Equivalently, we may seek to minimize the Sobolev quotient

$$S_\lambda(u;\Omega) = \frac{\int_\Omega \left(|\nabla u|^2 - \lambda|u|^2\right)\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{2/2^*}}\ , \qquad u \neq 0\ .$$

Note that for $\lambda = 0$, as in Section I.4.4,

$$S(\Omega) = \inf_{\substack{u\in H_0^{1,2}\\ u\neq 0}} S_0(u;\Omega) = \inf_{\substack{u\in H_0^{1,2}\\ u\neq 0}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{2/2^*}}$$

is related to the (best) Lipschitz constant for the Sobolev embedding $H_0^{1,2}(\Omega) \to L^{2^*}(\Omega)$.

Recall that for any $u \in H_0^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ the ratio $S_0(u;\mathbb{R}^n)$ is invariant under scaling $u \mapsto u_R(x) = u(x/R)$; that is, we have

$$(1.4) \qquad S_0(u;\mathbb{R}^n) = S_0(u_R;\mathbb{R}^n), \qquad \text{for all } R > 0\ .$$

Hence, in particular, we have (see Remark I.4.5):

1.1 Lemma. *$S(\Omega) = S$ is independent of Ω.*

Moreover, this implies (see Remark I.4.7):

1.2 Theorem. *S is never attained on a domain $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$.*

Hence, for $\lambda = 0$, the proof of Theorem I.2.1 necessarily fails in the limit case $p = 2^*$. More generally, we have the following uniqueness result, due to Pohožaev [1]:

1.3 Theorem. *Suppose $\Omega \neq \mathbb{R}^n$ is a smooth (possibly unbounded) domain in $\mathbb{R}^n$, $n \geq 3$, which is strictly star-shaped with respect to the origin in $\mathbb{R}^n$, and let $\lambda \leq 0$. Then any solution $u \in H_0^{1,2}(\Omega)$ of the boundary value problem (1.1), (1.3) vanishes identically.*

The proof is based on the following "Pohožaev identity":

1.4 Lemma. *Let $g: \mathbb{R} \to \mathbb{R}$ be continuous with primitive $G(u) = \int_0^u g(v)\, dv$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the equation*

$$-\Delta u = g(u) \quad \text{in } \Omega \tag{1.5}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.6}$$

in a domain $\Omega \subset\subset \mathbb{R}^n$. Then there holds

$$\frac{n-2}{2} \int_\Omega |\nabla u|^2\, dx - n \int_\Omega G(u)\, dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 x \cdot \nu\, do = 0\ ,$$

where ν denotes the exterior unit normal.

Proof of Theorem 1.3. Let $g(u) = \lambda u + u|u|^{2^*-2}$ with primitive

$$G(u) = \frac{\lambda}{2}|u|^2 + \frac{1}{2^*}|u|^{2^*}\ .$$

By Theorem I.2.2 and Lemma B.3 of the appendix, any solution of (1.1), (1.3) is smooth on $\overline{\Omega}$. Hence from Pohožaev's identity we infer that

$$\begin{aligned}\int_\Omega |\nabla u|^2\, dx - 2^* \int_\Omega G(u)\, dx + \frac{1}{n-2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 x \cdot \nu\, do \\ = \int_\Omega \left(|\nabla u|^2 - |u|^{2^*}\right) dx + \frac{n|\lambda|}{n-2} \int_\Omega |u|^2\, dx \\ + \frac{1}{n-2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 x \cdot \nu\, do = 0\ .\end{aligned}$$

However, testing the equation (1.1) with u, we infer that

§2. Non-existence and Rellich-Pohozaev identity

In the previous section, we see that if $f(u) = u^p$, $1 < p < n^*$, then (I.4) possesses a solution. It is natural to ask, "what happens if $p \geq n^*$?" Unfortunately, we do not have a complete answer up to now. We only know, for instance, that for $f(u) = u^p$, $p > 1$,

(A) (I.4) has no solution for $p \geq n^*$ if Ω is star-shaped,

(B) (I.4) has a radial solution for every $p > 1$ if Ω is an annulus,

(C) (I.4) has a solution if $p = n^*$, $n = 3$ and Ω is not contractiable to a point.

(A) is an easy consequence of the following well-known Rellich-Pohozaev identity.

LEMMA. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$ *and* u *be a classical solution of the equation* $\Delta u + f(x,u) = 0$. *Then*

$$\text{(I.15)} \quad \int_\Omega [nF(x,u) - \frac{n-2}{2} uf(x,u) + x \cdot F_x(x,u)]dx$$

$$= \int_{\partial\Omega} [(x \cdot \nabla u)\frac{\partial u}{\partial \nu} - (x \cdot \nu)\frac{|\nabla u|^2}{2} + (x \cdot \nu)F(x,u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu}]ds$$

where

$$F(x,u) = \int_0^u f(x,t)dt,$$

F_x *is the gradient of* F *with respect to* x, ds *is the volume element of* $\partial\Omega$ *and* ν *is the unit outer normal to* $\partial\Omega$.

Proof. Let

$$V(x) = (x\cdot\nabla u(x))\nabla u(x) - \frac{|\nabla u|^2}{2}x + xF_x(x,u(x)) + \frac{n-2}{2}u(x)\nabla u(x).$$

We compute, using the equation,

$$\operatorname{div} V = nF(x,u(x)) - \frac{n-2}{2}u(x)f(x,u(x)) + x\cdot F_x(x,u(x)).$$

Our assertion then follows from the Divergence Theorem.

Note that no boundary condition is imposed in the above lemma. Applying this lemma to solutions of (I.4), we obtain

$$\text{(I.16)} \quad \int_\Omega [nF(u) - \frac{n-2}{2}uf(u)]dx = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2 ds$$

since $\nu = \nabla u/|\nabla u|$ on $\partial\Omega$. Now if Ω is star-shaped, then $x\cdot\nu \geq 0$ and $\not\equiv 0$ on $\partial\Omega$. Moreover, if $f(0) \geq 0$, then $|\nabla u| > 0$ on $\partial\Omega$ by Hopf's boundary point lemma. Thus the right-hand side of (I.16) is strictly positive. If, in addition, $f(u) = u^p$, then left-hand side of (I.16) becomes

$$\int_\Omega (\frac{n}{p+1} - \frac{n-2}{2})u^{p+1}$$

which is non-positive if $p \geq n^*$, thus (A) follows.

REMARKS. (i) A different way to derive (I.16) is to

$$\int_\Omega \left(|\nabla u|^2 - \lambda |u|^2 - |u|^{2^*}\right) dx = 0 ,$$

whence

$$2|\lambda| \int_\Omega |u|^2 \, dx + \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu \, do = 0 .$$

Moreover, since Ω is strictly star-shaped with respect to $0 \in \mathbb{R}^n$, we have $x \cdot \nu > 0$ for all $x \in \partial\Omega$. Thus $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, and hence $u \equiv 0$ by the principle of unique continuation. □

Proof of Lemma 1.4. Multiply (1.5) by $x \cdot \nabla u$ and compute

$$\begin{aligned} 0 = \big(\Delta u + g(u)\big)(x \cdot \nabla u) = \\ &= \operatorname{div}\big(\nabla u (x \cdot \nabla u)\big) - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2}\right) + x \cdot \nabla G(u) \\ &= \operatorname{div}\left(\nabla u (x \cdot \nabla u) - x \frac{|\nabla u|^2}{2} + x G(u)\right) + \frac{n-2}{2} |\nabla u|^2 - n G(u) . \end{aligned}$$

Upon integrating this identity over Ω and taking account of the fact that by (1.6) we have

$$x \cdot \nabla u = x \cdot \nu \frac{\partial u}{\partial \nu} \qquad \text{on } \partial\Omega ,$$

the lemma follows. □

1.5 Interpretation. Theorem 1.3 goes beyond Theorem 1.2, as the former applies to any solution, whereas the latter is limited to minima of $S_0(\cdot; \Omega)$. However, Theorem 1.2 applies to any domain.

The connection between the scale invariance of $S = S_0$ and Theorem 1.3 is given by the fact that the function $x \cdot \nabla u = \frac{d}{dR} u_R$ used in the proof of Lemma 1.4 is the generator of the family of scaled maps $\{u_R \ ; \ 0 < R < \infty\}$. We interpret Theorem 1.3 as reflecting the non-compactness of the multiplicative group $\mathbb{R}_+ = \{R \ ; \ 0 < R < \infty\}$ acting on S via scaling. Note that this group action is manifest for $S_\lambda(\cdot; \Omega)$ only if $\lambda = 0$ and $\Omega = \mathbb{R}^n$. In case of a bounded domain Ω not all scalings $u \to u_R$ will map $H_0^{1,2}(\Omega)$ into itself. For instance, if Ω is an annular region $\Omega = \{x \ ; \ a < |x| < b\}$, in fact, $H_0^{1,2}(\Omega)$ does not admit any of these scalings as symmetries. (In Section 3 we will see that in this case (1.1)–(1.3) does have nontrivial solutions.) However, if Ω is star-shaped with respect to the origin, all scalings $u \to u_R$, $R \leq 1$ will be symmetries of $H_0^{1,2}(\Omega)$, and compactness is lost as $R \to 0$. The effect is shown in Theorem 1.3.

Remark that it is also possible to characterize solutions $u \in H_0^{1,2}(\Omega)$ of equation (1.1) as critical points of a functional E_λ on $H_0^{1,2}(\Omega)$ given by

$$E_\lambda(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx . \tag{1.7}$$

By continuity of the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$, the functional E_λ is Fréchet differentiable on $H_0^{1,2}(\Omega)$. Moreover, for $\lambda < \lambda_1$, the first Dirichlet eigenvalue of the operator $-\Delta$, E_λ satisfies the conditions (1°)–(3°) of the mountain pass lemma Theorem II.6.1; compare the proof of Theorem I.2.1. In view of Theorem II.6.1, the absence of a critical point $u > 0$ of E_λ for any $\lambda \le 0$ proves that E_λ for such λ cannot satisfy the Palais-Smale condition (P.-S.) on a star-shaped domain. Again the non-compact action $R \mapsto u_R(x) = u(Rx)$ can be held responsible.

2. The Brezis-Nirenberg Result

In contrast to Theorem 1.3, for $\lambda > 0$ problem (1.1)–(1.3) may admit non-trivial solutions. However, a subtle dependence on the dimension n is observed.

The first result in this direction is due to Brezis and Nirenberg [2]; their approach is related to ideas of Trudinger [1] and Aubin [2].

2.1 Theorem. *Suppose Ω is a domain in $\mathbb{R}^n$, $n \ge 3$, and let $\lambda_1 > 0$ denote the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions.*
(1°) If $n \ge 4$, then for any $\lambda \in]0, \lambda_1[$ there exists a (positive) solution of (1.1)–(1.3).
(2°) If $n = 3$, there exists $\lambda_* \in [0, \lambda_1[$ such that for any $\lambda \in]\lambda_*, \lambda_1[$ problem (1.1)–(1.3) admits a solution.
(3°) If $n = 3$ and $\Omega = B_1(0) \subset \mathbb{R}^3$, then $\lambda_* = \frac{\lambda_1}{4}$ and for $\lambda \le \frac{\lambda_1}{4}$ there is no solution to (1.1)–(1.3).

As we have seen in Section 1, there are (at least) two different approaches to this theorem. The first, which is the one primarly chosen by Brezis and Nirenberg [2], involves the quotient

$$S_\lambda(u;\Omega) = \frac{\int_\Omega (|\nabla u| - \lambda |u|^2)\, dx}{\left(\int_\Omega |u|^{2^*}\, dx\right)^{2/2^*}} .$$

A second proof can be given along the lines of Theorem II.6.1, applied to the "free" functional E_λ

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda |u^2\right) dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\, dx$$

defined earlier. Recall that $E_\lambda \in C^1(H_0^{1,2}(\Omega))$. As we shall see, while it is not true that E_λ satisfies the Palais-Smale condition "globally", some compactness will hold in an energy range determined by the best Sobolev constant S; see Lemma 2.3 below. A similar compactness property holds for the functional S_λ. We will first pursue the approach involving S_λ.

Constrained Minimization

Denote
$$S_\lambda(\Omega) = \inf_{u \in H_0^{1,2}(\Omega)\setminus\{0\}} S_\lambda(u;\Omega) .$$

Note that $S_\lambda(\Omega) \le S$ for all $\lambda \ge 0$ (in fact, for all $\lambda \in \mathbb{R}$), and $S_\lambda(\Omega)$ in general is not attained. Similar to Theorem I.4.2 now there holds:

2.2 Lemma. *If Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 3$, and if*
$$S_\lambda(\Omega) < S ,$$
then there exists $u \in H_0^{1,2}(\Omega)$, $u > 0$, such that $S_\lambda(\Omega) = S_\lambda(u;\Omega)$.

Proof. Consider a minimizing sequence (u_m) for S_λ in $H_0^{1,2}(\Omega)$; normalize $\|u_m\|_{L^{2^*}} = 1$. Replacing u_m by $|u_m|$, if necessary, we may assume that $u_m \ge 0$. Since by Hölder's inequality
$$S_\lambda(u_m;\Omega) = \int_\Omega \left(|\nabla u_m|^2 - \lambda|u_m|^2\right) dx \ge \int_\Omega |\nabla u_m|^2 \, dx - c ,$$
we also may assume that $u_m \to u$ weakly in $H_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ as $m \to \infty$.

To proceed, observe that like (I.4.4) by Vitali's convergence theorem we have
$$\begin{aligned}
&\int_\Omega \left(|u_m|^{2^*} - |u_m - u|^{2^*}\right) dx = \\
&\quad = \int_\Omega \int_0^1 \frac{d}{dt}\left|u_m + (t-1)u\right|^{2^*} dt\, dx \\
(2.1)\qquad &\quad = 2^* \int_0^1 \int_\Omega \left(u_m + (t-1)u\right)\left|u_m + (t-1)u\right|^{2^*-2} u \, dx\, dt \\
&\quad \to 2^* \int_0^1 \int_\Omega t\,u|t\,u|^{2^*-2} u \, dx\, dt = \int_\Omega |u|^{2^*} dx \quad \text{as } m \to \infty .
\end{aligned}$$

Also note that
$$\int_\Omega |\nabla u_m|^2 \, dx = \int_\Omega |\nabla(u_m - u)|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx + o(1) , \tag{2.2}$$
where $o(1) \to 0$ as $m \to \infty$. Hence we obtain:
$$\begin{aligned}
S_\lambda(\Omega) = S_\lambda(u_m;\Omega) + o(1) &= \int_\Omega |\nabla(u_m - u)|^2 \, dx + \int_\Omega \left(|\nabla u|^2 - \lambda|u|^2\right) dx + o(1) \\
&\ge S\|u_m - u\|_{L^{2^*}}^2 + S_\lambda(\Omega)\|u\|_{L^{2^*}}^2 + o(1) \\
&\ge S\|u_m - u\|_{L^{2^*}}^{2^*} + S_\lambda(\Omega)\|u\|_{L^{2^*}}^{2^*} + o(1) \\
&\ge \left(S - S_\lambda(\Omega)\right)\|u_m - u\|_{L^{2^*}}^{2^*} + S_\lambda(\Omega) + o(1) .
\end{aligned}$$

Since $S > S_\lambda(\Omega)$ by assumption, this implies that $u_m \to u$ in $L^{2^*}(\Omega)$; that is $u \in M$, and by weak lower semi-continuity of the norm in $H_0^{1,2}(\Omega)$ it follows that

$$S_\lambda(u;\Omega) \leq \lim_{m\to\infty} S_\lambda(u_m;\Omega) = S_\lambda(\Omega) ,$$

as desired.

Computing the first variation of $S_\lambda(u;\Omega)$, as in the proof of Theorem I.2.1 we see that a positive multiple of u satisfies (1.1), (1.3). Since $u \geq 0$, $u \not\equiv 0$, from the strong maximum principle (Theorem B.4 of the appendix) we infer that $u > 0$ in Ω. The proof is complete. □

The Unconstrained Case: Local Compactness

Postponing the complete proof of Theorem 2.1 for a moment, we now also indicate the second approach, based on a careful study of the compactness properties of the free functional E_λ. Note that in the case of Theorem 2.1 both approaches are completely equivalent – and the final step in the proof of Theorem 2.1 actually is identical in both cases. However, for more general nonlinearities with critical growth it is not always possible to reduce a boundary value problem like (II.6.1),(II.6.2) to a constrained minimization problem and we will have to use the free functional instead. Moreover, this second approach will bring out the peculiarities of the limiting case more clearly. Our presentation follows Cerami-Fortunato-Struwe [1]. An indication of Lemma 2.3 below is also given by Brezis-Nirenberg [2; p.463].

2.3 Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$. Then for any $\lambda \in \mathbb{R}$, any sequence (u_m) in $H_0^{1,2}(\Omega)$ such that*

$$E_\lambda(u_m) \to \beta < \frac{1}{n} S^{n/2}, \qquad DE_\lambda(u_m) \to 0 ,$$

as $m \to \infty$, is relatively compact.

Proof. To show boundedness of (u_m), compute

$$\begin{aligned} o(1)(1 + \|u_m\|_{H_0^{1,2}}) + \frac{2}{n} S^n &\geq 2E_\lambda(u_m) - \langle u_m, DE_\lambda(u_m)\rangle \\ &= \left(1 - \frac{2}{2^*}\right) \int_\Omega |u_m|^{2^*}\, dx \geq c \left(\int_\Omega |u_m|^2\, dx\right)^{2^*/2} , \end{aligned}$$

where $c > 0$ and $o(1) \to 0$ as $m \to \infty$. Hence

$$\begin{aligned} \|u_m\|^2_{H_0^{1,2}} &= 2E_\lambda(u_m) + \lambda \int_\Omega |u_m|^2\, dx + \frac{2}{2^*} \int_\Omega |u_m|^{2^*}\, dx \\ &\leq C + o(1)\|u_m\|_{H_0^{1,2}} , \end{aligned}$$

and it follows that (u_m) is bounded.

Hence we may assume that $u_m \to u$ weakly in $H_0^{1,2}(\Omega)$, and therefore also strongly in $L^p(\Omega)$ for all $p < 2^*$ by the Rellich-Kondrakov theorem; see Theorem A.5 of the appendix.

In particular, for any $\varphi \in C_0^\infty(\Omega)$ we obtain that

$$\begin{aligned}\langle \varphi, DE_\lambda(u_m)\rangle &= \int_\Omega \left(\nabla u_m \nabla \varphi - \lambda u_m \varphi - u_m |u_m|^{2^*-2}\varphi\right)\, dx \\ &\to \int_\Omega \left(\nabla u \nabla \varphi - \lambda u \varphi - u|u|^{2^*-2}\varphi\right)\, dx = \langle \varphi, DE_\lambda(u)\rangle = 0\end{aligned}$$

as $m \to \infty$. Hence, $u \in H_0^{1,2}(\Omega)$ weakly solves (1.1). Moreover, choosing $\varphi = u$, we have

$$0 = \langle u, DE_\lambda(u)\rangle = \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2 - |u|^{2^*}\right)\, dx ,$$

and hence

$$E_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |u|^{2^*}\, dx = \frac{1}{n}\int_\Omega |u|^{2^*}\, dx \geq 0 .$$

To proceed, note that by (2.1) and (2.2) we have

$$\begin{aligned}\int_\Omega |\nabla u_m|^2\, dx &= \int_\Omega |\nabla(u_m - u)|^2\, dx + \int_\Omega |\nabla u|^2\, dx + o(1) , \\ \int_\Omega |u_m|^{2^*}\, dx &= \int_\Omega |(u_m - u)|^{2^*}\, dx + \int_\Omega |u|^{2^*}\, dx + o(1) ,\end{aligned}$$

and similarly, again using (2.1),

$$\begin{aligned}&\int_\Omega \left(u_m|u_m|^{2^*-2} - u|u|^{2^*-2}\right)(u_m - u)\, dx \\ &\qquad = \int_\Omega \left(|u_m|^{2^*} - u_m|u_m|^{2^*-2}u\right)\, dx + o(1) \\ &\qquad = \int_\Omega \left(|u_m|^{2^*} - |u|^{2^*}\right)\, dx + o(1) = \int_\Omega |u_m - u|^{2^*}\, dx + o(1) ,\end{aligned}$$

where $o(1) \to 0$ $(m \to \infty)$. Hence

$$\begin{aligned}E_\lambda(u_m) &= E_\lambda(u) + E_0(u_m - u) + o(1) , \\ o(1) = \langle u_m - u, DE_\lambda(u_m)\rangle &= \langle u_m - u, DE_\lambda(u_m) - DE_\lambda(u)\rangle \\ &= \int_\Omega \left(|\nabla(u_m - u)|^2 - |u_m - u|^{2^*}\right)\, dx + o(1) .\end{aligned}$$

In particular, from the last equation

$$E_0(u_m - u) = \frac{1}{n}\int_\Omega |\nabla(u_m - u)|^2\, dx + o(1) ,$$

while

$$E_0(u_m - u) = E_\lambda(u_m) - E_\lambda(u) + o(1)$$
$$\leq E_\lambda(u_m) + o(1) \leq c < \frac{1}{n} S^{n/2} \qquad \text{for } m \geq m_0 .$$

Therefore

$$\|u_m - u\|^2_{H_0^{1,2}} \leq c < S^{n/2} \qquad \text{for } m \geq m_0 .$$

But then Sobolev's inequality

$$\|u_m - u\|^2_{H_0^{1,2}} \left(1 - S^{-2^*/2} \|u_m - u\|^{2^*-2}_{H_0^{1,2}}\right) \leq$$
$$\leq \int_\Omega \left(|\nabla(u_m - u)|^2 - |u_m - u|^{2^*}\right) dx = o(1)$$

shows that $u_m \to u$ strongly in $H_0^{1,2}(\Omega)$, as desired. □

Lemma 2.3 motivates to introduce the following variant of (P.-S.), which seems to appear first in Brezis-Coron-Nirenberg [1].

2.4 Definition. *Let V be a Banach space, $E \in C^1(V)$, be $\in \mathbb{R}$. E satisfies condition (P.-S.)$_\beta$, if any sequence (u_m) in V such that $E(u_m) \to \beta$ while $DE(u_m) \to 0$ as $m \to \infty$ is relatively compact. (Such sequences in the sequel for brevity will be referred to as (P.-S.)$_\beta$-sequences.)*

Now recall that E_λ for $\lambda < \lambda_1$ satisfies conditions (1°)–(3°) of Theorem II.6.1.

By Lemma 2.3, therefore, the proof of the first two parts of Theorem 2.1 will be complete if we can show that for $\lambda > 0$ (respectively $\lambda > \lambda_*$) there holds

$$\beta = \inf_{p \in P} \sup_{u \in p} E_\lambda(u) < \frac{1}{n} S^{n/2} , \tag{2.3}$$

where, for a suitable function u_1 satisfying $E(u_1) \leq 0$, we let

$$P = \{p \in C^0([0,1] \, ; \, H_0^{1,2}(\Omega)) \, ; \, p(0) = 0, \, p(1) = u_1\} ,$$

as in Theorem 6.1.

Of course, (2.3) and the condition $S_\lambda(\Omega) < S$ of Lemma 2.2 are related. Given $u \in H_0^{1,2}(\Omega)$, $\|u\|_{L^{2^*}} = 1$, we may let $p(t) = t\,u$, $u_1 = t_1 u$ for sufficiently large t_1 to obtain

$$\beta \leq \sup_{0 \leq t < \infty} E_\lambda(t\,u) = \sup_{0 \leq t < \infty} \left(\frac{t^2}{2} S_\lambda(u;\Omega) - \frac{t^{2^*}}{2^*}\right) = \frac{1}{n} S_\lambda^{n/2}(u;\Omega) .$$

Likewise, for $p \in P$ there exists $u \in p$ such that $u \neq 0$ and

$$\langle u, DE_\lambda(u)\rangle = \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2 - |u|^{2^*}\right) dx = 0 .$$

Indeed, since $\lambda < \lambda_1$, for $u = p(t)$ with t close to 0 we have $\langle u, DE_\lambda(u)\rangle > 0$, while for $u = p(1) = u_1$ we have

$$\langle u_1, DE_\lambda(u_1)\rangle < 2E_\lambda(u_1) \le 0 \ ,$$

and by the intermediate value theorem there exists u, as claimed. But for such u we easily compute

$$\begin{aligned} S_\lambda(u;\Omega) &= \left(\int_\Omega |\nabla u|^2 - \lambda |u|^2 \, dx\right)^{1-2/2^*} \\ &= (n\, E_\lambda(u))^{2/n} \le \left(n \sup_{u\in p} E_\lambda(u)\right)^{2/n} . \end{aligned}$$

That is,

$$\beta = \inf_{p\in P} \sup_{u\in p} E_\lambda(u) = \frac{1}{n} S_\lambda^{n/2}(\Omega) \ , \tag{2.4}$$

and (2.3) and the condition $S_\lambda < S$ are in fact equivalent.

Proof of Theorem 2.1(1°). It suffices to show that $S_\lambda < S$. Consider the family

$$u_\varepsilon^*(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}, \ \varepsilon > 0 \ , \tag{2.5}$$

of functions $u_\varepsilon^* \in D^{1,2}(\mathbb{R}^n)$. Note that $u_\varepsilon^*(u) = \varepsilon^{\frac{2-n}{2}} u_1^*\left(\frac{x}{\varepsilon}\right)$, and u_ε^* satisfies the equation

$$-\Delta u_\varepsilon^* = u_\varepsilon^* |u_\varepsilon^*|^{2^*-2} \qquad \text{in } \mathbb{R}^n \ , \tag{2.6}$$

as is easily verified by a direct computation. We claim that $S_0(u_\varepsilon^*;\mathbb{R}^n) = S$; that is, the best Sobolev constant is achieved by the family u_ε^*, $\varepsilon > 0$. Indeed, let $u \in D^{1,2}(\mathbb{R}^n)$ satisfy $S_0(u;\mathbb{R}^n) = S$. (The existence of such a function u can be deduced for instance from Theorem I.4.9.) Using Schwarz-symmetrization we may assume that u is radially symmetric; that is, $u(x) = u(|x|)$. Moreover, u solves (2.6). Choose $\varepsilon > 0$ such that $u_\varepsilon^*(0) = u(0)$. Then u and u_ε^* both are solutions of the ordinary differential equation of second order in $r = |x|$,

$$r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) u = u|u|^{2^*-2} \qquad \text{for } r > 0 \ ,$$

sharing the initial data $u(0) = u_\varepsilon^*(0)$, $\partial_r u(0) = \partial_r u_\varepsilon^*(0) = 0$. It is not hard to prove that this initial value problem admits a unique solution, and thus $u = u_\varepsilon^*$, which implies that $S_0(u_\varepsilon^*;\mathbb{R}^n) = S_0(u;\mathbb{R}^n) = S$.

In particular,

$$\|u_\varepsilon^*\|^2_{H_0^{1,2}} = \|u_\varepsilon^*\|^{2^*}_{L^{2^*}} = S^{n/2}, \qquad \text{for all } \varepsilon > 0 \ .$$

We may suppose that $0 \in \Omega$. Let $\eta \in C_0^\infty(\Omega)$ be a fixed cut-off function, $\eta \equiv 1$ in a neighborhood $B_\rho(0)$ of 0. Let $u_\varepsilon = \eta\, u_\varepsilon^*$ and compute

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx = \int_\Omega |\nabla u_\varepsilon^*|^2 \eta^2\, dx + O(\varepsilon^{n-2})$$
$$= \int_{\mathbb{R}^n} |\nabla u_\varepsilon^*|^2\, dx + O(\varepsilon^{n-2}) = S^{n/2} + O(\varepsilon^{n-2})\ .$$
$$\int_\Omega |u_\varepsilon|^{2^*}\, dx = \int_{\mathbb{R}^n} |u_\varepsilon^*|^{2^*}\, dx + O(\varepsilon^n) = S^{n/2} + O(\varepsilon^n)$$
$$\int_\Omega |u_\varepsilon|^2\, dx = \int_{B_\rho(0)} |u_\varepsilon^*|^2\, dx + O(\varepsilon^{n-2})$$
$$\geq \int_{B_\varepsilon(0)} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[2\varepsilon^2]^{n-2}}\, dx \tag{2.7}$$
$$+ \int_{B_\rho(0)\setminus B_\varepsilon(0)} \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{2}}}{[2|x|^2]^{n-2}}\, dx + O(\varepsilon^{n-2})$$
$$= c_1 \cdot \varepsilon^2 + c_2 \varepsilon^{n-2} \int_\varepsilon^\rho r^{3-n}\, dr + O(\varepsilon^{n-2})$$
$$= \begin{cases} c\,\varepsilon^2 + O(\varepsilon^{n-2}), & \text{if } n > 4 \\ c\,\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2), & \text{if } n = 4 \\ c\,\varepsilon + O(\varepsilon^2), & \text{if } n = 3 \end{cases}$$

with positive constants $c, c_1, c_2 > 0$. Thus, if $n \geq 5$

$$S_\lambda(u_\varepsilon) \leq \frac{(S^{n/2} - c\lambda\varepsilon^2 + O(\varepsilon^{n-2}))}{(S^{n/2} + O(\varepsilon^n))^{2/2^*}}$$
$$= S - c\lambda\varepsilon^2 + O(\varepsilon^{n-2}) < S\ ,$$

if $\varepsilon > 0$ is sufficiently small. Similarly, if $n = 4$, we have

$$S_\lambda(u_\varepsilon) \leq S - c\lambda\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) < S$$

for $\varepsilon > 0$ sufficiently small.

Remark on Theorem 2.1.(2°),(3°). If $n = 3$, estimate (2.7) shows that the "gain" due to the presence of λ and the "loss" due to truncation of u_ε^* may be of the same order in ε; hence S_λ can only be expected to be smaller than S for "large" λ. To see that $\lambda_* < \lambda_1$, choose the first eigenfunction $u = \varphi_1$ of $(-\Delta)$ as comparison function. The non-existence result for $\Omega = B_1(0)$, $\lambda \leq \frac{\lambda_1}{4}$ follows from a weighted estimate similar to Lemma 1.4; see Brezis-Nirenberg [2; Lemma 1.4]. We omit the details. □

Theorem 2.1 should be viewed together with the global bifurcation result of Rabinowitz [1; p. 195 f.]. Intuitively, Theorem 2.1 indicates that the branch of

We claim that

$$\int_0^1 u^2(\lambda\psi' + \tfrac{1}{4}\psi''')r^2\,dr = \tfrac{2}{3}\int_0^1 u^6(r\psi - r^2\psi')\,dr + \tfrac{1}{2}|u'(1)|^2\psi(1) \tag{1.37}$$

for every smooth function ψ such that $\psi(0)=0$.[5] Indeed, we first multiply (1.35) by $r^2\psi u'$ and obtain

$$\begin{aligned}&\int_0^1 |u'|^2(\tfrac{1}{2}r^2\psi' - r\psi)\,dr - \tfrac{1}{2}|u'(1)|^2\psi(1)\\ &\qquad = -\tfrac{1}{6}\int_0^1 u^6(2r\psi + r^2\psi')\,dr - \tfrac{1}{2}\lambda\int_0^1 u^2(2r\psi + r^2\psi')\,dr.\end{aligned} \tag{1.38}$$

Next we multiply (1.35) by $(\frac{1}{2}r^2\psi' - r\psi)u$ and obtain

$$\begin{aligned}&\int_0^1 |u'|^2(\tfrac{1}{2}r^2\psi' - r\psi)\,dr - \tfrac{1}{4}\int_0^1 u^2r^2\psi'''\,dr\\ &\qquad = \int_0^1 u^6(\tfrac{1}{2}r^2\psi' - r\psi)\,dr + \lambda\int_0^1 u^2(\tfrac{1}{2}r^2\psi' - r\psi)\,dr.\end{aligned} \tag{1.39}$$

Combining (1.38) and (1.39) we obtain (1.37). We already know that there is no solution of (1.23) for $\lambda \leqq 0$; thus we may assume that $0 < \lambda \leqq \frac{1}{4}\pi^2$. In (1.37) we choose $\psi(r) = \sin((4\lambda)^{1/2}r)$ so that $\psi(1) \geqq 0$,

$$\lambda\psi' + \tfrac{1}{4}\psi''' = 0,$$

and

$$r\psi - r^2\psi' = r\sin((4\lambda)^{1/2}r) - r^2(4\lambda)^{1/2}\cos((4\lambda)^{1/2}r) > 0 \quad \text{on} \quad (0, 1]$$

(since $\sin\theta - \theta\cos\theta > 0$ for all $\theta \in (0, \pi]$) and we obtain a contradiction.

Proof of Theorem 1.2 concluded: If $\lambda > \frac{1}{4}\lambda_1$ we know that $S_\lambda < S$ (see Lemma 1.3). We may proceed exactly as in the proof of Theorem 1.1 (Lemma 1.2) and conclude that the infimum in (1.24) is achieved. Thus we obtain some $u \in H_0^1$ with $u \geqq 0$ on Ω, $\|u\|_6 = 1$ and

$$-\Delta u - \lambda u = S_\lambda u^5.$$

If, in addition, $\lambda < \lambda_1$, then $S_\lambda > 0$ and after stretching, we obtain a solution of (1.23).

1.3. Additional properties, miscellaneous remarks and open problems.

(1). REGULARITY OF SOLUTIONS. The solution u of (1.1) given by Theorem 1.1 (respectively Theorem 1.2) lies in $H_0^1(\Omega)$. In fact, u belongs to

[5] Note that Pohozaev's identity corresponds to the case where $\psi(r) = r$.

A Global Compactness Result

Remark that by Theorem 1.2, for $\lambda = 0$ no non-trivial solution $u \in H_0^{1,2}(\Omega)$ of (1.1) can satisfy $S_\lambda(u;\Omega) \leq S$. Hence the local compactness of Lemma 2.3 will not suffice to produce such solutions and we must study the compactness properties of E_λ, respectively S_λ, at higher energy levels as well. The next result can be viewed as an extension of P.-L. Lions' concentration-compactness method for minimization problems (see Section I.4) to problems of minimax type. The idea of analyzing the behavior of a (P.-S.)-sequence near points of concentration by "blowing up" the singularities seems to appear first in papers by Sacks and Uhlenbeck [1] and Wente [5] where variants of the local compactness condition Lemma 2.3 are obtained (see Sacks-Uhlenbeck [1; Lemma 4.2]). In the next result, due to Struwe [8], we systematically employ the blow-up technique to characterize *all* energy values β of a variational problem where $(P.\text{-}S.)_\beta$ may fail in terms of "critical points at infinity".

3.1 Theorem. *Suppose Ω is a bounded domain in $\mathbb{R}^n, n \geq 3$, and for $\lambda \in \mathbb{R}$ let (u_m) be a (P.-S.)-sequence for E_λ in $H_0^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$. Then there exist an index $k \in \mathbb{N}_0$, sequences (R_m^j), (x_m^j), $1 \leq j \leq k$, of radii $R_m^j \to \infty$ $(m \to \infty)$ and points $x_m^j \in \Omega$, a solution $u^0 \in H_0^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ to (1.1), (1.3) and non-trivial solutions $u^j \in D^{1,2}(\mathbb{R}^n)$, $1 \leq j \leq k$, to the "limiting problem" associated with (1.1) and (1.3),*

$$-\Delta u = u|u|^{2^*-2} \quad \text{in } \mathbb{R}^n \ , \tag{3.1}$$

such that a subsequence (u_m) satisfies

$$\left\| u_m - u^0 - \sum_{j=1}^k u_m^j \right\|_{D^{1,2}(\mathbb{R}^n)} \to 0 \ .$$

Here u_m^j denotes the rescaled function

$$u_m^j(x) = (R_m^j)^{\frac{n-2}{2}} u^j\big(R_m^j(x - x_m^j)\big), \ 1 \leq j \leq k, \ m \in \mathbb{N} \ .$$

Moreover,

$$E_\lambda(u_m) \to E_\lambda(u^0) + \sum_{j=1}^k E_0(u^j) \ .$$

3.2 Remark. In particular, if Ω is a ball $\Omega = B_R(0), u_m \in H_{0,rad}^{1,2}(\Omega)$, from the uniqueness of the family $(u_\varepsilon^*)_{\varepsilon>0}$ of radial solutions to (3.1) – see the proof of Theorem 2.1.(1°) – it follows that each u^j is of the form (2.5) with $E_0(u^j) = \frac{1}{n}S^{n/2} =: \beta^*$. Hence in this case $(P.\text{-}S.)_\beta$ holds for E_λ for all levels β which cannot be decomposed

$$\beta = \beta_0 + k\beta^* \ ,$$

where $k \geq 1$ and $\beta_0 = E_\lambda(u^0)$ is the energy of some radial solution of (1.1), (1.3). Similarly, if Ω is an arbitrary bounded domain and $u_m \geq 0$ for all m, then also $u^j \geq 0$ for all j, and by a result of Gidas-Ni-Nirenberg [1; p. 210 f.] and Obata [1] again each function u^j will be radially symmetric about some point x^j. Therefore also in this case each u^j is of the form $u^j = u^*_\varepsilon(\cdot - x^j)$ for some $\varepsilon > 0$, and (P.-S.)$_\beta$ holds for all β which are not of the form

$$\beta = \beta_0 + k\beta^* ,$$

where $k \geq 1$ and $\beta_0 = E_\lambda(u_0)$ is the energy of some non-negative solution u^0 of (1.1), (1.3).

For some time it was believed that the family (2.5) gives all non-trivial solutions of (3.1). Surprisingly, Ding [1] was able to establish that (3.1) also admits infinitely many solutions of changing sign which are distinct modulo scaling.

In general, decomposing a solution v of (3.1) into positive and negative parts $v = v_+ + v_-$, where $v_\pm = \pm \max\{\pm v, 0\}$, upon testing (3.1) with $v_\pm$ from Sobolev's inequality we infer that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \left(-\Delta v - v|v|^{2^*-2}\right) v_\pm \, dx \\ &= \int_{\mathbb{R}^n} \left(|\nabla v_\pm|^2 - |v_\pm|^{2^*}\right) dx \geq \left(1 - S^{-2^*/2} \|v_\pm\|_{D^{1,2}}^{2^*-2}\right) \|v_\pm\|_{D^{1,2}}^2 . \end{aligned}$$

Hence $v_\pm \equiv 0$ or

$$E_0(v_\pm) = \frac{1}{n} \|v_\pm\|_{D^{1,2}}^2 \geq \frac{1}{n} S^{n/2} = \beta^* ,$$

and therefore any solution v of (3.1) that changes sign satisfies

$$E_0(v) = E_0(v_+) + E_0(v_-) \geq 2\beta^* .$$

In fact, $E_0(v) > 2\beta^*$; otherwise S would be achieved at v_+ and v_-, which would contradict Theorem 1.2. Thus, in Theorem 3.1 we can assert that $E_0(u^j) \in \{\beta^*\} \cup]2\beta^*, \infty[$.

In particular, if (1.1), (1.3) does not admit any solution but the trivial solution $u \equiv 0$, the local Palais-Smale condition (P.-S.)$_\beta$ will hold for all $\beta < 2\beta^*$, except for $\beta = \beta^*$.

Proof of Theorem 3.1. First recall that as in the proof of Lemma 2.3 any (P.-S.)-sequence for E_λ is bounded. Hence we may assume that $u_m \rightharpoonup u^0$ weakly in $H_0^{1,2}(\Omega)$, and u^0 solves (1.1), (1.3). Moreover, if we let $v_m = u_m - u^0$ we have $v_m \to 0$ strongly in $L^2(\Omega)$, and by (2.1), (2.2) also that

$$\int_\Omega |v_m|^{2^*} \, dx = \int_\Omega |u_m|^{2^*} \, dx - \int_\Omega |u^0|^{2^*} \, dx + o(1) ,$$

$$\int_\Omega |\nabla v_m|^2 \, dx = \int_\Omega |\nabla u_m|^2 \, dx - \int_\Omega |\nabla u^0|^2 \, dx + o(1) ,$$

where $o(1) \to 0$ $(m \to \infty)$. Hence, in particular, we obtain that

$$E_\lambda(u_m) = E_\lambda(u^0) + E_0(v_m) + o(1) .$$

Also note that

$$DE_\lambda(u_m) = DE_\lambda(u^0) + DE_0(v_m) + o(1) = DE_0(v_m) + o(1) ,$$

where $o(1) \to 0$ in $H^{-1}(\Omega)$ $(m \to \infty)$. Using the following lemma, we can now proceed by induction:

3.3 Lemma. *Suppose (v_m) is a (P.-S.)-sequence for $E = E_0$ in $H_0^{1,2}(\Omega)$ such that $v_m \rightharpoonup 0$ weakly. Then there exists a sequence (x_m) of points $x_m \in \Omega$, a sequence (R_m) of radii $R_m \to \infty$ $(m \to \infty)$, a non-trivial solution v^0 to the limiting problem (3.1) and a (P.-S.)-sequence (w_m) for E_0 in $H_0^{1,2}(\Omega)$ such that for a subsequence (v_m) there holds*

$$w_m = v_m - R_m^{\frac{n-2}{2}} v^0\big(R_m(\cdot - x_m)\big) + o(1) ,$$

where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$ as $m \to \infty$. In particular, $w_m \rightharpoonup 0$ weakly. Furthermore,

$$E_0(w_m) = E_0(v_m) - E_0(v^0) + o(1) .$$

Moreover,

$$R_m \, dist(x_m, \partial\Omega) \to \infty .$$

Finally, if $E_0(v_m) \to \beta < \beta^*$, the sequence (v_m) is relatively compact and hence $v_m \to 0$, $E_0(v_m) \to \beta = 0$.

Proof of Theorem 3.1 (completed). Apply Lemma 3.3 to the sequences $v_m^1 = u_m - u^0$, $v_m^j = u_m - u^0 - \sum_{i=1}^{j-1} u_m^i = v_m^{j-1} - u_m^{j-1}$, $j > 1$, where

$$u_m^i(x) = (R_m^i)^{\frac{n-2}{2}} u^i\big(R_m^i(x - x_m^i)\big) .$$

By induction

$$\begin{aligned} E_0(v_m^j) &= E_\lambda(u_m) - E_\lambda(u^0) - \sum_{i=1}^{j-1} E_0(u^i) \\ &\le E_\lambda(u_m) - (j-1)\beta^* . \end{aligned}$$

Since the latter will be negative for large j, by Lemma 3.3 the induction will terminate after some index $k \ge 0$. Moreover, for this index we have

$$v_m^{k+1} = u_m - u^0 - \sum_{j=1}^{k} u_m^i \to 0$$

strongly in $D^{1,2}(\mathbb{R}^n)$, and

$$E_\lambda(u_m) - E_\lambda(u^0) - \sum_{j=1}^{k} E_0(u^j) \to 0 \ ,$$

as desired. □

Proof of Lemma 3.3. If $E_0(v_m) \to \beta < \beta^*$, by Lemma 2.3 the sequence (v_m) is strongly relatively compact and hence $v_m \to 0$, $\beta = 0$. Therefore, we may assume that $E_0(v_m) \to \beta \geq \beta^* = \frac{1}{n}S^{n/2}$. Moreover, since $DE_0(v_m) \to 0$ we also have

$$\frac{1}{n}\int_\Omega |\nabla v_m|^2 \, dx = E_0(v_m) - \frac{1}{2^*}\langle v_m, DE_0(v_m)\rangle \to \beta \geq \frac{1}{n}S^{n/2}$$

and hence that

$$\liminf_{m\to\infty} \int_\Omega |\nabla v_m|^2 \, dx = n\beta \geq S^{n/2} \ . \tag{3.2}$$

Denote

$$Q_m(r) = \sup_{x\in\Omega} \int_{B_r(x)} |\nabla v_m|^2 \, dx$$

the concentration function of v_m, introduced in Section I.4.3. Choose $x_m \in \overline{\Omega}$ and scale

$$v_m \mapsto \tilde{v}_m(x) = R_m^{\frac{2-n}{2}} v_m(x/R_m + x_m)$$

such that

$$\tilde{Q}_m(1) = \sup_{\substack{x\in\mathbb{R}^n \\ x/R_m + x_m \in \Omega}} \int_{B_1(x)} |\nabla \tilde{v}_m|^2 \, dx = \int_{B_1(0)} |\nabla \tilde{v}_m|^2 \, dx = \frac{1}{2L}S^{n/2} \ ,$$

where L is a number such that $B_2(0)$ is covered by L balls of radius 1. Clearly, by (3.2) we have $R_m \geq R_0 > 0$, uniformly in m.

Considering $\tilde{\Omega}_m = \{x \in \mathbb{R}^n \ ; \ x/R_m + x_m \in \Omega\}$, we may regard $\tilde{v}_m \in H_0^{1,2}(\tilde{\Omega}_m) \subset D^{1,2}(\mathbb{R}^n)$.

Moreover,

$$\|\tilde{v}_m\|^2_{D^{1,2}} = \|v_m\|^2_{D^{1,2}} \to n\beta < \infty$$

and we may assume that $\tilde{v}_m \to v^0$ weakly in $D^{1,2}(\mathbb{R}^n)$. We claim that $\tilde{v}_m \to v^0$ strongly in $H^{1,2}(\Omega')$, for any $\Omega' \subset\subset \mathbb{R}^n$. It suffices to consider $\Omega' = B_1(x_0)$ for any $x_0 \in \mathbb{R}^n$. (For brevity $B_r(x_0) =: B_r$.) Indeed, by Fubini's theorem and since

$$\int_1^2 \left(\int_{\partial B_r} |\nabla \tilde{v}_m|^2 \, do\right) dr \leq \int_{B_2} |\nabla \tilde{v}_m|^2 \, dx \leq n\beta + o(1) \ ,$$

where $o(1) \to 0$ $(m \to \infty)$, there is a radius $\rho \in [1,2]$ such that

$$\int_{\partial B_\rho} |\nabla \tilde{v}_m|^2 \, do \leq 2n\beta$$

for infinitely many $m \in \mathbb{N}$. (Relabelling, we may assume that this estimate holds for all $m \in \mathbb{N}$.) By compactness of the embedding $H^{1,2}(\partial B_\rho) \hookrightarrow H^{1/2,2}(\partial B_\rho)$, we deduce that a subsequence $\tilde{v}_m \to \bar{v}^0$ strongly in $H^{1/2,2}(\partial B_\rho)$; see Theorem A.8 of the appendix. Moreover, since also the trace operator $H^{1,2}(B_2) \to L^2(\partial B_\rho)$ is compact, we conclude that $\bar{v}^0 = v^0$. Now let

$$\varphi_m = \begin{cases} \tilde{v}_m - v^0 & \text{in } B_\rho \\ \tilde{w}_m & \text{in } B_3 \setminus B_\rho \ , \end{cases}$$

where $\tilde{w}_m$ denotes the solution to the Dirichlet problem $\Delta \tilde{w}_m = 0$ in $B_3 \setminus B_\rho$, $\tilde{w}_m = \tilde{v}_m - v^0$ on ∂B_ρ, $\tilde{w}_m = 0$ on ∂B_3. By continuity of the solution operator to the Dirichlet problem on the annulus $B_3 \setminus B_\rho$ in the $H^{1/2,2}$-norm (see for instance Lions-Magenes [1; Theorem 8.2]), we have

$$\|\tilde{w}_m\|_{H^{1,2}(B_3 \setminus B_\rho)} \le c \|\tilde{v}_m - v^0\|_{H^{1/2,2}(\partial B_\rho)} \to 0 \ .$$

Hence $\varphi_m = \tilde{\varphi}_m + o(1) \in H_0^{1,2}(\tilde{\Omega}_m) + D^{1,2}(\mathbb{R}^n)$, where $\tilde{\varphi}_m \in H_0^{1,2}(\tilde{\Omega}_m)$ and $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$ as $m \to \infty$. Thus

$$\langle \varphi_m, DE_0(\tilde{v}_m; \mathbb{R}^n)\rangle = \langle \tilde{\varphi}_m, DE_0(\tilde{v}_m; \tilde{\Omega}_m)\rangle + o(1) \to 0 \ .$$

On the other hand, using convergence arguments familiar by now and Sobolev's inequality, we obtain

$$\begin{aligned} o(1) = \langle \varphi_m, DE_0(\tilde{v}_m; \mathbb{R}^n)\rangle &= \\ &= \int_{\mathbb{R}^n} \left(\nabla \tilde{v}_m \nabla \varphi_m - \tilde{v}_m |\tilde{v}_m|^{2^*-2} \varphi_m\right) dx \\ &= \int_{B_\rho} \left(|\nabla(\tilde{v}_m - v^0)|^2 - |\tilde{v}_m - v^0|^{2^*}\right) dx + o(1) \\ &= \int_{\mathbb{R}^n} \left(|\nabla \varphi_m|^2 - |\varphi_m|^{2^*}\right) dx + o(1) \\ &\ge \|\varphi_m\|^2_{D^{1,2}(\mathbb{R}^n)} \left(1 - S^{-2^*/2} \|\varphi_m\|^{2^*-2}_{D^{1,2}(\mathbb{R}^n)}\right) \ , \end{aligned} \tag{3.3}$$

where $o(1) \to 0$ as $m \to \infty$. But now we note that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \varphi_m|^2 \, dx = \int_{B_\rho} |\nabla(\tilde{v}_m - v^0)|^2 \, dx + o(1) &\le \int_{B_2} |\nabla \tilde{v}_m|^2 \, dx + o(1) \\ &\le L \, \tilde{Q}_m(1) = \frac{1}{2} S^{n/2} \ , \end{aligned}$$

from which via (3.3) we deduce that $\varphi_m \to 0$ in $D^{1,2}(\mathbb{R}^n)$; that is, $\tilde{v}_m \to v^0$ locally in $H^{1,2}$, as desired.

In particular,

$$\int_{B_1(0)} |\nabla v^0|^2 \, dx = \frac{1}{2L} S^{n/2} > 0 \ ,$$

$\int_{B_\rho} |\nabla \tilde{v}_m|^2 = \int_{B_\rho} |\nabla v^0|^2 + \int_{B_\rho} |\nabla(\tilde{v}_m - v^0)|^2 + o(1)$ Lieb's Lemma

and $v^0 \not\equiv 0$. Since the original sequence $v_m \to 0$ weakly, thus it also follows that $R_m \to \infty$ as $m \to \infty$. Now we distinguish two cases:
(1°) $R_m \, dist(x_m, \partial\Omega) \le c < \infty$, uniformly, in which case (after rotation of coordinates) we may assume that the sequence $\tilde\Omega_m$ exhausts the half-space

$$\tilde\Omega_\infty = \mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \, ; \, x_1 > 0\} ,$$

or
(2°) $R_m \, dist(x_m, \partial\Omega) \to \infty$, in which case $\tilde\Omega_m \to \tilde\Omega_\infty = \mathbb{R}^n$.

Since in each case for any $\varphi \in C_0^\infty(\tilde\Omega_\infty)$ we have that $\varphi \in C_0^\infty(\tilde\Omega_m)$ for large m, there holds

$$\langle \varphi, DE_0(v^0; \tilde\Omega_\infty)\rangle = \lim_{m\to\infty} \langle \varphi, DE_0(\tilde v_m; \tilde\Omega_m)\rangle = 0 ,$$

for all such φ, and $v^0 \in H_0^{1,2}(\tilde\Omega_\infty)$ is a weak solution of (3.1) on $\tilde\Omega_\infty$. But if $\tilde\Omega_\infty = \mathbb{R}^n{}_+$, by Theorem 1.3 then v^0 must vanish identically. Thus (1°) is impossible, and we are left with (2°).

To conclude the proof, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_1(0)$, $\varphi \equiv 0$ outside $B_2(0)$, and let

$$w_m(x) = v_m(x) - R_m^{\frac{n-2}{2}} v^0\big(R_m(x - x_m)\big) \cdot \varphi\big(\overline{R}_m(x - x_m)\big) \in H_0^{1,2}(\Omega) ,$$

where the sequence $(\overline{R}_m)$ is chosen such that $\tilde R_m := R_m(\overline{R}_m)^{-1} \to \infty$ while $\overline{R}_m \operatorname{dist}(x_m, \partial\Omega) \to \infty$ as $m \to \infty$; that is,

$$\tilde w_m(x) = R_m^{\frac{2-n}{2}} w_m(x/R_m + x_m) = \tilde v_m(x) - v^0(x)\varphi(x/\tilde R_m) .$$

Set $\varphi_m(x) = \varphi(x/\tilde R_m)$. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|\nabla\big(v^0(\varphi_m - 1)\big)\big|^2 \, dx \le \\
&\qquad \le C \int_{\mathbb{R}^n} |\nabla v^0|^2 (\varphi_m - 1)^2 \, dx + C \int_{\mathbb{R}^n} |v^0|^2 \big|\nabla(\varphi_m - 1)\big|^2 \, dx \\
&\qquad \le C \int_{\mathbb{R}^n \setminus B_{\tilde R_m}(0)} |\nabla v^0|^2 \, dx + C \tilde R_m^{-2} \int_{B_{2\tilde R_m}(0)\setminus B_{\tilde R_m}(0)} |v^0|^2 \, dx .
\end{aligned}$$

But $\nabla v^0 \in L^2(\mathbb{R}^n)$. Therefore the first term tends to 0 as $m \to \infty$, while by Hölder's inequality also the second term

$$\tilde R_m^{-2} \int_{B_{2\tilde R_m}(0)\setminus B_{\tilde R_m}(0)} |v^0|^2 \, dx \le C \left(\int_{B_{2\tilde R_m}(0)\setminus B_{\tilde R_m}(0)} |v^0|^{2^*} \, dx \right)^{2/2^*} \to 0$$

as $m \to \infty$. Thus we have $\tilde w_m = \tilde v_m - v^0 + o(1)$, where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$. Hence, as in the proof of Lemma 2.3, also

$$\begin{aligned} E_0(w_m) &= E_0(\tilde{w}_m) = E_0(\tilde{v}_m) - E_0(v^0) + o(1) , \\ \|DE_0(w_m; \Omega)\| &= \|DE_0(\tilde{w}_m; \tilde{\Omega}_m)\| \\ &\leq \|DE_0(\tilde{v}_m; \tilde{\Omega}_m\| + \|DE_0(v^0; \mathbb{R}^n)\| + o(1) \\ &= \|DE_0(v_m; \Omega)\| + o(1) \to 0 \qquad (m \to \infty) . \end{aligned}$$

This concludes the proof. □

Positive Solutions on Annular-Shaped Regions

With the aid of Theorem 3.2 we can now show the existence of solutions to (1.1),(1.3) on perturbed annular domains for $\lambda = 0$.

The following result is due to Coron [2]:

3.4 Theorem. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ satisfying the following condition: There exist constants $0 < R_1 < R_2 < \infty$ such that*

(1°) $$\Omega \supset \{x \in \mathbb{R}^n \; ; \; R_1 < |x| < R_2\} ,$$

(2°) $$\overline{\Omega} \not\supset \{x \in \mathbb{R}^n \; ; \; |x| < R_1\} .$$

Then, if R_2/R_1 is sufficiently large, problem (1.1), (1.3) for $\lambda = 0$ admits a positive solution to $u \in H_0^{1,2}(\Omega)$.

Again remark that the solution u must have an energy above the compactness threshold given by Lemma 2.3.

The idea of the proof is to argue by contradiction and to use a minimax method for $S = S_0(\,\cdot\,;\Omega)$ based on a set A of non-negative functions which is homeomorphic to a sphere Σ around 0 in Ω. Note that A is contractible in the positive cone in $H_0^{1,2}(\Omega)$. Moreover, if (1.1), (1.3) does not admit a positive solution, then under certain conditions such a contraction of A in $H_0^{1,2}(\Omega)$ will induce a contraction of Σ in Ω, and the desired contradiction will result.

Proof. We may assume $R_1 = (4R)^{-1} < 1 < 4R = R_2$. Consider the unit sphere

$$\Sigma = \{x \in \mathbb{R}^n \; ; \; |x| = 1\} .$$

For $\sigma \in \Sigma$, $x \in \mathbb{R}^n$, $0 \leq t < 1$ let

$$u_t^\sigma(x) = \left[\frac{1-t}{(1-t)^2 + |x - t\sigma|^2}\right]^{\frac{n-2}{2}} \in D^{1,2}(\mathbb{R}^n) .$$

Note that S is attained on any such function u_t^σ, and u_t^σ "concentrates" at σ as $t \to 1$. Moreover, letting $t \to 0$ we have

$$u_t^\sigma \to u_0 = \left[\frac{1}{1+|x|^2}\right]^{\frac{n-2}{2}} ,$$