7

Maximum Principles

7.1 Introduction

As a simple example of maximum principle, let's consider a C^2 function $u(x)$ of one independent variable x. It is well-known in calculus that at a local maximum point x_o of u, we must have

$$u''(x_o) \leq 0.$$

Based on this observation, then we have the following simplest version of maximum principle:

Assume that $u''(x) > 0$ in an open interval (a, b), then u can not have any interior maximum in the interval.

One can also see this geometrically. Since under the condition $u''(x) > 0$, the graph is concave up, it can not have any local maximum.

More generally, for a C^2 function $u(x)$ of n-independent variables $x = (x_1, \cdots, x_n)$, at a local maximum x^o, we have

$$D^2 u(x^o) := (u_{x_i x_j}(x^o)) \leq 0 \,,$$

that is, the symmetric matrix is non-positive definite at point x^o. Correspondingly, the simplest version maximum principle reads:

If

$$(a_{ij}(x)) \geq 0 \quad \textit{and} \quad \sum_{ij} a_{ij}(x) u_{x_i x_j}(x) > 0 \tag{7.1}$$

in an open bounded domain Ω, then u can not achieve its maximum in the interior of the domain.

An interesting special case is when $(a_{ij}(x))$ is an identity matrix, in which condition (7.1) becomes

$$\triangle u > 0.$$

Unlike its one-dimensional counterpart, condition (7.1) no longer implies that the graph of $u(x)$ is concave up. A simple counter example is

$$u(x_1, x_2) = x_1^2 - \frac{1}{2} x_2^2.$$

One can easily see from the graph of this function that $(0,0)$ is a saddle point.

In this case, the validity of the maximum principle comes from the simple algebraic fact:

For any two $n \times n$ matrices A and B, if $A \geq 0$ and $B \leq 0$, then $AB \leq 0$.

In this chapter, we will introduce various maximum principles, and most of them will be used in the method of moving planes in the next chapter. Besides this, there are numerous other applications. We will list some below.

i) *Providing Estimates*

Consider the boundary value problem

$$\begin{cases} -\triangle u = f(x), & x \in B_1(0) \subset R^n \\ u(x) = 0, & x \in \partial B_1(0). \end{cases} \tag{7.2}$$

If $a \leq f(x) \leq b$ in $B_1(0)$, then we can compare the solution u with the two functions

$$\frac{a}{2n}(1 - |x|^2) \quad \text{and} \quad \frac{b}{2n}(1 - |x|^2)$$

which satisfy the equation with $f(x)$ replaced by a and b, respectively, and which vanish on the boundary as u does. Now, applying the maximum principle for $\triangle$ operator (see Theorem 7.1.1 in the following), we obtain

$$\frac{a}{2n}(1 - |x|^2) \leq u(x) \leq \frac{b}{2n}(1 - |x|^2).$$

ii) *Proving Uniqueness of Solutions*

In the above example, if $f(x) \equiv 0$, then we can choose $a = b = 0$, and this implies that $u \equiv 0$. In other words, the solution of the boundary value problem (7.2) is unique.

iii) *Establishing the Existence of Solutions*

(a) For a linear equation such as (7.2) in any bounded open domain Ω, let

$$u(x) = \sup \phi(x)$$

where the sup is taken among all the functions that satisfy the corresponding differential inequality

$$\begin{cases} -\triangle \phi \leq f(x) \,, \; x \in \Omega \\ \phi(x) = 0 \,, \qquad x \in \partial \Omega. \end{cases}$$

Then u is a solution of (7.2).

(b) Now consider the nonlinear problem

$$\begin{cases} -\triangle u = f(u) \,, x \in \Omega \\ u(x) = 0 \,, x \in \partial \Omega. \end{cases} \tag{7.3}$$

Assume that $f(\cdot)$ is a smooth function with $f'(\cdot) \geq 0$. Suppose that there exist two functions $\underline{u}(x) \leq \bar{u}(x)$, such that

$$-\triangle \underline{u} \leq f(\underline{u}) \leq f(\bar{u}) \leq -\triangle \bar{u}.$$

These two functions are called sub (or lower) and super (or upper) solutions respectively.

To seek a solution of problem (7.3), we use successive approximations. Let

$$-\triangle u_1 = f(\underline{u}) \quad \text{and} \quad -\triangle u_{i+1} = f(u_i).$$

Then by maximum principle, we have

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq \cdots \leq \bar{u}.$$

Let u be the limit of the sequence $\{u_i\}$:

$$u(x) = \lim u_i(x),$$

then u is a solution of the problem (7.3).

In Section 7.2 , we introduce and prove the weak maximum principles.

Theorem 7.1.1 *(Weak Maximum Principle for $-\triangle$.)*

i) If

$$-\triangle u(x) \geq 0, \quad x \in \Omega,$$

then

$$\min_{\bar{\Omega}} u \geq \min_{\partial \Omega} u.$$

ii) If

$$-\triangle u(x) \leq 0, \quad x \in \Omega,$$

then

$$\max_{\bar{\Omega}} \; u \leq \max_{\partial \Omega} \; u.$$

This result can be extended to general uniformly elliptic operators. Let

$$D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}.$$

Define

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x).$$

Here we always assume that $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are bounded continuous functions in $\bar{\Omega}$. We say that L is uniformly elliptic if

$$a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \quad \text{for any } x \in \Omega, \quad \text{any } \xi \in R^n \text{ and for some } \delta > 0.$$

Theorem 7.1.2 *(Weak Maximum Principle for L) Let L be the uniformly elliptic operator defined above. Assume that $c(x) \equiv 0$.*

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u.$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega}.$$

These Weak Maximum Principles infer that the minima or maxima of u attain at some points on the boundary $\partial\Omega$. However, they do not exclude the possibility that the minima or maxima may also occur in the interior of Ω. Actually this can not happen unless u is constant, as we will see in the following.

Theorem 7.1.3 *(Strong Maximum Principle for L with $c(x) \equiv 0$) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \equiv 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u attains its minimum value only on $\partial\Omega$ unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u attains its maximum value only on $\partial\Omega$ unless u is constant.

This maximum principle (as well as the weak one) can also be applied to the case when $c(x) \geq 0$ with slight modifications.

Theorem 7.1.4 *(Strong Maximum Principle for L with $c(x) \geq 0$) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \geq 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u can not attain its non-positive minimum in the interior of Ω unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u can not attain its non-negative maximum in the interior of Ω unless u is constant.

We will prove these Theorems in Section 7.3 by using the Hopf Lemma.

Notice that in the previous Theorems, we all require that $c(x) \geq 0$. Roughly speaking, maximum principles hold for 'positive' operators. $-\triangle$ is 'positive', and obviously so does $-\triangle + c(x)$ if $c(x) \geq 0$. However, as we will see in the next chapter, in practical problems it occurs frequently that the condition $c(x) \geq 0$ can not be met. Do we really need $c(x) \geq 0$? The answer is 'no'. Actually, if $c(x)$ is not 'too negative', then the operator '-△+ c(x)' can still remain 'positive' to ensure the maximum principle. These will be studied in Section 7.4, where we prove the 'Maximum Principles Based on Comparisons'.

Let ϕ be a positive function on $\bar{\Omega}$ satisfying

$$-\triangle\phi + \lambda(x)\phi \geq 0. \tag{7.4}$$

Let u be a function such that

$$\begin{cases} -\triangle u + c(x)u \geq 0 & x \in \Omega \\ u \geq 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Theorem 7.1.5 *(Maximum Principle Based on Comparison)*

Assume that Ω is a bounded domain. If

$$c(x) > \lambda(x) \; , \forall x \in \Omega,$$

then $u \geq 0$ in Ω.

Also in Section 7.4, as consequences of Theorem 7.1.5, we derive the 'Narrow Region Principle' and the 'Decay at Infinity Principle'. These principles can be applied very conveniently in the 'Method of Moving Planes' to establish the symmetry of solutions for semi-linear elliptic equations, as we will see in later sections.

In Section 7.5, we establish a maximum principle for integral inequalities.

7.2 Weak Maximum Principles

In this section, we prove the weak maximum principles.

Theorem 7.2.1 *(Weak Maximum Principle for $-\triangle$.)*
i) If

$$-\triangle u(x) \geq 0, \quad x \in \Omega, \tag{7.6}$$

then

$$\min_{\bar{\Omega}} u \geq \min_{\partial \Omega} u. \tag{7.7}$$

ii) If

$$-\triangle u(x) \leq 0, \quad x \in \Omega, \tag{7.8}$$

then

$$\max_{\bar{\Omega}} u \leq \max_{\partial \Omega} u. \tag{7.9}$$

Proof. Here we only present the proof of part i). The entirely similar proof also works for part ii).

To better illustrate the idea, we will deal with one dimensional case and higher dimensional case separately.

First, let Ω be the interval (a, b). Then condition (7.6) becomes $u''(x) \leq 0$. This implies that the graph of $u(x)$ on (a, b) is concave downward, and therefore one can roughly see that the values of $u(x)$ in (a, b) are large or equal to the minimum value of u at the end points (See Figure 2).

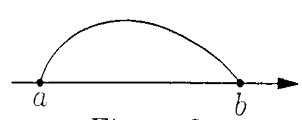


Figure 2

To prove the above observation rigorously, we first carry it out under the stronger assumption that

$$-u''(x) > 0. \tag{7.10}$$

Let $m = \min_{\partial\Omega} u$. Suppose in contrary to (7.7), there is a minimum $x^o \in (a, b)$ of u, such that $u(x^o) < m$. Then by the second order Taylor expansion of u around x^o, we must have $-u''(x^o) \leq 0$. This contradicts with the assumption (7.10).

Now for u only satisfying the weaker condition (7.6), we consider a perturbation of u:

$$u_\epsilon(x) = u(x) - \epsilon x^2.$$

Obviously, for each $\epsilon > 0$, $u_\epsilon(x)$ satisfies the stronger condition (7.10), and hence

$$\min_{\bar{\Omega}} u_\epsilon \geq \min_{\partial\Omega} u_\epsilon .$$

Now letting $\epsilon \to 0$, we arrive at (7.7).

To prove the theorem in dimensions higher than one, we need the following

Lemma 7.2.1 *(Mean Value Inequality) Let x^o be a point in Ω. Let $B_r(x^o) \subset \Omega$ be the ball of radius r center at x^o, and $\partial B_r(x^o)$ be its boundary.*

i) If $-\triangle u(x) > (=)\ 0$ for $x \in B_{r_o}(x^o)$ with some $r_o > 0$, then for any $r_o > r > 0$,

$$u(x^o) > (=) \frac{1}{|\partial B_r(x^o)|} \int_{\partial B_r(x^o)} u(x) dS. \tag{7.11}$$

It follows that, if x^o is a minimum of u in Ω, then

$$-\triangle u(x^o) \leq 0. \tag{7.12}$$

ii) If $-\triangle u(x) < 0$ for $x \in B_{r_o}(x^o)$ with some $r_o > 0$, then for any $r_o > r > 0$,

$$u(x^o) < \frac{1}{|\partial B_r(x^o)|} \int_{\partial B_r(x^o)} u(x) dS. \tag{7.13}$$

It follows that, if x^o is a maximum of u in Ω, then

$$-\triangle u(x^o) \geq 0. \tag{7.14}$$

We postpone the proof of the Lemma for a moment. This Lemma tell us that, if $-\triangle u(x) > 0$, then the value of u at the center of the small ball $B_r(x^o)$ is larger than its average value on the boundary $\partial B_r(x^o)$. Roughly speaking, the graph of u is locally somewhat concave downward. Now based on this Lemma, to prove the theorem, we first consider $u_\epsilon(x) = u(x) - \epsilon |x|^2$. Obviously,

$$-\triangle u_\epsilon = -\triangle u + 2\epsilon n > 0. \tag{7.15}$$

Hence we must have

$$\min_{\Omega} u_\epsilon \geq \min_{\partial\Omega} u_\epsilon \tag{7.16}$$

Otherwise, if there exists a minimum x^o of u in Ω, then by Lemma 7.2.1, we have $-\triangle u_\epsilon(x^o) \leq 0$. This contradicts with (7.15). Now in (7.16), letting $\epsilon \to 0$, we arrive at the desired conclusion (7.7).

This completes the proof of the Theorem.

The Proof of Lemma 7.2.1. By the Divergence Theorem,

$$\int_{B_r(x^o)} \triangle u(x) dx = \int_{\partial B_r(x^o)} \frac{\partial u}{\partial \nu} dS = r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(x^o + r\omega) dS_\omega, \tag{7.17}$$

where dS_ω is the area element of the $n-1$ dimensional unit sphere $S^{n-1} = \{\omega \mid |\omega| = 1\}$.

If $\triangle u < 0$, then by (7.17),

$$\frac{\partial}{\partial r}\{\int_{S^{n-1}} u(x^o + r\omega) dS_\omega\} < 0. \tag{7.18}$$

Integrating both sides of (7.18) from 0 to r yields

$$\int_{S^{n-1}} u(x^o + r\omega) dS_\omega - u(x^o)|S^{n-1}| < 0,$$

where $|S^{n-1}|$ is the area of S^{n-1}. It follows that

$$u(x^o) > \frac{1}{r^{n-1}|S^{n-1}|} \int_{\partial B_r(x^o)} u(x) dS.$$

This verifies (7.11).

To see (7.12), we suppose in contrary that $-\triangle u(x^o) > 0$. Then by the continuity of $\triangle u$, there exists a $\delta > 0$, such that

$$-\triangle u(x) > 0, \ \ \forall x \in B_\delta(x^o).$$

Consequently, (7.11) holds for any $0 < r < \delta$. This contradicts with the assumption that x^o is a minimum of u.

This completes the proof of the Lemma.

From the proof of Theorem 7.2.1, one can see that if we replace $-\triangle$ operator by $-\triangle + c(x)$ with $c(x) \geq 0$, then the conclusion of Theorem 7.2.1 is still true (with slight modifications). Furthermore, we can replace the Laplace operator $-\triangle$ with general uniformly elliptic operators. Let

$$D_i = \frac{\partial}{\partial x_i}, \ \ D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}.$$

Define

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x). \tag{7.19}$$

Here we always assume that $a_{ij}(x)$, $b_i(x)$, and $c(x)$ are bounded continuous functions in $\bar{\Omega}$. We say that L is uniformly elliptic if

$$a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2 \ \ \text{for any } x \in \Omega, \ \ \text{any } \ \xi \in R^n \ \text{ and for some } \delta > 0.$$

Theorem 7.2.2 *(Weak Maximum Principle for L with $c(x) \equiv 0$). Let L be the uniformly elliptic operator defined above. Assume that $c(x) \equiv 0$.*

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u. \tag{7.20}$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u.$$

For $c(x) \geq 0$, the principle still applies with slight modifications.

Theorem 7.2.3 *(Weak Maximum Principle for L with $c(x) \geq 0$). Let L be the uniformly elliptic operator defined above. Assume that $c(x) \geq 0$. Let*

$$u^-(x) = \min\{u(x), 0\} \quad \text{and} \quad u^+(x) = \max\{u(x), 0\}.$$

i) If $Lu \geq 0$ in Ω, then

$$\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u^- .$$

ii) If $Lu \leq 0$ in Ω, then

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+ .$$

Interested readers may find its proof in many standard books, say in [Ev], page 327.

7.3 The Hopf Lemma and Strong Maximum Principles

In the previous section, we prove a weak form of maximum principle. In the case $Lu \geq 0$, it concludes that the minimum of u attains at some point on the boundary $\partial\Omega$. However it does not exclude the possibility that the minimum may also attain at some point in the interior of Ω. In this section, we will show that this can not actually happen, that is, the minimum value of u can only be achieved on the boundary unless u is constant. This is called the "Strong Maximum Principle". We will prove it by using the following

Lemma 7.3.1 (**Hopf Lemma**). *Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Let*

$$L = -\sum_{ij} a_{ij}(x) D_{ij} + \sum_i b_i(x) D_i + c(x)$$

be uniformly elliptic in Ω with $c(x) \equiv 0$. Assume that

$$Lu \geq 0 \quad \text{in } \Omega. \tag{7.21}$$

Suppose there is a ball B contained in Ω with a point $x^o \in \partial\Omega \cap \partial B$ and suppose

$$u(x) > u(x^o), \ \forall x \in B. \tag{7.22}$$

Then for any outward directional derivative at x^o,

$$\frac{\partial u(x^o)}{\partial \nu} < 0. \tag{7.23}$$

In the case $c(x) \geq 0$, if we require additionally that $u(x^o) \leq 0$, then the same conclusion of the Hopf Lemma holds.

Proof. Without loss of generality, we may assume that B is centered at the origin with radius r. Define

$$w(x) = e^{-\alpha r^2} - e^{-\alpha |x|^2}.$$

Consider $v(x) = u(x) + \epsilon w(x)$ on the set $D = B_{\frac{r}{2}}(x^o) \cap B$ (See Figure 3).

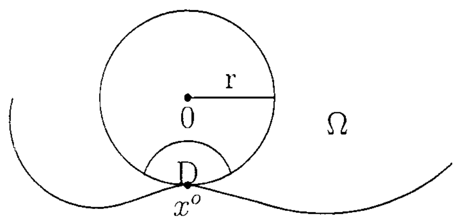


Figure 3

We will choose α and ϵ appropriately so that we can apply the Weak Maximum Principle to $v(x)$ and arrive at

$$v(x) \geq v(x^o) \quad \forall x \in D. \tag{7.24}$$

We postpone the proof of (7.24) for a moment. Now from (7.24), we have

$$\frac{\partial v}{\partial \nu}(x^o) \leq 0, \tag{7.25}$$

Noticing that

$$\frac{\partial w}{\partial \nu}(x^o) > 0$$

We arrive at the desired inequality

$$\frac{\partial u}{\partial \nu}(x^o) < 0.$$

Now to complete the proof of the Lemma, what left to verify is (7.24). We will carry this out in two steps. First we show that

$$Lv \geq 0. \tag{7.26}$$

Hence we can apply the Weak Maximum Principle to conclude that

$$\min_{\bar{D}} v \geq \min_{\partial D} . \tag{7.27}$$

Then we show that the minimum of v on the boundary ∂D is actually attained at x^o:

$$v(x) \geq v(x^o) \ \ \forall x \in \partial D. \tag{7.28}$$

Obviously, (7.27) and (7.28) imply (7.24).

To see (7.26), we directly calculate

$$\begin{aligned} Lw &= e^{-\alpha|x|^2}\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n}[a_{ii}(x) - b_i(x)x_i] - c(x)\} + c(x)e^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n}[a_{ii}(x) - b_i(x)x_i] - c(x)\} \end{aligned} \tag{7.29}$$

By the ellipticity assumption, we have

$$\sum_{i,j=1}^{n} a_{ij}(x)x_i x_j \geq \delta|x|^2 \geq \delta(\frac{r}{2})^2 > 0 \quad \text{in } D. \tag{7.30}$$

Hence we can choose α sufficiently large, such that $Lw \geq 0$. This, together with the assumption $Lu \geq 0$ implies $Lv \geq 0$, and (7.27) follows from the Weak Maximum Principle.

To verify (7.28), we consider two parts of the boundary ∂D separately.

(i) On $\partial D \cap B$, since $u(x) > u(x^o)$, there exists a $c_o > 0$, such that $u(x) \geq u(x^o) + c_o$. Take ϵ small enough such that $\epsilon|w| \leq \delta$ on $\partial D \cap B$. Hence

$$v(x) \geq u(x^o) = v(x^o) \ \ \forall x \in \partial D \cap B.$$

(ii) On $\partial D \cap \partial B$, $w(x) = 0$, and by the assumption $u(x) \geq u(x^o)$, we have $v(x) \geq v(x^o)$.

This completes the proof of the Lemma.

Now we are ready to prove

Theorem 7.3.1 *(Strong Maximum Principle for L with $c(x) \equiv 0$.) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \equiv 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u attains its minimum only on $\partial\Omega$ unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u attains its maximum only on $\partial\Omega$ unless u is constant.

Proof. We prove part i) here. The proof of part ii) is similar. Let m be the minimum value of u in Ω. Set $\Sigma = \{x \in \Omega \mid u(x) = m\}$. It is relatively closed in Ω. We show that either Σ is empty or $\Sigma = \Omega$.

We argue by contradiction. Suppose Σ is a nonempty proper subset of Ω. Then we can find an open ball $B \subset \Omega \setminus \Sigma$ with a point on its boundary belonging to Σ. Actually, we can first find a point $p \in \Omega \setminus \Sigma$ such that $d(p, \Sigma) < d(p, \partial\Omega)$, then increase the radius of a small ball center at p until it hits Σ (before hitting $\partial\Omega$). Let x^o be the point at $\partial B \cap \Sigma$. Obviously we have in B

$$Lu \geq 0 \quad \text{and } u(x) > u(x^o).$$

Now we can apply the Hopf Lemma to conclude that the normal outward derivative

$$\frac{\partial u}{\partial \nu}(x^o) < 0. \tag{7.31}$$

On the other hand, x^o is an interior minimum of u in Ω, and we must have $Du(x^o) = 0$. This contradicts with (7.31) and hence completes the proof of the Theorem.

In the case when $c(x) \geq 0$, the strong principle still applies with slight modifications.

Theorem 7.3.2 *(Strong Maximum Principle for L with $c(x) \geq 0$.) Assume that Ω is an open, bounded, and connected domain in R^n with smooth boundary $\partial\Omega$. Let u be a function in $C^2(\Omega) \cap C(\bar{\Omega})$. Assume that $c(x) \geq 0$ in Ω.*

i) If

$$Lu(x) \geq 0, \quad x \in \Omega,$$

then u can not attain its non-positive minimum in the interior of Ω unless u is constant.

ii) If

$$Lu(x) \leq 0, \quad x \in \Omega,$$

then u can not attain its non-negative maximum in the interior of Ω unless u is constant.

Remark 7.3.1 *In order that the maximum principle to hold, we assume that the domain Ω be bounded. This is essential, since it guarantees the existence of maximum and minimum of u in $\bar{\Omega}$. A simple counter example is when Ω is the half space $\{x \in R^n \mid x_n > 0\}$, and $u(x, y) = x_n$. Obviously, $\triangle u = 0$, but u does not obey the maximum principle:*

$$\max_{\Omega} u \leq \max_{\partial\Omega} u.$$

Equally important is the non-negativeness of the coefficient $c(x)$. For example, set $\Omega = \{(x, y) \in R^2 \mid -\frac{\pi}{2} < x < \frac{\pi}{2},\ -\frac{\pi}{2} < y < \frac{\pi}{2}\}$. Then $u = \cos x \cos y$ satisfies

$$\begin{cases} -\triangle u - 2u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

But, obviously, there are some points in Ω at which $u < 0$.

However, if we impose some sign restriction on u, say $u \geq 0$, then both conditions can be relaxed. A simple version of such result will be present in the next theorem.

Also, as one will see in the next section, $c(x)$ is actually allowed to be negative, but not 'too negative'.

Theorem 7.3.3 *(Maximum Principle and Hopf Lemma for not necessarily bounded domain and not necessarily non-negative $c(x)$.)*

Let Ω be a domain in R^n with smooth boundary. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and satisfies

$$\begin{cases} -\triangle u + \sum_{i=1}^n b_i(x) D_i u + c(x) u \geq 0, \ u(x) \geq 0, & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{7.32}$$

with bounded functions $b_i(x)$ and $c(x)$. Then

i) if u vanishes at some point in Ω, then $u \equiv 0$ in Ω; and

ii) if $u \not\equiv 0$ in Ω, then on $\partial\Omega$, the exterior normal derivative $\dfrac{\partial u}{\partial \nu} < 0$.

To prove the Theorem, we need the following Lemma concerning eigenvalues.

Lemma 7.3.2 *Let λ_1 be the first positive eigenvalue of*

$$\begin{cases} -\triangle \phi = \lambda_1 \phi(x) & x \in B_1(0) \\ \phi(x) = 0 & x \in \partial B_1(0) \end{cases} \tag{7.33}$$

with the corresponding eigenfunction $\phi(x) > 0$. Then for any $\rho > 0$, the first positive eigenvalue of the problem on $B_\rho(0)$ is $\dfrac{\lambda_1}{\rho^2}$. More precisely, if we let $\psi(x) = \phi(\frac{x}{\rho})$, then

$$\begin{cases} -\triangle \psi = \dfrac{\lambda_1}{\rho^2} \psi(x) & x \in B_\rho(0) \\ \psi(x) = 0 & x \in \partial B_\rho(0) \end{cases} \tag{7.34}$$

The proof is straight forward, and will be left for the readers.

The Proof of Theorem 7.3.3.

i) Suppose that $u = 0$ at some point in Ω, but $u \not\equiv 0$ on Ω. Let

$$\Omega_+ = \{x \in \Omega \mid u(x) > 0\}.$$

Then by the regularity assumption on u, Ω_+ is an open set with C^2 boundary. Obviously,

$$u(x) = 0 \,, \quad \forall \, x \in \partial\Omega_+.$$

Let x^o be a point on $\partial\Omega_+$, but not on $\partial\Omega$. Then for $\rho > 0$ sufficiently small, one can choose a ball $B_{\rho/2}(\bar{x}) \subset \Omega_+$ with x^o as its boundary point. Let ψ be the positive eigenfunction of the eigenvalue problem (7.34) on $B_\rho(x^o)$ corresponding to the eigenvalue $\frac{\lambda_1}{\rho^2}$. Obviously, $B_\rho(x^o)$ completely covers $B_{\rho/2}(\bar{x})$. Let $v = \frac{u}{\psi}$. Then from (7.32), it is easy to deduce that

$$\begin{aligned} 0 &\leq -\triangle v - 2\bigtriangledown v \cdot \frac{\bigtriangledown \psi}{\psi} + \sum_{i=1}^{n} b_i(x) D_i v + \left(\frac{-\triangle \psi}{\psi} + \sum_{i=1}^{n} \frac{D_i \psi}{\psi} + c(x) \right) v \\ &\equiv -\triangle v + \sum_{i=1}^{n} \tilde{b}_i(x) D_i v + \tilde{c}(x) v. \end{aligned}$$

Let ϕ be the positive eigenfunction of the eigenvalue problem (7.33) on B_1, then

$$\tilde{c}(x) = \frac{\lambda_1}{\rho^2} + \frac{1}{\rho} \sum_{i=1}^{n} \frac{D_i \phi}{\phi} + c(x).$$

This allows us to choose ρ sufficiently small so that $\tilde{c}(x) \geq 0$. Now we can apply *Hopf Lemma* to conclude that, the outward normal derivative at the boundary point x^o of $B_{\rho/2}(\bar{x})$,

$$\frac{\partial v}{\partial \nu}(x^o) < 0, \tag{7.35}$$

because

$$v(x) > 0 \; \forall x \in B_{\rho/2}(\bar{x}) \;\text{ and }\; v(x^o) = \frac{u(x^o)}{\psi(x^o)} = 0.$$

On the other hand, since x^o is also a minimum of v in the interior of Ω, we must have

$$\bigtriangledown v(x^o) = 0.$$

This contradicts with (7.35) and hence proves part i) of the Theorem.

ii) The proof goes almost the same as in part i) except we consider the point x^o on $\partial\Omega$ and the ball $B_{\rho/2}(\bar{x})$ is in Ω with $x^o \in \partial B_{\rho/2}(\bar{x})$. Then for the outward normal derivative of u, we have

$$\frac{\partial u}{\partial \nu}(x^o) = \frac{\partial v}{\partial \nu}(x^o)\psi(x^o) + v(x^o)\frac{\partial \psi}{\partial \nu}(x^o) = \frac{\partial v}{\partial \nu}(x^o)\psi(x^o) < 0.$$

Here we have used a well-known fact that the eigenfunction ψ on $B_\rho(x^o)$ is radially symmetric about the center x^o, and hence $\bigtriangledown \psi(x^o) = 0$. This completes the proof of the Theorem.

7.4 Maximum Principles Based on Comparisons

In the previous section, we show that if $(-\triangle + c(x))u \geq 0$, then the maximum principle, i.e. (7.20), applies. There, we required $c(x) \geq 0$. We can think $-\triangle$

as a 'positive' operator, and the maximum principle holds for any 'positive' operators. For $c(x) \geq 0$, $-\triangle + c(x)$ is also 'positive'. Do we really need $c(x) \geq 0$ here? To answer the question, let us consider the Dirichlet eigenvalue problem of $-\triangle$:

$$\begin{cases} -\triangle\phi - \lambda\phi(x) = 0 & x \in \Omega \\ \phi(x) = 0 & x \in \partial\Omega. \end{cases} \tag{7.36}$$

We notice that the eigenfunction ϕ corresponding to the first positive eigenvalue λ_1 is either positive or negative in Ω. That is, the solutions of (7.36) with $\lambda = \lambda_1$ obey Maximum Principle, that is, the maxima or minima of ϕ are attained only on the boundary $\partial\Omega$. This suggests that, to ensure the Maximum Principle, $c(x)$ need not be nonnegative, it is allowed to be as negative as $-\lambda_1$. More precisely, we can establish the following more general maximum principle based on comparison.

Theorem 7.4.1 *Assume that Ω is a bounded domain. Let ϕ be a positive function on $\bar{\Omega}$ satisfying*

$$-\triangle\phi + \lambda(x)\phi \geq 0. \tag{7.37}$$

Assume that u is a solution of

$$\begin{cases} -\triangle u + c(x)u \geq 0 & x \in \Omega \\ u \geq 0 & on\ \partial\Omega. \end{cases} \tag{7.38}$$

If

$$c(x) > \lambda(x)\ , \quad \forall x \in \Omega, \tag{7.39}$$

then $u \geq 0$ in Ω.

Proof. We argue by contradiction. Suppose that $u(x) < 0$ somewhere in Ω. Let $v(x) = \dfrac{u(x)}{\phi(x)}$. Then since $\phi(x) > 0$, we must have $v(x) < 0$ somewhere in Ω. Let $x^o \in \Omega$ be a minimum of $v(x)$. By a direct calculation, it is easy to verify that

$$-\triangle v = 2\bigtriangledown v \cdot \frac{\bigtriangledown\phi}{\phi} + \frac{1}{\phi}(-\triangle u + \frac{\triangle\phi}{\phi}u). \tag{7.40}$$

On one hand, since x^o is a minimum, we have

$$-\triangle v(x^o) \leq 0 \quad \text{and } \bigtriangledown v(x^o) = 0. \tag{7.41}$$

While on the other hand, by (7.37), (7.38), and (7.39), and taking into account that $u(x^o) < 0$, we have, at point x^o,

$$\begin{aligned} -\triangle u + \frac{\triangle\phi}{\phi}u(x^o) &\geq -\triangle u + \lambda(x^o)u(x^o) \\ &> -\triangle u + c(x^o)u(x^o) \geq 0. \end{aligned}$$

This is an obvious contradiction with (7.40) and (7.41), and thus completes the proof of the Theorem.

Remark 7.4.1 *From the proof, one can see that conditions (7.37) and (7.39) are required only at the points where v attains its minimum, or at points where u is negative.*

The Theorem is also valid on an unbounded domains if u is "nonnegative" at infinity:

Theorem 7.4.2 *If Ω is an unbounded domain, besides condition (7.38), we assume further that*

$$\liminf_{|x|\longrightarrow\infty} u(x) \geq 0. \tag{7.42}$$

Then $u \geq 0$ in Ω.

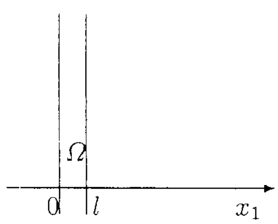
Proof. Still consider the same $v(x)$ as in the proof of Theorem 7.4.1. Now condition (7.42) guarantees that the minima of $v(x)$ do not "leak" away to infinity. Then the rest of the arguments are exactly the same as in the proof of Theorem 7.4.1.

For convenience in applications, we provide two typical situations where there exist such functions ϕ and $c(x)$ satisfying condition (7.37) and (7.39), so that the Maximum Principle Based on Comparison applies:
i) Narrow regions, and
ii) $c(x)$ decays fast enough at ∞.

i) Narrow Regions. When

$$\Omega = \{x \mid 0 < x_1 < l\}$$

is a narrow region with width l as shown:



We can choose $\phi(x) = \sin(\frac{x_1+\epsilon}{l})$. Then it is easy to see that $-\Delta\phi = (\frac{1}{l})^2\phi$, where $\lambda(x) = \dfrac{-1}{l^2}$ can be very negative when l is sufficiently small.

Corollary 7.4.1 (Narrow Region Principle.) *If u satisfies (7.38) with bounded function $c(x)$. Then when the width l of the region Ω is sufficiently small, $c(x)$ satisfies (7.39), i.e. $c(x) > \lambda(x) = \dfrac{-1}{l^2}$. Hence we can directly apply Theorem 7.4.1 to conclude that $u \geq 0$ in Ω, provided* $\liminf_{|x|\longrightarrow\infty} u(x) \geq 0$.

ii) Decay at Infinity. In dimension $n \geq 3$, one can choose some positive number $q < n-2$, and let $\phi(x) = \frac{1}{|x|^q}$ Then it is easy to verify that

$$-\triangle \phi = \frac{q(n-2-q)}{|x|^2}\phi.$$

In the case $c(x)$ decays fast enough near infinity, we can adapt the proof of Theorem 7.4.1 to derive

Corollary 7.4.2 (Decay at Infinity) *Assume there exist $R > 0$, such that*

$$c(x) > -\frac{q(n-2-q)}{|x|^2}, \quad \forall |x| > R. \tag{7.43}$$

Suppose

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|^q} = 0.$$

Let Ω be a region containing in $B_R^C(0) \equiv R^n \setminus B_R(0)$. If u satisfies (7.38) on $\bar{\Omega}$, then

$$u(x) \geq 0 \quad \textit{for all } x \in \Omega.$$

Remark 7.4.2 *From Remark 7.4.1, one can see that actually condition (7.43) is only required at points where u is negative.*

Remark 7.4.3 *Although Theorem 7.4.1 as well as its Corollaries are stated in linear forms, they can be easily applied to a nonlinear equation, for example,*

$$-\triangle u - |u|^{p-1}u = 0 \quad x \in R^n. \tag{7.44}$$

Assume that the solution u decays near infinity at the rate of $\frac{1}{|x|^s}$ with $s(p-1) > 2$. Let $c(x) = -|u(x)|^{p-1}$. Then for R sufficiently large, and for the region Ω as stated in Corollary 7.4.2, $c(x)$ satisfies (7.43) in Ω. If further assume that

$$u\mid_{\partial\Omega} \geq 0,$$

then we can derive from Corollary 7.4.2 that $u \geq 0$ in the entire region Ω.

7.5 A Maximum Principle for Integral Equations

In this section, we introduce a maximum principle for integral equations.

Let Ω be a region in R^n, may or may not be bounded. Assume

$$K(x,y) \geq 0, \quad \forall (x,y) \in \Omega \times \Omega.$$

Define the integral operator T by

$$(Tf)(x) = \int_\Omega K(x,y) f(y) dy.$$

or more generally, the integral inequality

$$f \leq Tf \quad \text{in } \Omega. \tag{7.50}$$

Further assume that

$$\|Tf\|_{L^p(\Omega)} \leq C\|c(y)\|_{L^\tau(\Omega)}\|f\|_{L^p(\Omega)}. \tag{7.51}$$

for some $p, \tau > 1$.

If we have some right integrability condition on $c(y)$, then we can derived, from Theorem 7.5.1, a maximum principle that will be applied to "*Narrow Regions*" and "*Near Infinity*". More precisely, we have

Corollary 7.5.1 *Assume that $c(y) \geq 0$ and $c(y) \in L^\tau(R^n)$. Let $f \in L^p(R^n)$ be a nonnegative function satisfying (7.50) and (7.51). Then there exist positive numbers R_o and ϵ_o depending on $c(y)$ only, such that*

$$\text{if } \mu(\Omega \cap B_{R_o}(0)) \leq \epsilon_o, \quad \text{then } f^+ \equiv 0 \quad \text{in } \Omega.$$

where $\mu(D)$ is the measure of the set D.

Proof. Since $c(y) \in L^\tau(R^n)$, by Lebesgue integral theory, when the measure of the intersection of Ω with $B_{R_o}(0)$ is sufficiently small, we can make the integral $\int_\Omega |c(y)|^\tau dy$ as small as we wish, and thus to obtain

$$C\|c(y)\|_{L^\tau(\Omega)} < 1.$$

Now it follows from Theorem 7.5.1 that

$$f^+(x) \equiv 0 \ , \forall x \in \Omega.$$

This completes the proof of the Corollary.

Remark 7.5.1 *One can see that the condition $\mu(\Omega \cap B_{R_o}(0)) \leq \epsilon_o$ in the Corollary is satisfied in the following two situations.*

i) Narrow Regions*: The width of Ω is very small.*

ii) Near Infinity*: Say, $\Omega = B_R^C(0)$, the complement of the ball $B_R(0)$, with sufficiently large R.*

As an immediate application of this "*Maximum Principle*", we study an integral equation in the next section. We will use the method of moving planes to obtain the radial symmetry and monotonicity of the positive solutions.

8

Methods of Moving Planes and of Moving Spheres

The Method of Moving Planes (MMP) was invented by the Soviet mathematician Alexanderoff in the early 1950s. Decades later, it was further developed by Serrin [Se], Gidas, Ni, and Nirenberg [GNN], Caffarelli, Gidas, and Spruck [CGS], Li [Li], Chen and Li [CL] [CL1], Chang and Yang [CY], and many others. This method has been applied to free boundary problems, semi-linear partial differential equations, and other problems. Particularly for semi-linear partial differential equations, there have seen many significant contributions. We refer to the paper of Frenkel [F] for more descriptions on the method.

The Method of Moving Planes and its variant–the Method of Moving Spheres–have become powerful tools in establishing symmetries and monotonicity for solutions of partial differential equations. They can also be used to obtain a priori estimates, to derive useful inequalities, and to prove non-existence of solutions.

From the previous chapter, we have seen the beauty and power of maximum principles. The MMP greatly enhances the power of maximum principles. Roughly speaking, the MMP is a continuous way of repeated applications of maximum principles. During this process, the maximum principle has been used infinitely many times; and the advantage is that each time we only need to use the maximum principle in a very narrow region. From the previous chapter, one can see that, in such a narrow region, even if the coefficients of the equation are not 'good,' the maximum principle can still be applied. In the authors' research practice, we also introduced a form of maximum principle at infinity to the MMP and therefore simplified many proofs and extended the results in more natural ways. We recommend the readers study this part carefully, so that they will be able to apply it to their own research.

It is well-known that by using a Green's function, one can change a differential equation into an integral equation, and under certain conditions, they are equivalent. To investigate the symmetry and monotonicity of integral equations, the authors, together with Ou, created an integral form of MMP. Instead of using local properties (say differentiability) of a differential equation, they employed the global properties of the solutions of integral equations.

In this chapter, we will apply the Method of Moving Planes and their variant–the Method of Moving Spheres– to study semi-linear elliptic equations and integral equations. We will establish symmetry, monotonicity, a priori estimates, and non-existence of the solutions. During the process of Moving Planes, the Maximum Principles introduced in the previous chapter are applied in innovative ways.

In Section 8.2, we will establish radial symmetry and monotonicity for the solutions of the following three semi-linear elliptic problems

$$\begin{cases} -\triangle u = f(u) & x \in B_1(0) \\ u = 0 & \text{on } \partial B_1(0); \end{cases}$$

$$-\triangle u = u^{\frac{n+2}{n-2}}(x) \quad x \in R^n \quad n \geq 3;$$

and

$$-\triangle u = e^{u(x)} \quad x \in R^2.$$

During the moving of planes, the Maximum Principles Base on Comparison will play a major role. In particular, the Narrow Region Principle and the Decay at Infinity Principle will be used repeatedly in dealing with the three examples.

In Section 8.3, we will apply the Method of Moving Planes in a 'local way' to obtain a priori estimates on the solutions of the prescribing scalar curvature equation on a compact Riemannian manifold M

$$-\frac{4(n-1)}{n-2}\triangle_o u + R_o(x)u = R(x)u^{\frac{n+2}{n-2}}, \quad \text{in } M.$$

We alow the function $R(x)$ to change signs. In this situation, the traditional blowing-up analysis fails near the set where $R(x) = 0$. We will use the Method of Moving Planes in an innovative way to obtain a priori estimates. Since the Method of Moving Planes can not be applied to the solution u directly, we introduce an auxiliary function to circumvent this difficulty.

In Section 8.4, we use the Method of Moving Spheres to prove a non-existence of solutions for the prescribing Gaussian and scalar curvature equations

$$-\triangle u + 2 = R(x)e^u,$$

and

$$-\triangle u + \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}}$$

on S^2 and on S^n $(n \geq 3)$, respectively. We prove that if the function $R(x)$ is rotationally symmetric and monotone in the region where it is positive, then both equations admit no solution. This provides a stronger necessary condition than the well known Kazdan-Warner condition, and it also becomes a sufficient condition for the existence of solutions in most cases.

In Section 8.5, as an application of the maximum principle for integral equations introduced in Section 7.5, we study the integral equation in R^n

$$u(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} u^{\frac{n+\alpha}{n-\alpha}}(y)dy,$$

for any real number α between 0 and n. It arises as an Euler-Lagrange equation for a functional in the context of the Hardy-Littlewood-Sobolev inequalities. Due to the different nature of the integral equation, the traditional Method of Moving Planes does not work. Hence we exploit its global property and develop a new idea–the Integral Form of the Method of Moving Planes to obtain the symmetry and monotonicity of the solutions. The Maximum Principle for Integral Equations established in Chapter 7 is combined with the estimates of various integral norms to carry on the moving of planes.

8.1 Outline of the Method of Moving Planes

To outline how the Method of Moving Planes works, we take the Euclidian space R^n for an example. Let u be a positive solution of a certain partial differential equation. If we want to prove that it is symmetric and monotone in a given direction, we may assign that direction as x_1 axis. For any real number λ, let

$$T_\lambda = \{x = (x_1, x_2, \cdots, x_n) \in R^n \mid x_1 = \lambda\}.$$

This is a plane perpendicular to x_1-axis and the plane that we will move with. Let Σ_λ denote the region to the left of the plane, i.e.

$$\Sigma_\lambda = \{x \in R^n \mid x_1 < \lambda\}.$$

Let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n),$$

the reflection of the point $x = (x_1, \cdots, x_n)$ about the plane T_λ (See Figure 1).

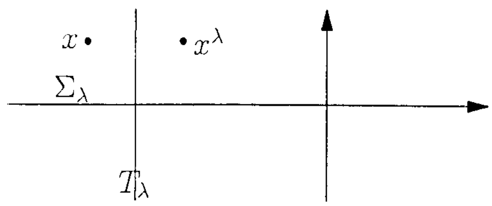


Figure 1

We compare the values of the solution u at point x and x^λ, and we want to show that u is symmetric about some plane T_{λ_o}. To this end, let

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

In order to show that, there exists some λ_o, such that

$$w_{\lambda_o}(x) \equiv 0 \ , \ \forall x \in \Sigma_{\lambda_o},$$

we generally go through the following two steps.

Step 1. We first show that for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0 \ , \ \forall x \in \Sigma_\lambda. \tag{8.1}$$

Then we are able to start off from this neighborhood of $x_1 = -\infty$, and move the plane T_λ along the x_1 direction to the right as long as the inequality (8.1) holds.

Step 2. We continuously move the plane this way up to its limiting position. More precisely, we define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \forall x \in \Sigma_\lambda\}.$$

We prove that u is symmetric about the plane T_{λ_o}, that is $w_{\lambda_o}(x) \equiv 0$ for all $x \in \Sigma_{\lambda_o}$. This is usually carried out by a contradiction argument. We show that if $w_{\lambda_o}(x) \not\equiv 0$, then there would exist $\lambda > \lambda_o$, such that (8.1) holds, and this contradicts with the definition of λ_o.

From the above illustration, one can see that the key to the Method of Moving Planes is to establish inequality (8.1), and for partial differential equations, maximum principles are powerful tools for this task. While for integral equations, we use a different idea. We estimate a certain norm of w_λ on the set

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$$

where the inequality (8.1) is violated. We show that this norm must be zero, and hence Σ_λ^- is empty.

8.2 Applications of the Maximum Principles Based on Comparisons

In this section, we study some semi-linear elliptic equations. We will apply the method of moving planes to establish the symmetry of the solutions. The essence of the method of moving planes is the application of various maximum principles. In the proof of each theorem, the readers will see vividly how the Maximum Principles Based on Comparisons are applied to *narrow regions* and to *solutions with decay at infinity*.

8.2.1 Symmetry of Solutions in a Unit Ball

We first begin with an elegant result of Gidas, Ni, and Nirenberg [GNN1]:

Theorem 8.2.1 *Assume that $f(\cdot)$ is a Lipschitz continuous function such that*

$$|f(p) - f(q)| \leq C_o|p-q| \tag{8.2}$$

for some constant C_o. Then every positive solution u of

$$\begin{cases} -\triangle u = f(u) & x \in B_1(0) \\ u = 0 & on\ \partial B_1(0). \end{cases} \tag{8.3}$$

is radially symmetric and monotone decreasing about the origin.

Proof.

As shown on Figure 4 below, let $T_\lambda = \{x \mid x_1 = \lambda\}$ be the plane perpendicular to the x_1 axis. Let Σ_λ be the part of $B_1(0)$ which is on the left of the plane T_λ. For each $x \in \Sigma_\lambda$, let x^λ be the reflection of the point x about the plane T_λ, more precisely, $x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$.

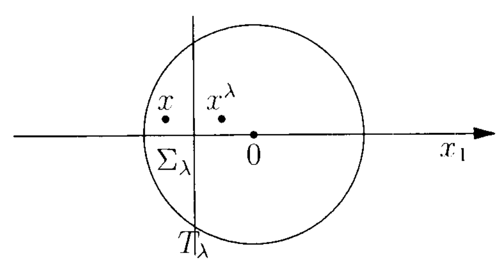


Figure 4

We compare the values of the solution u on Σ_λ with those on its reflection. Let

$$u_\lambda(x) = u(x^\lambda), \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x).$$

Then it is easy to see that u_λ satisfies the same equation as u does. Applying the Mean Value Theorem to $f(u)$, one can verify that w_λ satisfies

$$\triangle w_\lambda + C(x,\lambda) w_\lambda(x) = 0, \quad x \in \Sigma_\lambda,$$

where

$$C(x,\lambda) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)},$$

and by condition (8.2),

$$|C(x,\lambda)| \leq C_o. \tag{8.4}$$

Step 1: Start Moving the Plane.

We start from the near left end of the region. Obviously, for λ sufficiently close to -1, Σ_λ is a narrow (in x_1 direction) region, and on $\partial\Sigma_\lambda$, $w_\lambda(x) \geq 0$. (On T_λ, $w_\lambda(x) = 0$; while on the curve part of $\partial\Sigma_\lambda$, $w_\lambda(x) > 0$ since $u > 0$ in $B_1(0)$.)

Now we can apply the "*Narrow Region Principle*" (Corollary 7.4.1) to conclude that, for λ close to -1,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda. \tag{8.5}$$

This provides a starting point for us to move the plane T_λ

Step 2: Move the Plane to Its Right Limit.

We now increase the value of λ continuously, that is, we move the plane T_λ to the right as long as the inequality (8.5) holds. We show that, by moving this way, the plane will not stop before hitting the origin. More precisely, let

$$\bar{\lambda} = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\},$$

we first claim that

$$\bar{\lambda} \geq 0. \tag{8.6}$$

Otherwise, we will show that the plane can be further moved to the right by a small distance, and this would contradicts with the definition of $\bar{\lambda}$. In fact, if $\bar{\lambda} < 0$, then the image of the curved surface part of $\partial\Sigma_{\bar{\lambda}}$ under the reflection about $T_{\bar{\lambda}}$ lies inside $B_1(0)$, where $u(x) > 0$ by assumption. It follows that, on this part of $\partial\Sigma_{\bar{\lambda}}$, $w_{\bar{\lambda}}(x) > 0$. By the Strong Maximum Principle, we deduce that

$$w_{\bar{\lambda}}(x) > 0$$

in the interior of $\Sigma_{\bar{\lambda}}$.

Let d_o be the maximum width of narrow regions that we can apply the "*Narrow Region Principle*". Choose a small positive number δ, such that $\delta \leq \frac{d_o}{2}, -\bar{\lambda}$. We consider the function $w_{\bar{\lambda}+\delta}(x)$ on the narrow region (See Figure 5):

$$\Omega_\delta = \Sigma_{\bar{\lambda}+\delta} \cap \{x \mid x_1 > \bar{\lambda} - \frac{d_o}{2}\}.$$

It satisfies

$$\begin{cases} \triangle w_{\bar\lambda+\delta} + C(x, \bar\lambda + \delta) w_{\bar\lambda+\delta} = 0 & x \in \Omega_\delta \\ w_{\bar\lambda+\delta}(x) \geq 0 & x \in \partial\Omega_\delta. \end{cases} \tag{8.7}$$

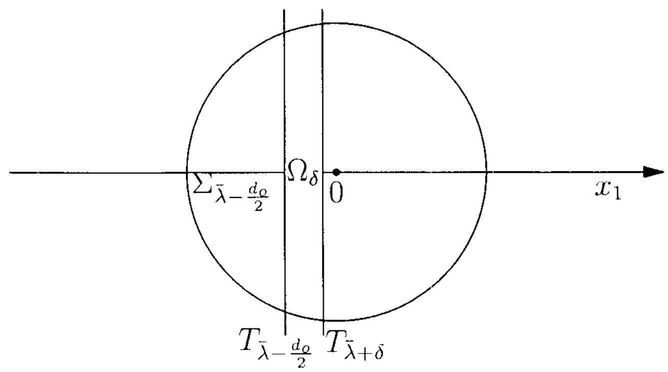


Figure 5

The equation is obvious. To see the boundary condition, we first notice that it is satisfied on the two curved parts and one flat part where $x_1 = \bar\lambda + \delta$ of the boundary $\partial\Omega_\delta$ due to the definition of $w_{\bar\lambda+\delta}$. To see that it is also true on the rest of the boundary where $x_1 = \bar\lambda - \frac{d_o}{2}$, we use continuity argument. Notice that on this part, $w_{\bar\lambda}$ is positive and bounded away from 0. More precisely and more generally, there exists a constant $c_o > 0$, such that

$$w_{\bar\lambda}(x) \geq c_o \ , \quad \forall x \in \Sigma_{\bar\lambda-\frac{d_o}{2}} .$$

Since w_λ is continuous in λ, for δ sufficiently small, we still have

$$w_{\bar\lambda+\delta}(x) \geq 0 \ , \quad \forall x \in \Sigma_{\bar\lambda-\frac{d_o}{2}} .$$

Hence in particular, the boundary condition in (8.7) holds for such small δ. Now we can apply the "*Narrow Region Principle*" to conclude that

$$w_{\bar\lambda+\delta}(x) \geq 0, \quad \forall x \in \Omega_\delta.$$

And therefore,

$$w_{\bar\lambda+\delta}(x) \geq 0, \quad \forall x \in \Sigma_{\bar\lambda+\delta}.$$

This contradicts with the definition of $\bar\lambda$ and thus establishes (8.6).

(8.6) implies that

$$u(-x_1, x') \leq u(x_1, x') \ , \forall x_1 \geq 0, \tag{8.8}$$

where $x' = (x_2, \cdots, x_n)$.

We then start from λ close to 1 and move the plane T_λ toward the left. Similarly, we obtain

$$u(-x_1, x') \geq u(x_1, x') \ , \forall x_1 \geq 0, \tag{8.9}$$

Combining two opposite inequalities (8.8) and (8.9), we see that $u(x)$ is symmetric about the plane T_0. Since we can place x_1 axis in any direction, we conclude that $u(x)$ must be radially symmetric about the origin. Also the monotonicity easily follows from the argument. This completes the proof.

8.2.2 Symmetry of Solutions of $-\triangle u = u^p$ in R^n

In an elegant paper of Gidas, Ni, and Nirenberg [2], an interesting results is the symmetry of the positive solutions of the semi-linear elliptic equation:

$$\triangle u + u^p = 0 \ , \quad x \in R^n, n \geq 3. \tag{8.10}$$

They proved

Theorem 8.2.2 *For $p = \frac{n+2}{n-2}$, all the positive solutions of (8.10) with reasonable behavior at infinity, namely*

$$u = O(\frac{1}{|x|^{n-2}}),$$

are radially symmetric and monotone decreasing about some point, and hence assume the form

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^o|^2)^{\frac{n-2}{2}}} \quad \text{for } \lambda > 0 \text{ and for some } x^o \in R^n.$$

This uniqueness result, as was pointed out by R. Schoen, is in fact equivalent to the geometric result due to Obata [O]: A Riemannian metric on S^n which is conformal to the standard one and having the same constant scalar curvature is the pull back of the standard one under a conformal map of S^n to itself. Recently, Caffarelli, Gidas and Spruck [CGS] removed the decay assumption $u = O(|x|^{2-n})$ and proved the same result. In the case that $1 \leq p < \frac{n+2}{n-2}$, Gidas and Spruck [GS] showed that the only non-negative solution of (8.10) is identically zero. Then, in the authors paper [CL1], a simpler and more elementary proof was given for almost the same result:

Theorem 8.2.3 i) *For $p = \frac{n+2}{n-2}$, every positive C^2 solution of (8.10) must be radially symmetric and monotone decreasing about some point, and hence assumes the form*

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x - x^o|^2)^{\frac{n-2}{2}}} \quad \text{for some } \lambda > 0 \text{ and } x^o \in R^n.$$

ii) *For $p < \frac{n+2}{n-2}$, the only nonnegative solution of (8.10) is identically zero.*

The proof of Theorem 8.2.2 is actually included in the more general proof of the first part of Theorem 8.2.3. However, to better illustrate the idea, we will first present the proof of Theorem 8.2.2 (mostly in our own idea). And the readers will see vividly, how the "*Decay at Infinity*" principle is applied here.

Proof of Theorem 8.2.2.
Define

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) \in R^n \mid x_1 < \lambda\}, \; T_\lambda = \partial \Sigma_\lambda$$

and let x^λ be the reflection point of x about the plane T_λ, i.e.

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n).$$

(See the previous Figure 1.)
Let

$$u_\lambda(x) = u(x^\lambda), \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x).$$

The proof consists of three steps. In the first step, we start from the very left end of our region R^n, that is near $x_1 = -\infty$. We will show that, for λ sufficiently negative,

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda. \tag{8.11}$$

Here, the "*Decay at Infinity*" is applied.

Then in the second step, we will move our plane T_λ in the the x_1 direction toward the right as long as inequality (8.11) holds. The plane will stop at some limiting position, say at $\lambda = \lambda_o$. We will show that

$$w_{\lambda_o}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_o}.$$

This implies that the solution u is symmetric and monotone decreasing about the plane T_{λ_o}. Since x_1-axis can be chosen in any direction, we conclude that u must be radially symmetric and monotone about some point.

Finally, in the third step, using the uniqueness theory in Ordinary Differential Equations, we will show that the solutions can only assume the given form.

Step 1. Prepare to Move the Plane from Near $-\infty$.

To verify (8.11) for λ sufficiently negative, we apply the maximum principle to $w_\lambda(x)$. Write $\tau = \frac{n+2}{n-2}$. By the definition of u_λ, it is easy to see that, u_λ satisfies the same equation as u does. Then by the *Mean Value Theorem*, it is easy to verify that

$$-\triangle w_\lambda = u_\lambda^\tau(x) - u^\tau(x) = \tau \psi_\lambda^{\tau-1}(x) w_\lambda(x). \tag{8.12}$$

where $\psi_\lambda(x)$ is some number between $u_\lambda(x)$ and $u(x)$. Recalling the "*Maximum Principle Based on Comparison*" (Theorem 7.4.1), we see here $c(x) =$

$-\tau\psi_\lambda^{\tau-1}(x)$. By the "*Decay at Infinity*" argument (Corollary 7.4.2), it suffice to check the decay rate of $\psi_\lambda^{\tau-1}(x)$, and more precisely, only at the points $\tilde{x}$ where w_λ is negative (see Remark 7.4.2). Apparently at these points,

$$u_\lambda(\tilde{x}) < u(\tilde{x}),$$

and hence

$$0 \leq u_\lambda(\tilde{x}) \leq \psi_\lambda(\tilde{x}) \leq u(\tilde{x}).$$

By the decay assumption of the solution

$$u(x) = O\left(\frac{1}{|x|^{n-2}}\right),$$

we derive immediately that

$$\psi_\lambda^{\tau-1}(\tilde{x}) = O\left((\frac{1}{|\tilde{x}|})^{\frac{4}{n-2}}\right) = O\left(\frac{1}{|\tilde{x}|^4}\right).$$

Here the power of $\frac{1}{|\tilde{x}|}$ is greater than two, which is what (actually more than) we desire for. Therefore, we can apply the "*Maximum Principle Based on Comparison*" to conclude that for λ sufficiently negative ($|\tilde{x}|$ sufficiently large), we must have (8.11). This completes the preparation for the moving of planes.

Step 2. Move the Plane to the Limiting Position to Derive Symmetry.

Now we can move the plane T_λ toward right, i.e., increase the value of λ, as long as the inequality (8.11) holds. Define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\}.$$

Obviously, $\lambda_o < +\infty$, due to the asymptotic behavior of u near $x_1 = +\infty$. We claim that

$$w_{\lambda_o}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_o}. \tag{8.13}$$

Otherwise, by the "*Strong Maximum Principle*" on unbounded domains (see Theorem 7.3.3), we have

$$w_{\lambda_o}(x) > 0 \ \text{ in the interior of } \Sigma_{\lambda_o}. \tag{8.14}$$

We show that the plane T_{λ_o} can still be moved a small distance to the right. More precisely, there exists a $\delta_o > 0$ such that, for all $0 < \delta < \delta_o$, we have

$$w_{\lambda_o+\delta}(x) \geq 0\ , \quad \forall x \in \Sigma_{\lambda_o+\delta}. \tag{8.15}$$

This would contradict with the definition of λ_o, and hence (8.13) must hold.

Recall that in the last section, we use the *"Narrow Region Principle "* to derive (8.15). Unfortunately, it can not be applied in this situation, because the "narrow region" here is unbounded, and we are not able to guarantee that w_{λ_o} is bounded away from 0 on the left boundary of the "narrow region".

To overcome this difficulty, we introduce a new function

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)},$$

where

$$\phi(x) = \frac{1}{|x|^q} \text{ with } 0 < q < n-2.$$

Then it is a straight forward calculation to verify that

$$-\triangle \bar{w}_\lambda = 2\bigtriangledown \bar{w}_\lambda \cdot \frac{\bigtriangledown \phi}{\phi} + \left(-\triangle w_\lambda + \frac{\triangle \phi}{\phi} w_\lambda\right)\frac{1}{\phi} \tag{8.16}$$

We have

Lemma 8.2.1 *There exists a $R_o > 0$ (independent of λ), such that if x^o is a minimum point of $\bar{w}_\lambda$ and $\bar{w}_\lambda(x^o) < 0$, then $|x^o| < R_o$.*

We postpone the proof of the Lemma for a moment. Now suppose that (8.15) is violated for any $\delta > 0$. Then there exists a sequence of numbers $\{\delta_i\}$ tending to 0 and for each i, the corresponding negative minimum x^i of $w_{\lambda_o+\delta_i}$. By Lemma 8.2.1, we have

$$|x^i| \leq R_o\,, \quad \forall i = 1, 2, \cdots.$$

Then, there is a subsequence of $\{x^i\}$ (still denoted by $\{x^i\}$) which converges to some point $x^o \in R^n$. Consequently,

$$\bigtriangledown \bar{w}_{\lambda_o}(x^o) = \lim_{i\to\infty} \bigtriangledown \bar{w}_{\lambda_o+\delta_i}(x^i) = 0 \tag{8.17}$$

and

$$\bar{w}_{\lambda_o}(x^o) = \lim_{i\to\infty} \bar{w}_{\lambda_o+\delta_i}(x^i) \leq 0.$$

However, we already know $\bar{w}_{\lambda_o} \geq 0$, therefore, we must have $\bar{w}_{\lambda_o}(x^o) = 0$. It follows that

$$\bigtriangledown w_{\lambda_o}(x^o) = \bigtriangledown \bar{w}_{\lambda_o}(x^o)\phi(x^o) + \bar{w}_{\lambda_o}(x^o)\bigtriangledown \phi = 0 + 0 = 0. \tag{8.18}$$

On the other hand, by (8.14), since $w_{\lambda_o}(x^o) = 0$, x^o must be on the boundary of Σ_{λ_o}. Then by the *Hopf Lemma* (see Theorem 7.3.3), we have, the outward normal derivative

$$\frac{\partial w_{\lambda_o}}{\partial \nu}(x^o) < 0.$$

This contradicts with (8.18). Now, to verify (8.15), what left is to prove the Lemma.

Proof of Lemma 8.2.1. Assume that x^o is a negative minimum of $\bar{w}_\lambda$. Then

$$-\triangle\bar{w}_\lambda(x^o) \leq 0 \text{ and } \bigtriangledown\bar{w}_\lambda(x^o) = 0. \tag{8.19}$$

On the other hand, as we argued in *Step 1*, by the asymptotic behavior of u at infinity, if $|x^o|$ is sufficiently large,

$$c(x^o) := -\tau\psi^{\tau-1}(x^o) > -\frac{q(n-2-q)}{|x^o|} \equiv \frac{\triangle\phi(x^o)}{\phi(x^o)}.$$

It follows from (8.12) that

$$\left(-\triangle w_\lambda + \frac{\triangle\phi}{\phi}w_\lambda\right)(x^o) > 0.$$

This, together with (8.19) contradicts with (8.16), and hence completes the proof of the Lemma.

Step 3. In the previous two steps, we show that the positive solutions of (8.10) must be radially symmetric and monotone decreasing about some point in R^n. Since the equation is invariant under translation, without loss of generality, we may assume that the solutions are symmetric about the origin. Then they satisfies the following ordinary differential equation

$$\begin{cases} -u''(r) - \frac{n-1}{r}u'(r) = u^\tau \\ u'(0) = 0 \\ u(0) = \frac{[n(n-2)]^{(n-2)/4}}{\lambda^{\frac{n-2}{2}}} \end{cases}$$

for some $\lambda > 0$. One can verify that $u(r) = \dfrac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2+r^2)^{\frac{n-2}{2}}}$ is a solution, and by the uniqueness of the ODE problem, this is the only solution. Therefore, we conclude that every positive solution of (8.10) must assume the form

$$u(x) = \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2+|x-x^o|^2)^{\frac{n-2}{2}}}$$

for $\lambda > 0$ and some $x^o \in R^n$. This completes the proof of the Theorem.

Proof of Theorem 8.2.3.

i) The general idea in proving this part of the Theorem is almost the same as that for Theorem 8.2.2. The main difference is that we have no decay assumption on the solution u at infinity, hence the method of moving planes can not be applied directly to u. So we first make a Kelvin transform to define a new function

$$v(x) = \frac{1}{|x|^{n-2}}u\left(\frac{x}{|x|^2}\right).$$

Then obviously, $v(x)$ has the desired decay rate $\frac{1}{|x|^{n-2}}$ at infinity, but has a possible singularity at the origin. It is easy to verify that v satisfies the same equation as u does, except at the origin:

$$\triangle v + v^{\tau}(x) = 0 \quad x \in R^n \setminus \{0\}, n \geq 3. \tag{8.20}$$

We will apply the method of moving planes to the function v, and show that v is radially symmetric and monotone decreasing about some point. If the center point is the origin, then by the definition of v, u is also symmetric and monotone decreasing about the origin. If the center point of v is not the origin, then v has no singularity at the origin, and hence u has the desired decay at infinity. Then the same argument in the proof of Theorem 8.2.2 would imply that u is symmetric and monotone decreasing about some point.

Define

$$v_\lambda(x) = v(x^\lambda) \ , \quad w_\lambda(x) = v_\lambda(x) - v(x).$$

Because $v(x)$ may be singular at the origin, correspondingly w_λ may be singular at the point $x_\lambda = (2\lambda, 0, \cdots, 0)$. Hence instead of on Σ_λ, we consider w_λ on $\tilde{\Sigma}_\lambda = \Sigma_\lambda \setminus \{x_\lambda\}$. And in our proof, we treat the singular point carefully. Each time we show that the points of interest are away from the singularities, so that we can carry on the method of moving planes to the end to show the existence of a λ_o such that $w_{\lambda_o}(x) \equiv 0$ for $x \in \tilde{\Sigma}_{\lambda_o}$ and v is strictly increasing in the x_1 direction in $\tilde{\Sigma}_{\lambda_o}$.

As in the proof of Theorem 8.2.2, we see that v_λ satisfies the same equation as v does, and

$$-\triangle w_\lambda = \tau \psi_\lambda^{\tau-1}(x) w_\lambda(x).$$

where $\psi_\lambda(x)$ is some number between $v_\lambda(x)$ and $v(x)$.

Step 1. We show that, for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0, \quad \forall x \in \tilde{\Sigma}_\lambda. \tag{8.21}$$

By the asymptotic behavior

$$v(x) \sim \frac{1}{|x|^{n-2}},$$

we derive immediately that, at a negative minimum point x^o of w_λ,

$$\psi_\lambda^{\tau-1}(x^o) \sim \left(\frac{1}{|x^o|^{n-2}}\right)^{\tau-1} = \frac{1}{|x^o|^4},$$

the power of $\frac{1}{|x^o|}$ is greater than two, and we have the desired decay rate for $c(x) := -\tau\psi_\lambda^{\tau-1}(x)$, as mentioned in Corollary 7.4.2. Hence we can apply the "*Decay at Infinity*" to $w_\lambda(x)$. The difference here is that, w_λ has a singularity at x_λ, hence we need to show that, the minimum of w_λ is away from x_λ. Actually, we will show that,

$$\text{If } \inf_{\tilde{\Sigma}_\lambda} w_\lambda(x) < 0, \text{ then the infimum is achieved in } \Sigma_\lambda \setminus B_1(x_\lambda) \tag{8.22}$$

To see this, we first note that for $x \in B_1(0)$,

$$v(x) \geq \min_{\partial B_1(0)} v(x) = \epsilon_0 > 0$$

due to the fact that $v(x) > 0$ and $\triangle v \leq 0$.

Then let λ be so negative, that $v(x) \leq \epsilon_0$ for $x \in B_1(x_\lambda)$. This is possible because $v(x) \to 0$ as $|x| \to \infty$.

For such λ, obviously $w_\lambda(x) \geq 0$ on $B_1(x_\lambda) \setminus \{x_\lambda\}$. This implies (8.22). Now similar to the *Step 1* in the proof of Theorem 8.2.2, we can deduce that $w_\lambda(x) \geq 0$ for λ sufficiently negative.

Step 2. Again, define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \tilde{\Sigma}_\lambda\}.$$

We will show that

$$\text{If } \lambda_o < 0, \quad \text{then } w_{\lambda_o}(x) \equiv 0, \ \forall x \in \tilde{\Sigma}_{\lambda_o}.$$

Define $\bar{w}_\lambda = \dfrac{w_\lambda}{\phi}$ the same way as in the proof of Theorem 8.2.2. Suppose that $\bar{w}_{\lambda_o}(x) \not\equiv 0$, then by *Maximum Principle* we have

$$\bar{w}_{\lambda_o}(x) > 0\,, \quad \text{for } x \in \tilde{\Sigma}_{\lambda_o}.$$

The rest of the proof is similar to the step 2 in the proof of Theorem 8.2.2 except that now we need to take care of the singularities. Again let $\lambda_k \searrow \lambda_o$ be a sequence such that $\bar{w}_{\lambda_k}(x) < 0$ for some $x \in \tilde{\Sigma}_{\lambda_k}$. We need to show that for each k, $\inf_{\tilde{\Sigma}_{\lambda_k}} \bar{w}_{\lambda_k}(x)$ can be achieved at some point $x^k \in \tilde{\Sigma}_{\lambda_k}$ and that the sequence $\{x^k\}$ is bounded away from the singularities x_{λ_k} of w_{λ_k}. This can be seen from the following facts

a) There exists $\epsilon > 0$ and $\delta > 0$ such that

$$\bar{w}_{\lambda_o}(x) \geq \epsilon \text{ for } x \in B_\delta(x_{\lambda_o}) \setminus \{x_{\lambda_o}\}.$$

b) $\lim_{\lambda \to \lambda_o} \inf_{x \in B_\delta(x_\lambda)} \bar{w}_\lambda(x) \geq \inf_{x \in B_\delta(x_{\lambda_o})} \bar{w}_{\lambda_o}(x) \geq \epsilon.$

Fact (a) can be shown by noting that $\bar{w}_{\lambda_o}(x) > 0$ on $\tilde{\Sigma}_{\lambda_o}$ and $\triangle w_{\lambda_o} \leq 0$, while fact (b) is obvious.

Now through a similar argument as in the proof of Theorem 8.2.2 one can easily arrive at a contradiction. Therefore $\bar{w}_{\lambda_o}(x) \equiv 0$.

If $\lambda_o = 0$, then we can carry out the above procedure in the opposite direction, namely, we move the plane in the negative x_1 direction from positive infinity toward the origin. If our planes T_λ stop somewhere before the origin, we derive the symmetry and monotonicity in x_1 direction by the above argument. If they stop at the origin again, we also obtain the symmetry and monotonicity in x_1 direction by combining the two inequalities obtained in the two opposite directions. Since the x_1 direction can be chosen arbitrarily, we conclude that the solution u must be radially symmetric about some point.

ii) To show the non-existence of solutions in the case $p < \frac{n+2}{n-2}$, we notice that after the Kelvin transform, v satisfies

$$\Delta v + \frac{1}{|x|^{n+2-p(n-2)}} v^p(x) = 0 \ , \quad x \in R^n \setminus \{0\}.$$

Due to the singularity of the coefficient of v^p at the origin, one can easily see that v can only be symmetric about the origin if it is not identically zero. Hence u must also be symmetric about the origin. Now given any two points x^1 and x^2 in R^n, since equation (8.10) is invariant under translations and rotations, we may assume that the origin is at the mid point of the line segment $\overline{x^1x^2}$. Then from the above argument, we must have $u(x^1) = u(x^2)$. It follows that u is constant. Finally, from the equation (8.10), we conclude that $u \equiv 0$. This completes the proof of the Theorem.

8.2.3 Symmetry of Solutions for $-\Delta u = e^u$ in R^2

When considering prescribing Gaussian curvature on two dimensional compact manifolds, if the sequence of approximate solutions "blows up", then by rescaling and taking limit, one would arrive at the following equation in the entire space R^2:

$$\begin{cases} \Delta u + \exp u = 0 \ , \quad x \in R^2 \\ \int_{R^2} \exp u(x) dx < +\infty \end{cases} \tag{8.23}$$

The classification of the solutions for this limiting equation would provide essential information on the original problems on manifolds, also it is interesting in its own right.

It is known that

$$\phi_{\lambda, x^o}(x) = \ln \frac{32\lambda^2}{(4 + \lambda^2 |x - x^o|^2)^2}$$

for any $\lambda > 0$ and any point $x^o \in R^2$ is a family of explicit solutions.

We will use the method of moving planes to prove:

Theorem 8.2.4 *Every solution of (8.23) is radially symmetric with respect to some point in R^2 and hence assumes the form of $\phi_{\lambda, x^o}(x)$.*

To this end, we first need to obtain some decay rate of the solutions near infinity.

Some Global and Asymptotic Behavior of the Solutions

The following Theorem gives the asymptotic behavior of the solutions near infinity, which is essential to the application of the method of moving planes.

Theorem 8.2.5 *If $u(x)$ is a solution of (8.23), then as $|x| \to +\infty$,*

$$\frac{u(x)}{\ln |x|} \to -\frac{1}{2\pi} \int_{R^2} \exp u(x) dx \leq -4$$

uniformly.

This Theorem is a direct consequence of the following two Lemmas.

Lemma 8.2.2 *(W. Ding) If u is a solution of*

$$-\triangle u = e^u, \quad x \in R^2$$

and

$$\int_{R^2} \exp u(x) dx < +\infty,$$

then

$$\int_{R^2} \exp u(x) dx \geq 8\pi.$$

Proof. For $-\infty < t < \infty$, let $\Omega_t = \{x \mid u(x) > t\}$, one can obtain

$$\int_{\Omega_t} \exp u(x) dx = -\int_{\Omega_t} \Delta u = \int_{\partial \Omega_t} |\nabla u| ds$$

$$-\frac{d}{dt} |\Omega_t| = \int_{\partial \Omega_t} \frac{ds}{|\nabla u|}$$

By the Schwartz inequality and the isoperimetric inequality,

$$\int_{\partial \Omega_t} \frac{ds}{|\nabla u|} \cdot \int_{\partial \Omega_t} |\nabla u| \geq |\partial \Omega_t|^2 \geq 4\pi |\Omega_t|.$$

Hence

$$-(\frac{d}{dt} |\Omega_t|) \cdot \int_{\Omega_t} \exp u(x) dx \geq 4\pi |\Omega_t|$$

and so

$$\frac{d}{dt} (\int_{\Omega_t} \exp u(x) dx)^2 = 2 \exp t \cdot (\frac{d}{dt} |\Omega_t|) \cdot \int_{\Omega_t} \exp u(x) dx \leq -8\pi |\Omega_t| e^t.$$

Integrating from $-\infty$ to ∞ gives

$$-(\int_{R^2} \exp u(x) dx)^2 \leq -8\pi \int_{R^2} \exp u(x) dx$$

which implies $\int_{R^2} \exp u(x) dx \geq 8\pi$ as desired.

Lemma 8.2.2 enables us to obtain the asymptotic behavior of the solutions at infinity.

Lemma 8.2.3 . *If $u(x)$ is a solution of (8.23), then as $|x| \to +\infty$,*

$$\frac{u(x)}{\ln |x|} \to -\frac{1}{2\pi}\int_{R^2} \exp u(x)dx \text{ uniformly.}$$

Proof.

By a result of Brezis and Merle [BM], we see that the condition $\int_{R^2} \exp u(x)dx < \infty$ implies that the solution u is bounded from above.

Let

$$w(x) = \frac{1}{2\pi}\int_{R^2}(\ln |x-y| - \ln(|y|+1)) \exp u(y)dy.$$

Then it is easy to see that

$$\triangle w(x) = \exp u(x)\ , \quad x \in R^2$$

and we will show

$$\frac{w(x)}{\ln |x|} \to \frac{1}{2\pi}\int_{R^2} \exp u(x)dx \text{ uniformly as } |x| \to +\infty. \tag{8.24}$$

To see this, we need only to verify that

$$I := \int_{R^2} \frac{\ln |x-y| - \ln(|y|+1) - \ln |x|}{\ln |x|} e^{u(y)}dy \to 0$$

as $|x| \to \infty$. Write $I = I_1 + I_2 + I_3$, where I_1, I_2 and I_3 are the integrals on the three regions

$$D_1 = \{y \mid |x-y| \leq 1\},$$

$$D_2 = \{y \mid |x-y| > 1 \text{ and } |y| \leq K\}$$

and

$$D_3 = \{y \mid |x-y| > 1 \text{ and } |y| > K\}$$

respectively. We may assume that $|x| \geq 3$.

a) To estimate I_1, we simply notice that

$$I_1 \leq C\int_{|x-y|\leq 1} e^{u(y)}dy - \frac{1}{\ln |x|}\int_{|x-y|\leq 1} \ln |x-y| e^{u(y)}dy$$

Then by the boundedness of $e^{u(y)}$ and $\int_{R^2} e^{u(y)}dy$, we see that $I_1 \to 0$ as $|x| \to \infty$.

b) For each fixed K, in region D_2, we have, as $|x| \to \infty$,

$$\frac{\ln |x-y| - \ln(|y|+1) - \ln |x|}{\ln |x|} \to 0$$

hence $I_2 \to 0$.

c)To see $I_3 \to 0$, we use the fact that for $|x-y|>1$

$$\left|\frac{\ln|x-y| - \ln(|y|+1) - \ln|x|}{\ln|x|}\right| \leq C$$

Then let $K \to \infty$. This verifies (8.24).

Consider the function $v(x) = u(x) + w(x)$. Then $\triangle v \equiv 0$ and

$$v(x) \leq C + C_1 \ln(|x|+1)$$

for some constant C and C_1. Therefore v must be a constant. This completes the proof of our Lemma.

Combining Lemma 8.2.2 and Lemma 8.2.3, we obtain our Theorem 8.2.5.

The Method of Moving Planes and Symmetry

In this subsection, we apply the method of moving planes to establish the symmetry of the solutions. For a given solution, we move the family of lines which are orthogonal to a given direction from negative infinity to a critical position and then show that the solution is symmetric in that direction about the critical position. We also show that the solution is strictly increasing before the critical position. Since the direction can be chosen arbitrarily, we conclude that the solution must be radially symmetric about some point. Finally by the uniqueness of the solution of the following O.D.E. problem

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) = f(u) \\ u'(0) = 0 \\ u(0) = 1 \end{cases}$$

we see that the solutions of (8.23) must assume the form $\phi_{\lambda,x^0}(x)$.

Assume that $u(x)$ is a solution of (8.23). Without loss of generality, we show the monotonicity and symmetry of the solution in the x_1 direction.

For $\lambda \in R^1$, let

$$\Sigma_\lambda = \{(x_1, x_2) \mid x_1 < \lambda\}$$

and

$$T_\lambda = \partial \Sigma_\lambda = \{(x_1, x_2) \mid x_1 = \lambda\}.$$

Let

$$x^\lambda = (2\lambda - x_1, x_2)$$

be the reflection point of $x = (x_1, x_2)$ about the line T_λ. (See the previous Figure 1.)

Define

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

A straight forward calculation shows

$$\triangle w_\lambda(x) + (\exp \psi_\lambda(x)) w_\lambda(x) = 0 \tag{8.25}$$

where $\psi_\lambda(x)$ is some number between $u(x)$ and $u_\lambda(x)$.

Step 1. As in the previous examples, we show that, for λ sufficiently negative, we have

$$w_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda.$$

By the "*Decay at Infinity*" argument, the key is to check the decay rate of $c(x) := -e^{\psi_\lambda(x)}$ at points x^o where $w_\lambda(x)$ is negative. At such point

$$u(x^{o\lambda}) < u(x^o), \quad \text{and hence } \psi_\lambda(x^o) \leq u(x^o).$$

By virtue of Theorem 8.2.5, we have

$$e^{\psi_\lambda(x^o)} = O(\frac{1}{|x^o|^4}) \tag{8.26}$$

Notice that we are in dimension two, while the key function $\phi = \frac{1}{|x|^q}$ given in Corollary 7.4.2 requires $0 < q < n-2$, hence it does not work here. As a modification, we choose

$$\phi(x) = \ln(|x| - 1).$$

Then it is easy to verify that

$$\frac{\triangle\phi}{\phi}(x) = \frac{-1}{|x|(|x|-1)^2 \ln(|x|-1)}.$$

It follows from this and (8.26) that

$$e^{\psi(x^o)} + \frac{\triangle\phi}{\phi}(x^o) < 0 \quad \text{for sufficiently large } |x^o|. \tag{8.27}$$

This is what we desire for.

Then similar to the argument in Subsection 5.2, we introduce the function

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)}.$$

It satisfies

$$\triangle \bar{w}_\lambda + 2\bigtriangledown \bar{w}_\lambda \frac{\bigtriangledown\phi}{\phi} + \left(e^{\psi_\lambda(x)} + \frac{\triangle\phi}{\phi}\right)\bar{w}_\lambda = 0. \tag{8.28}$$

Moreover, by the asymptotic behavior of the solution u near infinity (see Lemma 8.2.3), we have, for each fixed λ,

$$\bar{w}_\lambda(x) = \frac{u(x^\lambda) - u(x)}{\ln(|x|-1)} \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \tag{8.29}$$

Now, if $w_\lambda(x) < 0$ somewhere, then by (8.29), there exists a point x^o, which is a negative minimum of $\bar{w}_\lambda(x)$. At this point, one can easily derive a contradiction from (8.27) and (8.28). This completes *Step 1.*

Step 2 . Define

$$\lambda_o = \sup\{\lambda \mid w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda\}.$$

We show that

$$\text{If } \lambda_o < 0, \ \text{ then } w_{\lambda_o}(x) \equiv 0, \ \forall x \in \Sigma_{\lambda_o}.$$

The argument is entirely similar to that in the *Step 2* of the proof of Theorem 8.2.2 except that we use $\phi(x) = \ln(-x_1 + 2)$. This completes the proof of the Theorem.

8.3 Method of Moving Planes in a Local Way

8.3.1 The Background

Let M be a Riemannian manifold of dimension $n \geq 3$ with metric g_o. Given a function $R(x)$ on M, one interesting problem in differential geometry is to find a metric g that is point-wise conformal to the original metric g_o and has scalar curvature R(x). This is equivalent to finding a positive solution of the semi-linear elliptic equation

$$-\frac{4(n-1)}{n-2}\triangle_o u + R_o(x)u = R(x)u^{\frac{n+2}{n-2}} \ , \ \ x \in M, \tag{8.30}$$

where $\triangle_o$ is the Beltrami-Laplace operator of (M, g_o) and $R_o(x)$ is the scalar curvature of g_o.

In recent years, there have seen a lot of progress in understanding equation (8.30). When (M, g_o) is the standard sphere S^n, the equation becomes

$$-\triangle u + \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}R(x)u^{\frac{n+2}{n-2}}, \ \ u > 0, \ x \in S^n. \tag{8.31}$$

It is the so-called critical case where the lack of compactness occurs. In this case, the well-known Kazdan-Warner condition gives rise to many examples of R in which there is no solution. In the last few years, a tremendous amount of effort has been put to find the existence of the solutions and many interesting results have been obtained (see [Ba] [BC] [Bi] [BE] [CL1] [CL3] [CL4] [CL6] [CY1] [CY2] [ES] [KW1] [Li2] [Li3] [SZ] [Lin1] and the references therein).

One main ingredient in the proof of existence is to obtain a priori estimates on the solutions. For equation (8.31) on S^n, to establish a priori estimates, a useful technique employed by most authors was a 'blowing-up' analysis. However, it does not work near the points where $R(x) = 0$. Due to this limitation, people had to assume that R was positive and bounded away from 0. This technical assumption became somewhat standard and has been used by many authors for quite a few years. For example, see articles by Bahri