By dominated convergence, letting $k \to \infty$, it follows that

$$-\int_{\mathbb{R}^{n-1}} U^p(x')\,\varphi(x')\,dx' = -\int_{\mathbb{R}^{n-1}} U^p(x')\,\varphi(x') \int_0^1 \psi(s)\,ds\,dx'$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} U(x')\,\Delta\big(\varphi(x')\,\psi(s)\big)\,ds\,dx'.$$

But the RHS is equal to

$$\int_{\mathbb{R}^{n-1}} U(x')\,\Delta_{x'}\varphi(x')\,dx' \int_0^1 \psi(s)\,ds + \int_{\mathbb{R}^{n-1}} U(x')\,\varphi(x')\,dx' \int_0^1 \psi''(s)\,ds$$
$$= \int_{\mathbb{R}^{n-1}} U(x')\,\Delta_{x'}\varphi(x')\,dx'.$$

It follows that U solves (8.1) in $\mathbb{R}^{n-1}$ in the distribution sense, hence in the classical sense (this is a consequence of the boundedness of U and of Remark 47.4). The result is then a consequence of Theorem 8.1(i). □

9. Positive radial solutions of $\Delta u + u^p = 0$ in $\mathbb{R}^n$

In this section we study positive radial classical solutions of the equation

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^n. \tag{9.1}$$

Since this problem does not possess positive classical solutions if $1 < p < p_S$ due to Theorem 8.1, we restrict ourselves to the case $p \geq p_S$. Consequently, $n \geq 3$.

Positive radial classical solutions of (9.1) can be written in the form $u(x) = U(r)$, where $r = |x|$ and $U \in C^2([0,\infty))$ is a positive classical solution of

$$U'' + \frac{n-1}{r}U' + U^p = 0, \quad r \in (0,\infty), \qquad U'(0) = 0. \tag{9.2}$$

It is easily seen that prescribing initial values $U(0) = \alpha > 0$, $U'(0) = 0$, the equation in (9.2) has a unique solution for r small enough. In fact, this equation can be written in the form $(r^{n-1}U')' = -r^{n-1}U^p$ and, by integration we obtain the equivalent integral equation

$$U(r) = \alpha - \int_0^r \int_0^s \Big(\frac{t}{s}\Big)^{n-1} U^p(t)\,dt\,ds,$$

which can be solved by the Banach fixed point theorem.

Let $U_*(r) = c_p r^{-2/(p-1)}$ be the singular solution defined in (3.9) and set

$$p_{JL} := \begin{cases} +\infty & \text{if } n \leq 10, \\ 1 + 4\frac{n-4+2\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10. \end{cases} \tag{9.3}$$

The main result of this section is the following theorem.

Theorem 9.1. *Let $p \geq p_S$. Given $\alpha > 0$, problem (9.2) possesses a unique positive solution $U_\alpha \in C^2([0,\infty))$ satisfying $U_\alpha(0) = \alpha$. This solution is decreasing and we have*

$$U_\alpha(r) = \alpha U_1(\alpha^{(p-1)/2} r). \tag{9.4}$$

If $p > p_S$, then $r^{2/(p-1)} U_\alpha(r) \to c_p$ as $r \to \infty$. If $p = p_S$, then

$$U_1(r) = \left(\frac{n(n-2)}{n(n-2)+r^2}\right)^{(n-2)/2}. \tag{9.5}$$

Let $\alpha_1 > \alpha_2 > 0$. If $p \geq p_{JL}$, then $U_*(r) > U_{\alpha_1}(r) > U_{\alpha_2}(r)$ for all $r > 0$. If $p_S < p < p_{JL}$, then U_{α_1} and U_{α_2} intersect infinitely many times and U_{α_1}, U_* intersect infinitely many times as well. If $p = p_S$, then $U_{\alpha_1}, U_{\alpha_2}$ intersect once and U_{α_1}, U_* intersect twice.

Proof. Using the transformation

$$w(s) = r^{2/(p-1)} U(r), \qquad s = \log r, \tag{9.6}$$

problem (9.2) becomes

$$w'' + \beta w' + w^p - \gamma w = 0, \qquad s \in \mathbb{R}, \tag{9.7}$$

where

$$\beta := \frac{1}{p-1}\big((n-2)p - (n+2)\big) \geq 0, \qquad \gamma := c_p^{p-1} = \frac{2}{(p-1)^2}\big((n-2)p - n\big) > 0,$$

and we are looking for solutions w satisfying $w(s), w'(s) \to 0$ as $s \to -\infty$. Set

$$\mathcal{E}(w) = \mathcal{E}(w, w') := \frac{1}{2}|w'|^2 - \frac{\gamma}{2} w^2 + \frac{1}{p+1} w^{p+1}.$$

Then $\mathcal{E}$ is a Lyapunov functional for (9.7); more precisely,

$$\frac{d}{ds}\mathcal{E}\big(w(s)\big) = -\beta\big(w'(s)\big)^2 \leq 0. \tag{9.8}$$

Denoting $x := w$ and $y := w'$, problem (9.7) can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -\beta y - x^p + \gamma x \end{pmatrix} =: F(x,y) \tag{9.9}$$

where $x > 0$ and $(x,y) \to (0,0)$ as $s \to -\infty$. Problem (9.9) possesses two equilibria, $(0,0)$ and $(c_p, 0)$ lying in the half-space $\{(x,y) : x \geq 0\}$. Denote

$$A_1 := \nabla F(0,0) = \begin{pmatrix} 0 & 1 \\ \gamma & -\beta \end{pmatrix}, \qquad A_2 := \nabla F(c_p, 0) = \begin{pmatrix} 0 & 1 \\ -\gamma(p-1) & -\beta \end{pmatrix}.$$

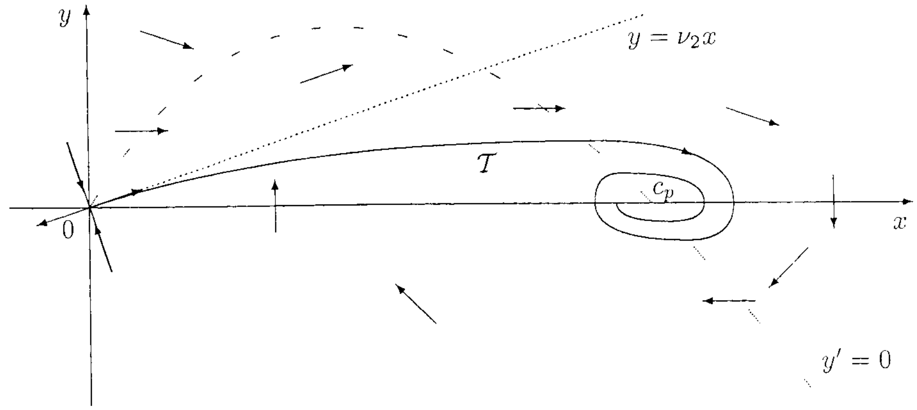


Figure 5: The flow generated by (9.9) for $p_S < p < p_{JL}$.

First consider the case $p > p_S$. Then $\beta > 0$ and the matrix A_1 has two real eigenvalues $\nu_{1,2} := -\frac{1}{2}(\beta \pm \sqrt{\beta^2 + 4\gamma})$ with $\nu_1 < 0 < \nu_2 = 2/(p-1)$. The corresponding eigenvectors (x_i, y_i) satisfy $y_i = \nu_i x_i$, $i = 1, 2$. The eigenvalues $\tilde{\nu}_{1,2} := -\frac{1}{2}(\beta \pm \sqrt{\beta^2 - 4\gamma(p-1)})$ of A_2 are real iff $\beta^2 \geq 4\gamma(p-1)$, that is iff $p \geq p_{JL}$.

Assume $p_S < p < p_{JL}$. In this case, the eigenvalues $\tilde{\nu}_1, \tilde{\nu}_2$ are complex and their real parts are negative so that the critical point $(c_p, 0)$ is a stable spiral. The flow for the planar system (9.9) is illustrated in Figure 5.

We are interested in the trajectory $\mathcal{T}$ emanating from the origin to the right half-space, since it represents the graph of any positive solution of (9.7) in the w-w' plane. This trajectory cannot hit the axis $x = 0$ again since the energy functional $\mathcal{E}$ is nonnegative on this axis, $\mathcal{E}(0,0) = 0$, $\beta > 0$ and (9.8) is true. Moreover, the corresponding solutions w exists for all $s \in \mathbb{R}$ and w, w' remain bounded for all $s \in \mathbb{R}$ due to (9.8). Consequently, $\mathcal{T}$ has to converge to the critical point $(c_p, 0)$ which corresponds to the singular solution $w_*(s) = r^{2/(p-1)}U_*(r) \equiv c_p$. Thus, if U_α is the unique local solution of (9.2) such that $U_\alpha(0) = \alpha > 0$, then its transform $w_\alpha(s) = r^{2/(p-1)}U_\alpha(r)$ exists globally and satisfies $w_\alpha(s) \to c_p$ as $s \to \infty$. Consequently, U_α exists globally and $r^{2/(p-1)}U_\alpha(r) \to c_p$ as $r \to \infty$. It is easily verified that the function $\tilde{U}_\alpha(r) := \alpha U_1(\alpha^{(p-1)/2}r)$ is a solution of (9.2) satisfying $\tilde{U}_\alpha(0) = \alpha$, hence $\tilde{U}_\alpha = U_\alpha$ by uniqueness. The graphs of w_α and w_1 in the w-w' plane are identical, so that there exists $s_\alpha \in \mathbb{R}$ such that $U_\alpha(e^s) = w_\alpha(s) = w_1(s - s_\alpha)$ for all $s \in \mathbb{R}$. Hence, given $\alpha_1 > \alpha_2 > 0$, $U_{\alpha_1}(r) = U_{\alpha_2}(r)$ for some $r > 0$ iff $w_1(s - s_{\alpha_1}) = w_1(s - s_{\alpha_2})$ for some $s \in \mathbb{R}$. This happens for infinitely many s since $\mathcal{T}$ spirals around the point $(c_p, 0)$. Similarly, $w_{\alpha_1}(s) = c_p$

for infinitely many s, hence U_{α_1} and U_* intersect infinitely many times.

Next consider the case $p \geq p_{JL}$. On the halfline $y = -\frac{\beta}{2}(x - c_p)$, $x < c_p$, we have for suitable $x_\theta \in (x, c_p)$:

$$\begin{aligned}\frac{y'}{x'} &= -\beta - \frac{x}{y}(x^{p-1} - \gamma) = -\beta + \frac{2x(x^{p-1} - c_p^{p-1})}{\beta(x - c_p)} \\ &= -\beta + \frac{2}{\beta}x(p-1)x_\theta^{p-2} < -\beta + \frac{2}{\beta}(p-1)\gamma \leq -\frac{\beta}{2}.\end{aligned}$$

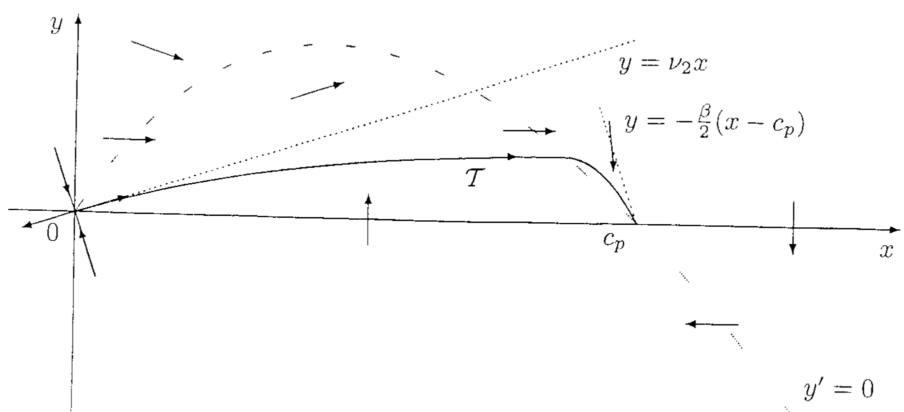


Figure 6: The flow generated by (9.9) for $p \geq p_{JL}$.

Consequently, the trajectory $\mathcal{T}$ ends up at $(c_p, 0)$ again but the x-coordinate is increasing along $\mathcal{T}$ (see Figure 6). Hence, the solutions U of (9.2) are ordered according to their values at $r = 0$, $U_* > U_{\alpha_1} > U_{\alpha_2}$ if $\alpha_1 > \alpha_2$.

Finally consider the case $p = p_S$. Then $\beta = 0$ and the energy functional $\mathcal{E}$ is constant along any solution. Since $\mathcal{E}(c_p, 0) < 0$ and $\mathcal{E}(0, y) > 0$ for $y \neq 0$, the trajectory $\mathcal{T}$ is a homoclinic orbit (see Figure 7).

Let w_α, s_α have the same meaning as above. Given $\alpha_1 \neq \alpha_2$, there exists a unique $s \in \mathbb{R}$ such that $w_1(s - s_{\alpha_1}) = w_1(s - s_{\alpha_2})$. Hence, the corresponding solutions $U_{\alpha_1}, U_{\alpha_2}$ of (9.2) intersect exactly once. Similarly, given $\alpha > 0$, we have $w_\alpha(s) = c_p$ for two values of s, so that U_α and U_* intersect twice. One can easily check that the function U_1 defined by (9.5) is a solution of (9.2) satisfying the initial condition $U_1(0) = 1$. $\square$

Remarks 9.2. (i) The exponent p_{JL} appeared for the first time in [293] where the authors studied mainly problems with the nonlinearities $f(u) = \lambda(1 + au)^p$ and $f(u) = \lambda e^u$, $\lambda, a > 0$.

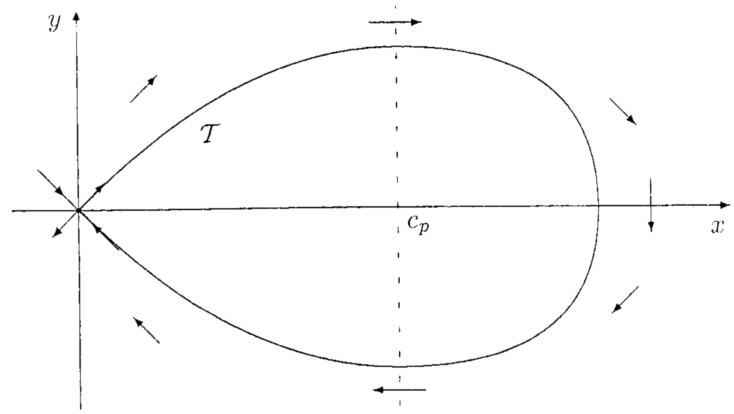


Figure 7: The flow generated by (9.9) for $p = p_S$.

(ii) The intersection properties of the solutions U in Theorem 9.1 play an important role in the study of stability and asymptotic behavior of solutions of the corresponding parabolic problem, see Sections 22, 23. □

Remark 9.3. Let $p = p_S$ and $a > 0$. For all $\alpha \geq M_0(a)$ with $M_0(a) > 0$ large enough, if V is a positive classical solution of

$$V'' + \frac{n-1}{r}V' + V^p = 0, \qquad 0 < r < a,$$

such that $V(a) = U_\alpha(a)$ and $\lim_{r\to 0} V(r) = \infty$, then V has to intersect U_α in $(0, a)$.

In fact, denoting $w_\alpha(s) := r^{2/(p-1)}U_\alpha(r)$, $s = \log r$, the rescaled function from the last proof, it suffices to chose $M_0(a)$ such that

$$w'_{M_0(a)}(\log a) < 0 \tag{9.10}$$

(hence $w'_\alpha(\log a) < 0$ for all $\alpha \geq M_0(a)$). Indeed the trajectory of $W(s) := r^{2/(p-1)}V(r)$, $s \in (-\infty, \log a)$, has to be a subset of a periodic orbit lying inside the trajectory $\mathcal{T}$ (see Figure 7). Due to (9.10) there exists $s_0 \in (-\infty, \log a)$ such that $w_\alpha(s_0) = W(s_0)$, hence $U_\alpha(e^{s_0}) = V(e^{s_0})$.

Note also that there exist infinitely many periodic orbits of (9.7) for $p = p_S$, corresponding to positive singular solutions of $u'' + \frac{n-1}{r}u' + u^p = 0$ for $r > 0$. □

Remark 9.4. Let $p > p_{JL}$. Since the trajectory $\mathcal{T}$ approaches the limit point $(c_p, 0)$ below the dotted line with slope $-\beta/2$ and $\tilde{\nu}_2 < -\beta/2 < \tilde{\nu}_1 < 0$, it has to converge along the eigenvector $(1, \tilde{\nu}_1)$ corresponding to the eigenvalue $\tilde{\nu}_1$, hence

$$\frac{y(s)}{x(s) - c_p} \to \tilde{\nu}_1 \quad \text{as} \quad s \to \infty.$$

Returning to the original variables and denoting $V(r) := U(r) - U_*(r)$ we obtain

$$\lim_{r\to\infty} \frac{rV'(r)}{V(r)} = \tilde{\nu}_1 - m, \tag{9.11}$$

where $m := 2/(p-1)$. Assuming that $V(r) = cr^{-\alpha} + h.o.t.$ for some $c \neq 0$ and $\alpha > m$, (9.11) guarantees $c < 0$ and $\alpha = m + \lambda_-$, where

$$\begin{aligned}\lambda_- := -\tilde{\nu}_1 &= \frac{1}{2}\big(\beta - \sqrt{\beta^2 - 4\gamma(p-1)}\big)\\ &= \frac{1}{2}\big(n-2-2m - \sqrt{(n-2-2m)^2 - 8(n-2-m)}\big).\end{aligned}$$

This expansion is indeed true: In fact, a more precise asymptotic expansion of V was established in [260] and [334]. □

10. A priori bounds via the method of Hardy-Sobolev inequalities

A priori estimates of solutions can be used for the proof of existence and multiplicity results. Unlike the variational methods in sections 6 and 7, this approach does not require any variational structure of the problem and enables one to prove the existence of continuous branches of solutions.

Due to Theorem 7.8(ii) one cannot hope for a priori estimates of all solutions. The bifurcation diagrams in Figure 2 suggest that there is some hope for such estimates if we restrict ourselves to positive solutions and to the subcritical case.[3]

In the present and the following three sections we introduce four different methods which are often used in the proofs of a priori bounds for positive solutions of superlinear elliptic problems. We will study mainly the scalar problem

$$\left.\begin{aligned} -\Delta u &= f(x,u,\nabla u), && x \in \Omega,\\ u &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{10.1}$$

where Ω is bounded and f is a sufficiently smooth function with superlinear growth in the u-variable. Some of the possible generalizations and modifications will be mentioned as remarks, others can be found in the subsequent chapters.

This section is devoted to the method of [99], which is based on a Hardy-type inequality and enables one to treat rather general nonlinearities f. On the

[3] In fact, in the subcritical case one can get a priori estimates of all solutions with bounded Morse indices (without the positivity assumption), see [49], [539], [32].

for $t \geq 0$, which implies that $e^{\lambda_1 t/2} g(t)$ is increasing and

$$e^{\lambda_1 t/2} g(t) \geq g(0) > 0$$

for all $t > 0$ which, in turn, implies that

$$g(t) \geq e^{\frac{-\lambda_1 t}{2}} g(0) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty,$$

a contradiction. Therefore u must be unstable.

It is generally believed that the diffusion process is a "smoothing" and "stabilizing" process. Thus in a closed system it seems reasonable to expect that *the only stable steady states are constants* (i.e. *spatially homogeneous*). It turns out that this is indeed the case *for single equations* (II.1) or (II.2) *provided that the domain* Ω *is nice*, e.g. *convex*. (For systems of equations with different diffusion coefficients, this is generally not true and we shall discuss this in §2.) This result was proved fairly recently by Casten and Holland [CH] in 1978, and by Matano [Ma] in 1979 independently. Matano also showed that this result also holds for other domains such as annuli $\{x \in \mathbb{R}^n | \ a < |x| < b\}$, and gave a counterexample showing that *for certain non-convex domains, nontrivial stable steady states of* (II.1) *or* (II.2) *do exist*. We shall prove the result of [CH] and

[Ma] and we shall follow Matano's proof. The role of convexity is contained in the following

Lemma. *Let* Ω *be a bounded smooth convex domain in* $\mathbb{R}^n$. *Suppose that* $v \in C^3(\bar{\Omega})$ *with* $\frac{\partial v}{\partial \nu} = 0$ *on* $\partial\Omega$. *Then*

$$\frac{\partial}{\partial \nu}|Dv|^2 \leq 0 \quad on \quad \partial\Omega$$

Proof. Let ρ be a "defining" function of Ω, i.e. ρ is smooth in a neighborhood of $\bar{\Omega}$ with the following properties

$$\begin{cases} \rho < 0 \text{ in } \Omega, \ \rho = 0 \text{ on } \partial\Omega, \ \rho > 0 \text{ outside } \bar{\Omega}, \text{ and} \\ \nabla\rho \text{ never vanishes on } \partial\Omega. \end{cases}$$

[A good example would be a smooth extension (beyond $\bar{\Omega}$) of the first eigenfunction of Δ with zero Dirichlet boundary condition.] Note that $\nu = \frac{\nabla\rho}{|\nabla\rho|}$ is the unit outer normal and by our hypothesis on v,

$$\frac{\partial v}{\partial \nu} = \frac{\nabla\rho}{|\nabla\rho|} \cdot \nabla v|_{\partial\Omega} = 0,$$

which implies that $\nabla\rho\cdot\nabla v = 0$ on $\partial\Omega$ and thus there exists $g \in C^1(\bar{\Omega})$ such that

(II.7) $$\nabla\rho\cdot\nabla v = g\rho$$

on $\bar{\Omega}$. Differentiating (II.7) with respect to x_i yields

$$\nabla\rho_i\cdot\nabla v + \nabla\rho\cdot\nabla v_i = g_i\rho + g\rho_i.$$

Multiplying by v_i and summing over i, we obtain

$$v_i \sum_j \rho_{ij} v_j + \nabla\rho \cdot \frac{1}{2} \nabla(v_i^2) = v_i \nabla\rho_i \cdot \nabla v + \nabla\rho \cdot (v_i \nabla v_i)$$

$$= \rho g_i v_i + g \rho_i v_i$$

i.e.

$$\sum_{i,j} \rho_{ij} v_i v_j + \frac{1}{2} \nabla\rho \cdot \nabla(|Dv|^2) = \rho \nabla g \cdot \nabla v + g \nabla\rho \cdot \nabla v$$

$$= \rho \nabla g \cdot \nabla v + g g \rho$$

by (II.7). Since $\rho = 0$ on $\partial\Omega$, we conclude that on $\partial\Omega$

$$\frac{1}{2} \nabla\rho \cdot \nabla(|Dv|^2) = - \sum_{i,j} \rho_{ij} v_i v_j .$$

Since

$$\frac{\partial}{\partial \nu}(|Dv|^2) = \nu \cdot \nabla(|Dv|^2) = \frac{\nabla\rho}{|\nabla\rho|} \cdot \nabla(|Dv|^2),$$

we have, on $\partial\Omega$, that

$$\text{(II.8)} \qquad \frac{\partial}{\partial \nu}(|Dv|^2) = - \frac{2}{|\nabla\rho|} \sum_{i,j} \rho_{ij} \, v_i v_j .$$

Now since Ω is convex, at each point $P \in \partial\Omega$, the matrix (ρ_{ij}) is positive semi-definite on the hyperplane tangent to $\partial\Omega$ at P. For those $v \in C^3(\bar{\Omega})$ with $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, we have $\nabla v \cdot \nu = 0$ on $\partial\Omega$, i.e. ∇v is perpendicular to ν (the normal to $\partial\Omega$) and ∇v must lie on the tangent space of $\partial\Omega$. Thus

$$\sum_{i,j} \rho_{ij} v_i v_j \geq 0$$

on $\partial\Omega$ and our proof is complete.

We are now ready for the main result.

<u>Theorem</u>. *If Ω is convex, then the only stable solutions of* (II.1) *are constants.*

Our approach is to show that if u is a non-constant solution of (II.1), then λ_1 (given by (II.5)) must be negative. We shall achieve this by choosing appropriate test functions in (II.3). However, it is natural to question a priori whether this approach would work. For, it seems that if $f' < 0$ on $\mathbb{R}$, then $\mathcal{H}(\varphi)$ is always positive for all $\varphi \not\equiv 0$ in $H^1(\Omega)$. It turns out that *if* $f' < 0$ *on* $\mathbb{R}$, *then* (II.1) *has no non-constant solutions*. To prove this, we let u be a solution of (II.1). Integrating the equation yields $\int_\Omega f(u(x))dx = 0$ and thus there exists a unique α such that $f(\alpha) = 0$ (since f is monotonically decreasing). Without loss of generality, we may assume that $\alpha = 0$, i.e. $f(0) = 0$. (For, we may set $v \equiv u - \alpha$, then $\Delta v + \tilde{f}(v) = 0$ and $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, where $\tilde{f}(v) = f(v+\alpha)$. Thus $\tilde{f}(0) = f(\alpha) = 0$.) Assume $u \not\equiv 0$, then $\{x \in \Omega |\ u(x) > 0\} \neq \phi$ and $\{x \in \Omega | u(x) < 0\} \neq \phi$. Let $u(P) = \max_{\bar{\Omega}} u$. Then $u(P) > 0$ and we have two cases:

(i) $P \in \Omega$. Since $f(u(P)) < 0$ ($f < 0$ on $\mathbb{R}_+$) we have $\Delta u(P) > 0$. On the other hand, u assumes its maximum at P, so $\Delta u(P) \le 0$, a contradiction.

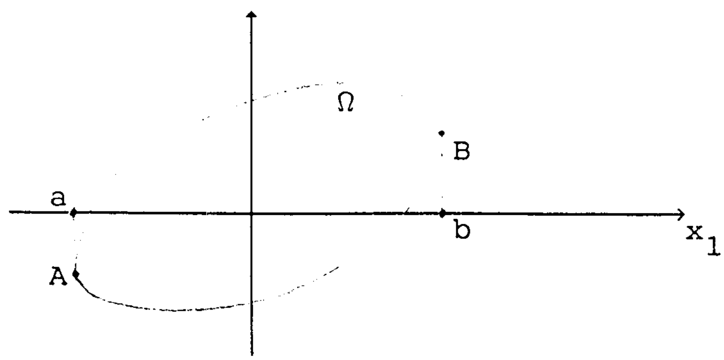
(ii) $P \in \partial\Omega$. Choose a ball $B \subseteq \Omega$ which is tangent to $\partial\Omega$ at P with $u > 0$ on $\bar{B}$. Then $f(u(x)) < 0$ on $\bar{B}$, and $\Delta u(x) > 0$ on B with $u(P) = \max_{\bar{B}} u$. by Hopf's boundary point lemma, $\frac{\partial u}{\partial \nu} > 0$ at P, which contradicts the boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

Coming back to our proof of the theorem, we choose $\varphi = u_i$. Then differentiating the equation in (II.1) gives $\Delta u_i + f'(u)u_i = 0$, and

$$\begin{aligned}
\sum_i \mathcal{H}(u_i) &= \sum_i \int_\Omega [|Du_i|^2 - f'(u)u_i^2] \\
&= \sum_i \int_\Omega [|Du_i|^2 + u_i \Delta u_i] \\
&= \sum_i \{\int_\Omega [|Du_i|^2 - |Du_i|^2] + \int_{\partial\Omega} u_i \frac{\partial u_i}{\partial \nu}\} \\
&= \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial \nu} |Du|^2 \\
&\leq 0
\end{aligned}$$

by our lemma above. If one of the $\mathcal{H}(u_i)$, $i = 1,2,\cdots,n$ is negative, then, we are done since $u_i \in H^1(\Omega)$. Therefore we only have to deal with the case that $\mathcal{H}(u_i) = 0$, $i = 1,2,\cdots,n$, and $\lambda_1 = 0$. We shall derive a contradiction. First of all, we note that under this assumption each u_i is an eigenfunction of λ_1. Since λ_1 is simple we see that for each i, there exists c_i such that $u_i = c_i \varphi_1$ where $\varphi_1 > 0$ is the normalized eigenfunction corresponding to λ_1 (i.e. $\|\varphi_1\|_{L^2(\Omega)} = 1$). Thus $Du = \vec{c}\,\varphi_1$ where $\vec{c} = (c_1,\cdots,c_n)$. This implies that u is constant when restricted to the hyperplane which is perpendicular to $\vec{c}$ (by Mean-Value Theorem); i.e. u is a function of one variable only.

If we rotate the coordinate system so that the new coordinate system, denoted by x', has its x_1'-axis pointing in the direction of $\vec{c}$, then u is a function of x_1' only. Since everything involved here are invariant under rotation, from now on, we shall be working with the new coordinate system x'. To keep the notations simple, we shall still denote this new coordinate by x, the domain by Ω. So we have $u(x) = u(x_1)$ where $x = (x_1,\cdots,x_n)$ and $Du(x) = (c\varphi_1(x), 0,\cdots,0)$ where $c = |\vec{c}|$. From the picture,



we see that on (a,b), $u'' + f(u) = 0$ since $\Delta u = u_{11} = u''$, and

$$-u'(a) = \frac{\partial u}{\partial \nu}(A) = 0 \quad \text{and} \quad u'(b) = \frac{\partial u}{\partial \nu}(B) = 0.$$

That is,

$$\begin{cases} u'' + f(u) = 0 & \text{in } (a,b) \\ u'(a) = u'(b) = 0. \end{cases}$$

Recall that $u_i = c_i\varphi_1$, this implies that in particular u_i

satisfies the homogeneous Neumann boundary condition

$$\frac{\partial u_i}{\partial \nu} = 0$$

on $\partial\Omega$, which in turn implies that at point A,

$$\frac{\partial u_1}{\partial \nu} = 0,$$

which is equivalent to $u_{11}(a) = 0$, i.e. $u''(a) = 0$. Now we have

$$\begin{cases} u'' + f(u) = 0 & \text{in } (a,b), \\ u'(a) = u''(a) = 0. \end{cases}$$

Thus at $x_1 = a$, $f(u(a)) = 0$ and $u \equiv u(a)$ is a solution of this problem. By the uniqueness of solutions of ordinary differential equations, $u \equiv u(a)$ is the only solution. Therefore u is also a constant in the x_1-direction which implies that u is identically a constant, this contradicts our assumption on u. Thus $\lambda_1 < 0$.

Remarks. (i) A similar question may be asked for solutions of Dirichlet problems. We know of rather little progress in this direction except Maginu's paper [Mg] for 1-dimensional case. In [LN1] this question was considered and the following statement was conjectured:

Conjecture. *A stable solution of the problem*

It is well known that (A) holds in bounded domains of $\mathbf{R}^n$, while (B) holds whenever φ is bounded away from 0 (i.e., $\varphi \geq \delta > 0$ on $\mathbf{R}^n$).

In [B-C-N], Berestycki, Caffarelli and Nirenberg verify that conjectures (A) and (B) hold in dimensions $n = 1, 2$ and inquire about their validity in higher dimension. In the next section, we give counterexamples that show the following:

Theorem 1.3: *Conjectures (A) and (B) are false for $n \geq 7$.*

We do not know the answer to the above conjectures in the intermediate dimensions $n = 3, \ldots, 6$, though we suspect that counterexamples are also in order.

2. The counterexamples

In [B-C-N], conjecture (A) is deduced from (B) which is then proved in dimensions 1 and 2. In this section, we include a direct proof for a version of (A) that is also valid in dimension 3 so that it can be applied in the proof of De Giorgi's problem. Some aspects of the proof are also relevant for the understanding of the counterexamples that will follow. The idea of using Ekeland's theorem to prove (A) seems to originate with H. Berestycki.

In the sequel, we shall let $L = -\Delta - V$ be a Schrödinger operator on $\mathbf{R}^n$ with a smooth and bounded potential V. Associate to L, its energy functional

$$\mathscr{L}(\psi) = \frac{\int_{\mathbf{R}^n}(|\nabla\psi|^2 - V|\psi|^2)dx}{\int_{\mathbf{R}^n}|\psi|^2dx}\ ; \text{ where } \psi \in C_0^\infty \text{ or } H^1(\mathbf{R}^n)\,.$$

We start with the following observation

Lemma 2.1: *If $u \in C^2$ is a bounded solution of $Lu = 0$, then $\lambda_1(L) \leq 0$.*

Proof: Let ℓ be any smooth function from $\mathbf{R}^+$ to $\mathbf{R}$ such that $\ell(t) = 1$ for $0 \leq t \leq 1$, $\ell(t) = 0$ for $2 < t$ and $|\ell'(t)| \leq 2$. For any $R > 0$, define on $\mathbf{R}^n$ the functions $\xi_R(x) = \ell(\frac{|x|}{R})$.

Since $\Delta u + Vu = 0$, multiply by $\xi_R^2 u$ and integrate by parts to get that:

$$\begin{aligned}\mathscr{L}(\xi_R u) &= \frac{\int_{\mathbf{R}^n}(|\nabla(\xi_R u)|^2 - V|(\xi_R u)|^2)dx}{\int_{\mathbf{R}^n}|(\xi_R u)|^2dx}\\ &= \frac{\int_{\mathbf{R}^n}|u|^2|\nabla\xi_R|^2dx}{\int_{\mathbf{R}^n}|(\xi_R u)|^2dx} \leq \frac{\frac{4}{R^2}\int_{B_{2R}\setminus B_R}|u|^2dx}{\int_{B_R}|u|^2dx}\,.\end{aligned}$$

Setting $K(R) = \int_{B_R}|u|^2dx$, we need to show that the infimum of $\alpha(R) := \frac{K(2R)-K(R)}{R^2K(R)}$ is zero, since then

$$\lambda_1(V) \leq \liminf_{R\to+\infty} \mathscr{L}(u\xi_R) = 0.$$

But if $\inf\alpha(R) \geq \delta > 0$, then $K(2R) \geq \delta R^2 K(R)$ and a straightforward iteration would then yield that for each integer m, $K(2^{m+1}) \geq C\delta^{m-1}R^{2(m-1)}$. From this follows that $K(R) \geq LR^{2\log_2 R}$, hence contradicting the fact that $K(R) \leq CR^n$ and the proposition is proved.

Remark 2.2: The above proposition may be extended as follows: If $u \in C^2$ satisfies $Lu = 0$ and $|u(x)e^{-A|x|}| \leq C|x|^m$ for some positive constants A, C, m, then $\lambda_1(V) \leq A$. This upper estimate for $\lambda_1(V)$ is actually optimal.

The following proposition is part of the folklore of the linear elliptic theory.

Proposition 2.3: *Let $L = -\Delta - V$ be a Schrödinger operator on $\mathbf{R}^n$ with a smooth and bounded potential V. Then $\lambda_1(V) < 0$ if and only if the equation $Lu = 0$ has no positive solutions.*

Proof: Suppose first that $\lambda_1(V) = 0$ and let (φ_R, λ_1^R) be the first eigenpair for the problem

$$\begin{aligned}(L - \lambda_1^R)\varphi_R &= 0 \text{ on } B_R \\ \varphi_R &= 0 \text{ on } \partial B_R\,.\end{aligned}$$

where B_R is the ball of radius R. The eigenfunctions can be chosen in such a way that $\varphi_R > 0$ on B_R and normalized so that $\varphi_R(0) = 1$.

Note that $\lambda_1^R \downarrow \lambda_1 = 0$ as $R \to \infty$. Harnack's inequality yields that for any compact subset K, $\frac{\max_K \varphi_R}{\min_K \varphi_R} \leq C(K)$ with the latter constant being independent of φ_R. Standard elliptic estimates also yield that the family $(\varphi_R)_R$ have also uniformly bounded derivatives on compact sets. It follows that for a subsequence $(R_k)_k$ going to infinity, $(\varphi_{R_k})_k$ converges in $C^2_{\text{loc}}(\mathbf{R}^n)$ to some $\varphi \in C^2$ and that $\varphi > 0$ on $\mathbf{R}^n$ while satisfying $L\varphi = 0$.

For the reverse implication, assume that $L\varphi = 0$ for some $\varphi > 0$ in C^2. If $\lambda_1(V) < 0$, there exists then a bounded domain Ω such that $\lambda_1(\Omega) := \lambda_1(V, \Omega) < 0$. Moreover, there exists $\varphi_\Omega > 0$ such that

$$\begin{aligned}(L - \lambda_1(\Omega))\varphi_\Omega &= 0 \text{ on } \Omega \\ \varphi_\Omega &= 0 \text{ on } \partial\Omega\,.\end{aligned}$$

Setting $w = \frac{\varphi_\Omega}{\varphi}$, one can immediately verify that

$$\begin{aligned}\nabla(\varphi^2\nabla w) + \varphi^2\lambda_1(\Omega)w &= 0 \text{ on } \Omega \\ w &= 0 \text{ on } \partial\Omega\,.\end{aligned}$$

Since $w > 0$ on Ω, this contradicts the fact that $\nabla(\varphi^2\nabla) + \varphi^2\lambda_1(\Omega)$ satisfies the maximum principle on Ω.

Theorem 2.4: *Let $L = -\Delta - V$ be a Schrödinger operator on $\mathbf{R}^n$ with a smooth and bounded potential V. Suppose that u is a bounded and sign-changing solution for $Lu = 0$.*

(1) If the dimension n is either 1 or 2, then $\lambda_1(V) < 0$.
(2) If $n = 3$ and if $|u(x)| \leq Ce^{-\alpha|x_3|}$ for $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ where C and α are positive constants, then $\lambda_1(V) < 0$.

Remarks 8. (i) The constant $C(N,p)$ is independent of both the given solution under consideration and of the domain Ω. We also note that we do not assume any boundary condition (actually, we do not require the existence of u on the boundary of Ω).

(ii) Let Ω be any proper domain containing the origin of $\mathbb{R}^N$. For any $N \geqslant 11$ and every $p \geqslant p_c(N)$, the radial functions

$$u_\alpha(x) = \alpha^{\frac{2}{p-1}} v(\alpha|x|), \quad \alpha > 0,$$

(v is defined in Theorem 5) are stable solutions of Eq. (1) in Ω. On the other hand, a direct computation yields: $u_\alpha(0) = \alpha^{\frac{2}{p-1}} \to +\infty$ as $\alpha \to +\infty$. This proves that universal estimate (11) cannot be true when $N \geqslant 11$ and $p \geqslant p_c(N)$, hence the upper bound on p in (10) is sharp.

(iii) In [15] N. Dancer proved a universal estimate similar to (11) for *positive* solutions of Eq. (1) in the subcritical case $1 < p < \frac{N+2}{N-2}$, $N \geqslant 3$ (without any stability assumption).

Theorem 13 can also be used to study the behaviour of stable solutions near an isolated singularity. More precisely we have:

Corollary 14. *Assume $R > 0$. Let $\Omega = B(0,2R) \setminus \{0\}$ and let $u \in C^2(\Omega)$ be a* stable *solution of* (1) *with*

$$\begin{cases} 1 < p < +\infty & \text{if } N \leqslant 10, \\ 1 < p < p_c(N) & \text{if } N \geqslant 11. \end{cases}$$

Then there exists a positive constant $C(N,p)$, depending only on p and N, such that

$$\forall x \in B(0,R) \quad |u(x)| \leqslant C(N,p)|x|^{-\frac{2}{p-1}}, \tag{13}$$

$$\forall x \in B(0,R) \quad |\nabla u(x)| \leqslant C(N,p)|x|^{-1-\frac{2}{p-1}}. \tag{14}$$

Remark 9. B. Gidas and J. Spruck [24], M.-F. Bidaut-Véron and L. Véron [6] proved the behaviour (13), near an isolated singularity, for *positive* solutions of Eq. (1) in the subcritical case $1 < p < \frac{N+2}{N-2}$, $N \geqslant 3$ (without any stability assumption).

2. Proofs of Propositions 4 and 6

This section is devoted to the proof of Propositions 4 and 6. These results are crucial for the present work.

Proof of Proposition 4. We split the proof into four steps:

Step 1. *For any $\varphi \in C_c^2(\Omega)$ we have*:

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 = \frac{(\gamma+1)^2}{4\gamma}\int_\Omega |u|^{p+\gamma}\varphi^2 + \frac{\gamma+1}{4\gamma}\int_\Omega |u|^{\gamma+1}\Delta(\varphi^2). \tag{2.1}$$

Multiply Eq. (1) by $|u|^{\gamma-1}u\varphi^2$ and integrate by parts to find

$$\int_\Omega \gamma|\nabla u|^2|u|^{\gamma-1}\varphi^2 + \int_\Omega \nabla u\nabla(\varphi^2)|u|^{\gamma-1}u = \int_\Omega |u|^{p+\gamma}\varphi^2,$$

therefore

$$\frac{\gamma}{(\frac{\gamma+1}{2})^2}\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + \int_\Omega \nabla\left(\frac{|u|^{\gamma+1}}{\gamma+1}\right)\nabla(\varphi^2)$$
$$= \frac{\gamma}{(\frac{\gamma+1}{2})^2}\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 - \int_\Omega \frac{|u|^{\gamma+1}}{\gamma+1}\Delta(\varphi^2) = \int_\Omega |u|^{p+\gamma}\varphi^2.$$

Identity (2.1) then follows by multiplying the latter identity by the factor $(\frac{\gamma+1}{2})^2/\gamma$.

Step 2. *For any* $\varphi \in C_c^2(\Omega)$ *we have*:

$$\left(p - \frac{(\gamma+1)^2}{4\gamma}\right)\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 + \left(\frac{\gamma+1}{4\gamma} - \frac{1}{2}\right)\int_\Omega |u|^{\gamma+1}\Delta(\varphi^2). \tag{2.2}$$

The function $\psi = |u|^{\frac{\gamma-1}{2}}u\varphi$ belongs to $C_c^1(\Omega)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives:

$$\begin{aligned} p\int_\Omega |u|^{p+\gamma}\varphi^2 &\leqslant \int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + \int_\Omega \left(|u|^{\frac{\gamma-1}{2}}u\right)^2|\nabla\varphi|^2 + \int_\Omega 2\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\nabla\varphi|u|^{\frac{\gamma-1}{2}}u\varphi \\ &= \int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 - \int_\Omega \frac{1}{2}|u|^{\gamma+1}\Delta(\varphi^2). \end{aligned} \tag{2.3}$$

Using (2.1) in the latter, we obtain:

$$p\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \frac{(\gamma+1)^2}{4\gamma}\int_\Omega |u|^{p+\gamma}\varphi^2 + \frac{\gamma+1}{4\gamma}\int_\Omega |u|^{\gamma+1}\Delta(\varphi^2) + \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 - \int_\Omega \frac{1}{2}|u|^{\gamma+1}\Delta(\varphi^2),$$

which immediately gives identity (2.2).

Step 3. *For any* $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ *and any integer* $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$ *there exists a constant* $C(p,m,\gamma)$, *depending only on* p, m *and* γ, *such that*

$$\int_\Omega |u|^{p+\gamma}\psi^{2m} \leqslant C(p,m,\gamma)\int_\Omega \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.4}$$

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} \leqslant C(p,m,\gamma)\int_\Omega \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.5}$$

for all test functions $\psi \in C_c^2(\Omega)$ *satisfying* $|\psi| \leqslant 1$ *in* Ω. *Moreover, the constant* $C(p,m,\gamma)$ *can be explicitly computed.*

From (2.2), we obtain that

$$\forall\varphi \in C_c^2(\Omega) \quad \alpha\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 + \beta\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi, \tag{2.6}$$

where we have set $\alpha = p - \frac{(\gamma+1)^2}{4\gamma}$ and $\beta = \frac{1-\gamma}{4\gamma}$. Notice that $\alpha > 0$ and $\beta \leqslant 0$, since $p > 1$ and $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$.

For any $\psi \in C_c^2(\Omega)$, with $|\psi| \leqslant 1$ in Ω, we set $\varphi = \psi^m$. The function φ belongs to $C_c^2(\Omega)$, since $m \geqslant 2$ and m is an integer, hence it can be used in (2.6). A direct computation gives:

$$\alpha\int_\Omega |u|^{p+\gamma}\psi^{2m} \leqslant \int_\Omega |u|^{\gamma+1}\psi^{2m-2}\left[m^2|\nabla\psi|^2 + \beta m(m-1)|\nabla\psi|^2 + \beta m\psi\Delta\psi\right], \tag{2.7}$$

hence

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\int_\Omega |u|^{\gamma+1}|\psi|^{2m-2}\left[|\nabla\psi|^2 + |\psi\Delta\psi|\right], \tag{2.8}$$

with $C_1 = \frac{m^2+\beta m(m-1)}{\alpha} > -\frac{\beta m}{\alpha} \geqslant 0$.

An application of Holder's inequality yields:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\left(\int_\Omega \left[|u|^{\gamma+1}|\psi|^{2m-2}\right]^{\frac{p+\gamma}{1+\gamma}}\right)^{\frac{1+\gamma}{p+\gamma}}\left(\int_\Omega \left[|\nabla\psi|^2 + |\psi\Delta\psi|\right]^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}. \tag{2.9}$$

At this point we notice that $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$ implies $(2m-2)(\frac{p+\gamma}{1+\gamma}) \geqslant 2m$ and thus $|\psi|^{(2m-2)(\frac{p+\gamma}{1+\gamma})} \leqslant |\psi|^{2m}$ in Ω, since $|\psi| \leqslant 1$ everywhere in Ω.

Therefore, we obtain:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\left(\int_\Omega |u|^{p+\gamma}|\psi|^{2m}\right)^{\frac{1+\gamma}{p+\gamma}}\left(\int_\Omega \left[|\nabla\psi|^2+|\psi\Delta\psi|\right]^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}. \tag{2.10}$$

The latter immediately implies:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1^{\frac{p+\gamma}{p-1}}\int_\Omega \left(|\nabla\psi|^2+|\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.11}$$

which proves inequality (2.4) with $C(p,m,\gamma) = C_1^{\frac{p+\gamma}{p-1}}$.

To prove (2.5) we combine (2.1) and (2.6). This leads to

$$\begin{aligned}\forall\varphi\in C_c^2(\Omega)\quad \int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 &\leqslant \frac{(\gamma+1)^2}{4\gamma}\left[\frac{1}{\alpha}\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+\frac{\beta}{\alpha}\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi\right]\\ &\quad+\frac{(\gamma+1)}{2\gamma}\left[\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi\right]\\ &= A\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+B\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi,\end{aligned} \tag{2.12}$$

where $A = \frac{(\gamma+1)^2}{4\gamma\alpha}+\frac{(\gamma+1)}{2\gamma} > 0$ and $B = \frac{\beta(\gamma+1)^2}{4\gamma\alpha}+\frac{(\gamma+1)}{2\gamma} \in \mathbb{R}$.

Now, we insert the test function $\varphi = \psi^m$ in the latter inequality to find,

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} \leqslant \int_\Omega |u|^{\gamma+1}\psi^{2m-2}\left[Am^2|\nabla\psi|^2+Bm(m-1)|\nabla\psi|^2+Bm\psi\Delta\psi\right], \tag{2.13}$$

and hence

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} \leqslant C_2\int_\Omega |u|^{\gamma+1}|\psi|^{2m-2}\left[|\nabla\psi|^2+|\psi\Delta\psi|\right], \tag{2.14}$$

with $C_2 = \max\{|Am^2+Bm(m-1)|, |Bm|\} > 0$. Using Holder's inequality in (2.14) yields:

$$\begin{aligned}\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} &\leqslant C_2\left(\int_\Omega \left[|u|^{\gamma+1}|\psi|^{2m-2}\right]^{\frac{p+\gamma}{1+\gamma}}\right)^{\frac{1+\gamma}{p+\gamma}}\left(\int_\Omega \left[|\nabla\psi|^2+|\psi\Delta\psi|\right]^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}\\ &\leqslant C_2\left(\int_\Omega |u|^{p+\gamma}|\psi|^{2m}\right)^{\frac{1+\gamma}{p+\gamma}}\left(\int_\Omega \left[|\nabla\psi|^2+|\psi\Delta\psi|\right]^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}.\end{aligned}$$

Finally, inserting (2.11) into the latter we obtain:

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} \leqslant C_2C_1^{\frac{1+\gamma}{p-1}}\int_\Omega \left(|\nabla\psi|^2+|\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.15}$$

which gives the desired inequality (2.5).

Step 4. *End of proof.* The desired conclusion follows immediately by adding inequality (2.4) to inequality (2.5). □

Proof of Proposition 6. Since Ω is smooth, $u \in C^2(\overline{\Omega})$, and u *vanishes on* $\partial\Omega$ we can proceed as in the proof of Proposition 4. Only some minor modifications are needed. Step 1 goes without any change if we remark that, for any

$\varphi \in C_c^2(\mathbb{R}^N \setminus \mathcal{K})$, the function $|u|^{\gamma-1}u\varphi^2$ belongs to $C^1(\overline{\Omega \setminus \mathcal{K}})$, has bounded support contained in $\overline{\Omega} \setminus \mathcal{K}$ and vanishes on $\partial\Omega$. In the same way, Step 2 can be carried over since, for any $\varphi \in C_c^2(\mathbb{R}^N \setminus \mathcal{K})$, the function $|u|^{\frac{\gamma-1}{2}}u\varphi$ belongs to $C^1(\overline{\Omega \setminus \mathcal{K}})$, has bounded support contained in $\overline{\Omega} \setminus \mathcal{K}$ and vanishes on $\partial\Omega$. In particular, it belongs to $H_0^1(\Omega)$ and hence it can be used as test function in the quadratic form Q_u. The rest of the proof is unchanged and for this reason we omit the details. □

3. Stable solutions

In this section we prove all the results concerning the classification of stable solutions, i.e., Theorems 1, 7 and Proposition 8. Let us start with:

Proof of Theorem 1. For every $R > 0$, we consider the function $\psi_R(x) = \varphi(\frac{|x|}{R})$, where $\varphi \in C_c^2(\mathbb{R})$, $0 \leqslant \varphi \leqslant 1$ everywhere on $\mathbb{R}$, and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| \geqslant 2. \end{cases}$$

Let us fix $p > 1$. We first observe that for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$, Proposition 4 yields,

$$\int\limits_{B(0,R)} \left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 + |u|^{p+\gamma}\right) \leqslant C_{p,m,\gamma} \int\limits_{\mathbb{R}^N} \left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{\frac{p+\gamma}{p-1}}$$
$$\leqslant C(p,\gamma,m,N,\varphi) R^{N-2(\frac{p+\gamma}{p-1})} \quad \forall R > 0, \tag{3.1}$$

where $B(0,R)$ denotes the open ball centered at the origin and with radius R, and $C(p,\gamma,m,N,\varphi)$ is a positive constant independent of R.

Next we claim that, under the assumptions on the exponent p assumed in Theorem 1, we can always choose $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that

$$N - 2\left(\frac{p+\gamma}{p-1}\right) < 0. \tag{3.2}$$

To this end, we set $\gamma_M(p) = 2p + 2\sqrt{p(p-1)} - 1$ and we consider separately the case $N \leqslant 10$ and the case $N \geqslant 11$.

First case: $N \leqslant 10$ *and* $p > 1$. In this case we have:

$$p + \gamma_M(p) > 3p - 1 + 2(p-1) > 5(p-1)$$

and therefore

$$N - 2\left(\frac{p + \gamma_M(p)}{p-1}\right) < N - 10 \leqslant 0. \tag{3.3}$$

The latter inequality and the continuity of the function $t \to N - 2(\frac{p+t}{p-1})$ immediately imply the existence of $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ satisfying (3.2).

Second case: $N \geqslant 11$ *and* $1 < p < p_c(N)$. In this case we consider the real-valued function $(1, +\infty) \ni t \to f(t) := 2(\frac{t+\gamma_M(t)}{t-1})$. Since f is a strictly decreasing function satisfying $\lim_{t\to 1^+} f(t) = +\infty$ and $\lim_{t\to+\infty} f(t) = 10$, there exists a unique $p_0 > 1$ such that $N = 2(\frac{p_0+\gamma_M(p_0)}{p_0-1})$. We claim that $p_0 = p_c(N)$. Indeed,

$$N = 2\left(\frac{p + \gamma_M(p)}{p-1}\right) \quad \Leftrightarrow \quad (N-2)(p-1) - 4p = 4\sqrt{p(p-1)},$$

which implies that p_0 satisfies:

$$\big((N-2)(N-10)\big)p_0^2 + \big(-2(N-2)^2 + 8N\big)p_0 + (N-2)^2 = 0, \tag{3.4}$$

and

$$(N-2)(p_0-1)-4p_0>4(p_0-1). \tag{3.5}$$

The roots of Eq. (3.4) are

$$p_1=\frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}=p_c(N), \tag{3.6}$$

$$p_2=\frac{(N-2)^2-4N-8\sqrt{N-1}}{(N-2)(N-10)}<p_c(N), \tag{3.7}$$

while (3.5) easily implies $p_0>\frac{N-6}{N-10}=\frac{(N-6)(N-2)}{(N-2)(N-10)}=\frac{(N-2)^2-4N+8}{(N-2)(N-10)}>p_2$. This proves that $p_0=p_c(N)$, as claimed.

Since we have just proven that $f(p_c(N))=N$ and f is a strictly decreasing function, it follows that

$$\forall\ 1<p<p_c(N)\quad N<f(p)=2\left(\frac{p+\gamma_M(p)}{p-1}\right) \tag{3.8}$$

Now we can conclude as in the first case, i.e, the continuity of $t\to N-2(\frac{p+t}{p-1})$ immediately implies the existence of $\gamma\in[1,2p+2\sqrt{p(p-1)}-1)$ satisfying (3.2).

We have proven that, under the assumptions of Theorem 1, there always exists a real $\gamma\in[1,2p+2\sqrt{p(p-1)}-1)$ satisfying (3.2). Therefore, by letting $R\to+\infty$ in (3.1), we deduce:

$$\int_{\mathbb{R}^N}\left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2+|u|^{p+\gamma}\right)=0,$$

which yields $u\equiv 0$. This result concludes the first part of the proof of Theorem 1.

To prove the second claim of Theorem 1 we invoke Theorem 5 (see case (b)). □

Proof of Theorem 7. Fix an integer $m\geqslant\max\{\frac{p+\gamma}{p-1},2\}$ and notice that, as in the proof of the first part of Theorem 1 we have (here, instead of Proposition 4 with test functions ψ_R, we use Proposition 6 with $\psi_{R,x_0}(x):=\varphi(\frac{|x-x_0|}{R})$ and $\mathcal{K}:=\emptyset$):

$$\begin{aligned}\int_{\Omega\cap B(x_0,\frac{R}{2})}\left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2+|u|^{p+\gamma}\right)&\leqslant C_{p,m,\gamma}\int_{\Omega\cap B(x_0,2R)}\left(|\nabla\psi_{R,x_0}|^2+|\psi_{R,x_0}||\Delta\psi_{R,x_0}|\right)^{(p+\gamma)/(p-1)}\\&\leqslant C(p,\gamma,m,N,\varphi)\left[\frac{\mathcal{L}_N(\Omega\cap B(x_0,2R))}{R^{2(\frac{p+\gamma}{p-1})}}\right],\end{aligned} \tag{3.9}$$

where $B(0,t)$ denotes the open ball centered at the origin and with radius t, and $C(p,\gamma,m,N)$ is a positive constant independent of R. The desired conclusion then follows by letting $R\to+\infty$ in (3.9) and using the assumption (4). □

Proof of Proposition 8. A direct computation proves that any of the cases considered in the statement of Proposition 8 implies the volume growth condition (4). □

4. Non-negative solutions

Here we prove Theorems 11 and 12.

Proof of Theorem 11. We claim that u is a stable solution of (6). Indeed, by the strong minimum principle either $u\equiv 0$, and then u is stable, or $u>0$ in Ω. In the latter case, since Ω is a coercive epigraph, a result of M.J. Esteban and P.-L. Lions (cf. Proposition II.1 on page 8 of [18]) implies that $\frac{\partial u}{\partial x_N}>0$ in Ω. Therefore $\frac{\partial u}{\partial x_N}$ is a positive solution of the linearized equation:

$$-\Delta s-pu^{p-1}s=0\quad\text{in }\Omega, \tag{4.1}$$