Remarks 8. (i) The constant $C(N,p)$ is independent of both the given solution under consideration and of the domain Ω. We also note that we do not assume any boundary condition (actually, we do not require the existence of u on the boundary of Ω).

(ii) Let Ω be any proper domain containing the origin of $\mathbb{R}^N$. For any $N \geqslant 11$ and every $p \geqslant p_c(N)$, the radial functions

$$u_\alpha(x) = \alpha^{\frac{2}{p-1}} v(\alpha|x|), \quad \alpha > 0,$$

(v is defined in Theorem 5) are stable solutions of Eq. (1) in Ω. On the other hand, a direct computation yields: $u_\alpha(0) = \alpha^{\frac{2}{p-1}} \to +\infty$ as $\alpha \to +\infty$. This proves that universal estimate (11) cannot be true when $N \geqslant 11$ and $p \geqslant p_c(N)$, hence the upper bound on p in (10) is sharp.

(iii) In [15] N. Dancer proved a universal estimate similar to (11) for *positive* solutions of Eq. (1) in the subcritical case $1 < p < \frac{N+2}{N-2}$, $N \geqslant 3$ (without any stability assumption).

Theorem 13 can also be used to study the behaviour of stable solutions near an isolated singularity. More precisely we have:

Corollary 14. *Assume $R > 0$. Let $\Omega = B(0,2R) \setminus \{0\}$ and let $u \in C^2(\Omega)$ be a* stable *solution of* (1) *with*

$$\begin{cases} 1 < p < +\infty & \text{if } N \leqslant 10, \\ 1 < p < p_c(N) & \text{if } N \geqslant 11. \end{cases}$$

Then there exists a positive constant $C(N,p)$, depending only on p and N, such that

$$\forall x \in B(0,R) \quad |u(x)| \leqslant C(N,p)|x|^{-\frac{2}{p-1}}, \tag{13}$$

$$\forall x \in B(0,R) \quad |\nabla u(x)| \leqslant C(N,p)|x|^{-1-\frac{2}{p-1}}. \tag{14}$$

Remark 9. B. Gidas and J. Spruck [24], M.-F. Bidaut-Véron and L. Véron [6] proved the behaviour (13), near an isolated singularity, for *positive* solutions of Eq. (1) in the subcritical case $1 < p < \frac{N+2}{N-2}$, $N \geqslant 3$ (without any stability assumption).

2. Proofs of Propositions 4 and 6

This section is devoted to the proof of Propositions 4 and 6. These results are crucial for the present work.

Proof of Proposition 4. We split the proof into four steps:

Step 1. *For any $\varphi \in C_c^2(\Omega)$ we have*:

$$\int_\Omega |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi^2 = \frac{(\gamma+1)^2}{4\gamma} \int_\Omega |u|^{p+\gamma} \varphi^2 + \frac{\gamma+1}{4\gamma} \int_\Omega |u|^{\gamma+1} \Delta(\varphi^2). \tag{2.1}$$

Multiply Eq. (1) by $|u|^{\gamma-1} u \varphi^2$ and integrate by parts to find

$$\int_\Omega \gamma |\nabla u|^2 |u|^{\gamma-1} \varphi^2 + \int_\Omega \nabla u \nabla(\varphi^2) |u|^{\gamma-1} u = \int_\Omega |u|^{p+\gamma} \varphi^2,$$

therefore

$$\frac{\gamma}{(\frac{\gamma+1}{2})^2} \int_\Omega |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi^2 + \int_\Omega \nabla\left(\frac{|u|^{\gamma+1}}{\gamma+1}\right) \nabla(\varphi^2)$$

$$= \frac{\gamma}{(\frac{\gamma+1}{2})^2} \int_\Omega |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi^2 - \int_\Omega \frac{|u|^{\gamma+1}}{\gamma+1} \Delta(\varphi^2) = \int_\Omega |u|^{p+\gamma} \varphi^2.$$

Identity (2.1) then follows by multiplying the latter identity by the factor $(\frac{\gamma+1}{2})^2/\gamma$.

Step 2. For any $\varphi \in C_c^2(\Omega)$ we have:

$$\left(p - \frac{(\gamma+1)^2}{4\gamma}\right)\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 + \left(\frac{\gamma+1}{4\gamma} - \frac{1}{2}\right)\int_\Omega |u|^{\gamma+1}\Delta(\varphi^2). \tag{2.2}$$

The function $\psi = |u|^{\frac{\gamma-1}{2}}u\varphi$ belongs to $C_c^1(\Omega)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives:

$$\begin{aligned} p\int_\Omega |u|^{p+\gamma}\varphi^2 &\leqslant \int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + \int_\Omega \left(|u|^{\frac{\gamma-1}{2}}u\right)^2|\nabla\varphi|^2 + \int_\Omega 2\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\nabla\varphi|u|^{\frac{\gamma-1}{2}}u\varphi \\ &= \int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\varphi^2 + \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 - \int_\Omega \frac{1}{2}|u|^{\gamma+1}\Delta(\varphi^2). \end{aligned} \tag{2.3}$$

Using (2.1) in the latter, we obtain:

$$p\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \frac{(\gamma+1)^2}{4\gamma}\int_\Omega |u|^{p+\gamma}\varphi^2 + \frac{\gamma+1}{4\gamma}\int_\Omega |u|^{\gamma+1}\Delta(\varphi^2) + \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 - \int_\Omega \frac{1}{2}|u|^{\gamma+1}\Delta(\varphi^2),$$

which immediately gives identity (2.2).

Step 3. For any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$ there exists a constant $C(p,m,\gamma)$, depending only on p, m and γ, such that

$$\int_\Omega |u|^{p+\gamma}\psi^{2m} \leqslant C(p,m,\gamma)\int_\Omega \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.4}$$

$$\int_\Omega \left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2\psi^{2m} \leqslant C(p,m,\gamma)\int_\Omega \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{\frac{p+\gamma}{p-1}}, \tag{2.5}$$

for all test functions $\psi \in C_c^2(\Omega)$ satisfying $|\psi| \leqslant 1$ in Ω. Moreover, the constant $C(p,m,\gamma)$ can be explicitly computed.

From (2.2), we obtain that

$$\forall\varphi \in C_c^2(\Omega) \quad \alpha\int_\Omega |u|^{p+\gamma}\varphi^2 \leqslant \int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2 + \beta\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi, \tag{2.6}$$

where we have set $\alpha = p - \frac{(\gamma+1)^2}{4\gamma}$ and $\beta = \frac{1-\gamma}{4\gamma}$. Notice that $\alpha > 0$ and $\beta \leqslant 0$, since $p > 1$ and $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$.

For any $\psi \in C_c^2(\Omega)$, with $|\psi| \leqslant 1$ in Ω, we set $\varphi = \psi^m$. The function φ belongs to $C_c^2(\Omega)$, since $m \geqslant 2$ and m is an integer, hence it can be used in (2.6). A direct computation gives:

$$\alpha\int_\Omega |u|^{p+\gamma}\psi^{2m} \leqslant \int_\Omega |u|^{\gamma+1}\psi^{2m-2}\left[m^2|\nabla\psi|^2 + \beta m(m-1)|\nabla\psi|^2 + \beta m\psi\Delta\psi\right], \tag{2.7}$$

hence

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\int_\Omega |u|^{\gamma+1}|\psi|^{2m-2}\left[|\nabla\psi|^2 + |\psi\Delta\psi|\right], \tag{2.8}$$

with $C_1 = \frac{m^2+\beta m(m-1)}{\alpha} > -\frac{\beta m}{\alpha} \geqslant 0$.

An application of Holder's inequality yields:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\left(\int_\Omega \left[|u|^{\gamma+1}|\psi|^{2m-2}\right]^{\frac{p+\gamma}{1+\gamma}}\right)^{\frac{1+\gamma}{p+\gamma}}\left(\int_\Omega \left[|\nabla\psi|^2 + |\psi\Delta\psi|\right]^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}. \tag{2.9}$$

At this point we notice that $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$ implies $(2m-2)(\frac{p+\gamma}{1+\gamma}) \geqslant 2m$ and thus $|\psi|^{(2m-2)(\frac{p+\gamma}{1+\gamma})} \leqslant |\psi|^{2m}$ in Ω, since $|\psi| \leqslant 1$ everywhere in Ω.

Therefore, we obtain:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1\Big(\int_\Omega |u|^{p+\gamma}|\psi|^{2m}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega \big[|\nabla\psi|^2+|\psi\Delta\psi|\big]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}. \tag{2.10}$$

The latter immediately implies:

$$\int_\Omega |u|^{p+\gamma}|\psi|^{2m} \leqslant C_1^{\frac{p+\gamma}{p-1}}\int_\Omega \big(|\nabla\psi|^2+|\psi||\Delta\psi|\big)^{\frac{p+\gamma}{p-1}}, \tag{2.11}$$

which proves inequality (2.4) with $C(p,m,\gamma) = C_1^{\frac{p+\gamma}{p-1}}$.

To prove (2.5) we combine (2.1) and (2.6). This leads to

$$\begin{aligned}\forall\varphi \in C_c^2(\Omega)\quad \int_\Omega \big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\varphi^2 &\leqslant \frac{(\gamma+1)^2}{4\gamma}\Big[\frac{1}{\alpha}\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+\frac{\beta}{\alpha}\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi\Big]\\ &\quad+\frac{(\gamma+1)}{2\gamma}\Big[\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi\Big]\\ &= A\int_\Omega |u|^{\gamma+1}|\nabla\varphi|^2+B\int_\Omega |u|^{\gamma+1}\varphi\Delta\varphi,\end{aligned} \tag{2.12}$$

where $A = \frac{(\gamma+1)^2}{4\gamma\alpha}+\frac{(\gamma+1)}{2\gamma} > 0$ and $B = \frac{\beta(\gamma+1)^2}{4\gamma\alpha}+\frac{(\gamma+1)}{2\gamma} \in \mathbb{R}$.

Now, we insert the test function $\varphi = \psi^m$ in the latter inequality to find,

$$\int_\Omega \big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2m} \leqslant \int_\Omega |u|^{\gamma+1}\psi^{2m-2}\big[Am^2|\nabla\psi|^2+Bm(m-1)|\nabla\psi|^2+Bm\psi\Delta\psi\big], \tag{2.13}$$

and hence

$$\int_\Omega \big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2m} \leqslant C_2\int_\Omega |u|^{\gamma+1}|\psi|^{2m-2}\big[|\nabla\psi|^2+|\psi\Delta\psi|\big], \tag{2.14}$$

with $C_2 = \max\{|Am^2+Bm(m-1)|, |Bm|\} > 0$. Using Holder's inequality in (2.14) yields:

$$\begin{aligned}\int_\Omega \big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2m} &\leqslant C_2\Big(\int_\Omega \big[|u|^{\gamma+1}|\psi|^{2m-2}\big]^{\frac{p+\gamma}{1+\gamma}}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega \big[|\nabla\psi|^2+|\psi\Delta\psi|\big]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}\\ &\leqslant C_2\Big(\int_\Omega |u|^{p+\gamma}|\psi|^{2m}\Big)^{\frac{1+\gamma}{p+\gamma}}\Big(\int_\Omega \big[|\nabla\psi|^2+|\psi\Delta\psi|\big]^{\frac{p+\gamma}{p-1}}\Big)^{\frac{p-1}{p+\gamma}}.\end{aligned}$$

Finally, inserting (2.11) into the latter we obtain:

$$\int_\Omega \big|\nabla\big(|u|^{\frac{\gamma-1}{2}}u\big)\big|^2\psi^{2m} \leqslant C_2C_1^{\frac{1+\gamma}{p-1}}\int_\Omega \big(|\nabla\psi|^2+|\psi||\Delta\psi|\big)^{\frac{p+\gamma}{p-1}}, \tag{2.15}$$

which gives the desired inequality (2.5).

Step 4. *End of proof.* The desired conclusion follows immediately by adding inequality (2.4) to inequality (2.5). □

Proof of Proposition 6. Since Ω is smooth, $u \in C^2(\overline{\Omega})$, and u *vanishes on* $\partial\Omega$ we can proceed as in the proof of Proposition 4. Only some minor modifications are needed. Step 1 goes without any change if we remark that, for any

$\varphi \in C_c^2(\mathbb{R}^N \setminus \mathcal{K})$, the function $|u|^{\gamma-1}u\varphi^2$ belongs to $C^1(\overline{\Omega \setminus \mathcal{K}})$, has bounded support contained in $\overline{\Omega} \setminus \mathcal{K}$ and vanishes on $\partial\Omega$. In the same way, Step 2 can be carried over since, for any $\varphi \in C_c^2(\mathbb{R}^N \setminus \mathcal{K})$, the function $|u|^{\frac{\gamma-1}{2}}u\varphi$ belongs to $C^1(\overline{\Omega \setminus \mathcal{K}})$, has bounded support contained in $\overline{\Omega} \setminus \mathcal{K}$ and vanishes on $\partial\Omega$. In particular, it belongs to $H_0^1(\Omega)$ and hence it can be used as test function in the quadratic form Q_u. The rest of the proof is unchanged and for this reason we omit the details. □

3. Stable solutions

In this section we prove all the results concerning the classification of stable solutions, i.e., Theorems 1, 7 and Proposition 8. Let us start with:

Proof of Theorem 1. For every $R > 0$, we consider the function $\psi_R(x) = \varphi(\frac{|x|}{R})$, where $\varphi \in C_c^2(\mathbb{R})$, $0 \leqslant \varphi \leqslant 1$ everywhere on $\mathbb{R}$, and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| \geqslant 2. \end{cases}$$

Let us fix $p > 1$. We first observe that for any $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and any integer $m \geqslant \max\{\frac{p+\gamma}{p-1}, 2\}$, Proposition 4 yields,

$$\int_{B(0,R)} \left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2 + |u|^{p+\gamma}\right) \leqslant C_{p,m,\gamma} \int_{\mathbb{R}^N} \left(|\nabla\psi_R|^2 + |\psi_R||\Delta\psi_R|\right)^{\frac{p+\gamma}{p-1}}$$
$$\leqslant C(p,\gamma,m,N,\varphi) R^{N-2(\frac{p+\gamma}{p-1})} \quad \forall R > 0, \tag{3.1}$$

where $B(0,R)$ denotes the open ball centered at the origin and with radius R, and $C(p,\gamma,m,N,\varphi)$ is a positive constant independent of R.

Next we claim that, under the assumptions on the exponent p assumed in Theorem 1, we can always choose $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ such that

$$N - 2\left(\frac{p+\gamma}{p-1}\right) < 0. \tag{3.2}$$

To this end, we set $\gamma_M(p) = 2p + 2\sqrt{p(p-1)} - 1$ and we consider separately the case $N \leqslant 10$ and the case $N \geqslant 11$.

First case: $N \leqslant 10$ *and* $p > 1$. In this case we have:

$$p + \gamma_M(p) > 3p - 1 + 2(p-1) > 5(p-1)$$

and therefore

$$N - 2\left(\frac{p + \gamma_M(p)}{p-1}\right) < N - 10 \leqslant 0. \tag{3.3}$$

The latter inequality and the continuity of the function $t \to N - 2(\frac{p+t}{p-1})$ immediately imply the existence of $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ satisfying (3.2).

Second case: $N \geqslant 11$ *and* $1 < p < p_c(N)$. In this case we consider the real-valued function $(1, +\infty) \ni t \to f(t) := 2(\frac{t+\gamma_M(t)}{t-1})$. Since f is a strictly decreasing function satisfying $\lim_{t\to 1^+} f(t) = +\infty$ and $\lim_{t\to+\infty} f(t) = 10$, there exists a unique $p_0 > 1$ such that $N = 2(\frac{p_0+\gamma_M(p_0)}{p_0-1})$. We claim that $p_0 = p_c(N)$. Indeed,

$$N = 2\left(\frac{p + \gamma_M(p)}{p-1}\right) \quad \Leftrightarrow \quad (N-2)(p-1) - 4p = 4\sqrt{p(p-1)},$$

which implies that p_0 satisfies:

$$\big((N-2)(N-10)\big)p_0^2 + \big(-2(N-2)^2 + 8N\big)p_0 + (N-2)^2 = 0, \tag{3.4}$$

and

$$(N-2)(p_0-1)-4p_0>4(p_0-1). \tag{3.5}$$

The roots of Eq. (3.4) are

$$p_1=\frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)}=p_c(N), \tag{3.6}$$

$$p_2=\frac{(N-2)^2-4N-8\sqrt{N-1}}{(N-2)(N-10)}<p_c(N), \tag{3.7}$$

while (3.5) easily implies $p_0>\frac{N-6}{N-10}=\frac{(N-6)(N-2)}{(N-2)(N-10)}=\frac{(N-2)^2-4N+8}{(N-2)(N-10)}>p_2$. This proves that $p_0=p_c(N)$, as claimed.

Since we have just proven that $f(p_c(N))=N$ and f is a strictly decreasing function, it follows that

$$\forall\ 1<p<p_c(N)\quad N<f(p)=2\left(\frac{p+\gamma_M(p)}{p-1}\right) \tag{3.8}$$

Now we can conclude as in the first case, i.e, the continuity of $t\to N-2(\frac{p+t}{p-1})$ immediately implies the existence of $\gamma\in[1,2p+2\sqrt{p(p-1)}-1)$ satisfying (3.2).

We have proven that, under the assumptions of Theorem 1, there always exists a real $\gamma\in[1,2p+2\sqrt{p(p-1)}-1)$ satisfying (3.2). Therefore, by letting $R\to+\infty$ in (3.1), we deduce:

$$\int_{\mathbb{R}^N}\left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2+|u|^{p+\gamma}\right)=0,$$

which yields $u\equiv 0$. This result concludes the first part of the proof of Theorem 1.

To prove the second claim of Theorem 1 we invoke Theorem 5 (see case (b)). □

Proof of Theorem 7. Fix an integer $m\geqslant\max\{\frac{p+\gamma}{p-1},2\}$ and notice that, as in the proof of the first part of Theorem 1 we have (here, instead of Proposition 4 with test functions ψ_R, we use Proposition 6 with $\psi_{R,x_0}(x):=\varphi(\frac{|x-x_0|}{R})$ and $\mathcal{K}=\emptyset$):

$$\int_{\Omega\cap B(x_0,\frac{R}{2})}\left(\left|\nabla\left(|u|^{\frac{\gamma-1}{2}}u\right)\right|^2+|u|^{p+\gamma}\right)\leqslant C_{p,m,\gamma}\int_{\Omega\cap B(x_0,2R)}\left(|\nabla\psi_{R,x_0}|^2+|\psi_{R,x_0}||\Delta\psi_{R,x_0}|\right)^{(p+\gamma)/(p-1)}$$

$$\leqslant C(p,\gamma,m,N,\varphi)\left[\frac{\mathcal{L}_N(\Omega\cap B(x_0,2R))}{R^{2(\frac{p+\gamma}{p-1})}}\right], \tag{3.9}$$

where $B(0,t)$ denotes the open ball centered at the origin and with radius t, and $C(p,\gamma,m,N)$ is a positive constant independent of R. The desired conclusion then follows by letting $R\to+\infty$ in (3.9) and using the assumption (4). □

Proof of Proposition 8. A direct computation proves that any of the cases considered in the statement of Proposition 8 implies the volume growth condition (4). □

4. Non-negative solutions

Here we prove Theorems 11 and 12.

Proof of Theorem 11. We claim that u is a stable solution of (6). Indeed, by the strong minimum principle either $u\equiv 0$, and then u is stable, or $u>0$ in Ω. In the latter case, since Ω is a coercive epigraph, a result of M.J. Esteban and P.-L. Lions (cf. Proposition II.1 on page 8 of [18]) implies that $\frac{\partial u}{\partial x_N}>0$ in Ω. Therefore $\frac{\partial u}{\partial x_N}$ is a positive solution of the linearized equation:

$$-\Delta s-pu^{p-1}s=0\quad\text{in }\Omega, \tag{4.1}$$

24] to study semilinear phase transitions problems. Farina [15], and later Farina, Sciunzi, and Valdinoci [16] for more general quasi-linear operators, have also used this method to establish some Liouville-type results.

In next section we prove our main estimate, Theorem 1.1. In Section 3 we establish Theorem 1.4 and, as a simple consequence, Theorem 1.2.

2 Proof of the Main Estimate

In this section we prove our main estimate, estimate (1.4) of Theorem 1.1. For this, we will use the following remarkable result. It is a Sobolev inequality due to Allard [1] and Michael and Simon [20]. It holds on every compact hypersurface of $\mathbb{R}^{m+1}$ without boundary, and its constant is independent of the geometry of the hypersurface.

THEOREM 2.1 (Allard [1], Michael and Simon [20]) *Let $M \subset \mathbb{R}^{m+1}$ be a C^∞-immersed, m-dimensional compact hypersurface without boundary. Then, for every $p \in [1, m)$, there exists a constant $C = C(m,p)$ depending only on the dimension m and exponent p such that, for every C^∞-function $v : M \to \mathbb{R}$,*

$$(2.1) \qquad \Big(\int_M |v|^{p^*}\,dV\Big)^{1/p^*} \le C(m,p)\Big(\int_M |\nabla v|^p + |Hv|^p\,dV\Big)^{1/p},$$

where H is the mean curvature of M and $p^* = mp/(m-p)$.

This inequality is stated in proposition 5.2 of [19], where references for it and related results are mentioned. In [5, sec. 28.5.2] it is stated and proved for $p = 1$.

The geometric Sobolev inequality (2.1) has been used in the PDE literature to obtain estimates for the extinction time of some geometric evolution flows; see, for instance, section F.2 of [14] and also [19].

In the proof of our main estimate in Theorem 1.1 we will use (2.1) with $M = \{u = s\}$ (a level set of u), $v = |\nabla u|^{1/2}$, and $p = 2$. The level sets of a solution u and their curvature appear in the following result of Sternberg and Zumbrun [23, 24]. Its statement is an inequality that follows from the semistability hypothesis (1.3) on the solution.

PROPOSITION 2.2 (Sternberg and Zumbrun [23, 24]) *Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain and u a smooth, positive, semistable solution of (1.2). Then, for every Lipschitz function η in $\overline{\Omega}$ with $\eta|_{\partial\Omega} \equiv 0$,*

$$(2.2) \qquad \int_{\Omega\cap\{|\nabla u|>0\}} (|\nabla_T|\nabla u||^2 + |A|^2|\nabla u|^2)\eta^2\,dx \le \int_\Omega |\nabla u|^2|\nabla\eta|^2\,dx,$$

where ∇_T denotes the tangential or Riemannian gradient along a level set of u (it is thus the orthogonal projection of the full gradient in $\mathbb{R}^n$ along a level set of u)

and where

$$|A|^2 = |A(x)|^2 = \sum_{l=1}^{n-1} \kappa_l^2,$$

with κ_l being the principal curvatures of the level set of u passing through x for a given $x \in \Omega \cap \{|\nabla u| > 0\}$.

This result (stated for a Neumann problem instead of a Dirichlet problem) is lemma 2.1 of [23] and theorem 4.1 of [24]. The authors conceived and used the result to study qualitative properties of phase transitions in Allen-Cahn equations. For the sake of completeness, we give an elementary proof of it here. See theorem 2.5 of [16] for a quasi-linear extension.

PROOF OF PROPOSITION 2.2: The semistability condition (1.3) also holds, by approximation, for every Lipschitz function ξ in $\overline{\Omega}$ with $\xi|_{\partial\Omega} \equiv 0$. Now, take $\xi = c\eta$ in (1.3), where c is a smooth function, η is Lipschitz in $\overline{\Omega}$, and $\eta|_{\partial\Omega} \equiv 0$. A simple integration by parts gives that

$$Q_u(c\eta) = \int_\Omega c^2|\nabla\eta|^2 - (\Delta c + f'(u)c)c\eta^2 \, dx \geq 0. \tag{2.3}$$

In contrast with [23, 24] (where they took $c = |\nabla u|$) and to avoid some considerations on the set $\{|\nabla u| = 0\}$, we take

$$c = \sqrt{|\nabla u|^2 + \varepsilon^2}$$

for a given $\varepsilon > 0$. Note that c is smooth.

Since $\Delta u + f(u) = 0$ in Ω, we have $\Delta u_j + f'(u)u_j = 0$ in Ω. We use the notation $u_j = \partial_{x_j} u$ and also $u_{ij} = \partial_{x_i x_j} u$. Using these equations, we can easily verify that

$$\Delta c = \frac{1}{|\nabla u|^2 + \varepsilon^2}\Bigg\{-f'(u)|\nabla u|^2\sqrt{|\nabla u|^2 + \varepsilon^2} + \sum_{i,j} u_{ij}^2 \sqrt{|\nabla u|^2 + \varepsilon^2} - \Bigg(\sum_i \Big(\sum_j u_{ij}u_j\Big)^2\Bigg)\frac{1}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\Bigg\},$$

and thus

$$(\Delta + f'(u))c = f'(u)\frac{\varepsilon^2}{\sqrt{|\nabla u|^2 + \varepsilon^2}} + \frac{1}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\Bigg\{\sum_{i,j} u_{ij}^2 - \sum_i \Bigg(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\Bigg)^2\Bigg\}.$$

Using this equality in (2.3), we deduce

$$\int_\Omega (|\nabla u|^2+\varepsilon^2)|\nabla\eta|^2\,dx \tag{2.4}$$

$$= \int_\Omega c^2|\nabla\eta|^2\,dx$$

$$\geq \int_\Omega (\Delta c + f'(u)c)c\eta^2\,dx$$

$$= \int_\Omega f'(u)\varepsilon^2\eta^2\,dx \tag{2.5}$$

$$+ \int_\Omega \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2\,dx. \tag{2.6}$$

The integrand in the last integral is nonnegative. Thus, we have

$$\int_\Omega \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2\,dx$$

$$\geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)^2\right\}\eta^2\,dx$$

$$\geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2\,dx.$$

From this and (2.4), (2.5), and (2.6), we arrive at

$$\int_\Omega (|\nabla u|^2+\varepsilon^2)|\nabla\eta|^2\,dx$$

$$\geq \int_\Omega f'(u)\varepsilon^2\eta^2\,dx$$

$$+ \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2\,dx.$$

We now let $\varepsilon \downarrow 0$ to obtain

$$\int_\Omega |\nabla u|^2|\nabla\eta|^2\,dx \geq \int_{\Omega\cap\{|\nabla u|>0\}} \left\{\sum_{i,j} u_{ij}^2 - \sum_i\left(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\right)^2\right\}\eta^2\,dx.$$

We conclude the claimed inequality (2.2) of the proposition since

$$\sum_{i,j} u_{ij}^2 - \sum_i \Big(\sum_j u_{ij} \frac{u_j}{|\nabla u|} \Big)^2 = \big|\nabla_T |\nabla u|\big|^2 + |A|^2 |\nabla u|^2 \tag{2.7}$$

at every point $x \in \Omega \cap \{|\nabla u| > 0\}$. This last equality can easily be checked assuming that $\nabla u(x) = (0, \ldots, 0, u_{x_n}(x))$ and looking at the quantities in (2.7) in the orthonormal basis $\{e_1, \ldots, e_{n-1}, (0, \ldots, 0, 1)\}$, where $\{e_1, \ldots, e_{n-1}\}$ are the principal directions of the level set of u through x. See also lemma 2.1 of [23] for a detailed proof of (2.7). □

Using Proposition 2.2 and Theorem 2.1, we can now establish Theorem 1.1.

PROOF OF THEOREM 1.1: By elliptic regularity, the solution u is smooth, that is, $u \in C^\infty(\overline{\Omega})$. Recall that $u > 0$ in Ω. Let us define

$$T := \max_{\Omega} u = \|u\|_{L^\infty(\Omega)}$$

and, for $s \in (0, T)$,

$$\Gamma_s := \{x \in \Omega : u(x) = s\}.$$

By Sard's theorem, almost every $s \in (0, T)$ is a regular value of u. By definition, if s is a regular value of u, then $|\nabla u(x)| > 0$ for all $x \in \Omega$ such that $u(x) = s$ (i.e., for all $x \in \Gamma_s$). In particular, if s is a regular value, Γ_s is a C^∞ immersed compact hypersurface of $\mathbb{R}^n$ without boundary. Later we will apply Theorem 2.1 with $M = \Gamma_s$. Note that Γ_s could have a finite number of connected components. However, inequality (2.1) for connected manifolds M leads to the same inequality (and with the same constant) for M with more than one component.

Since u is a semistable solution, we can use Proposition 2.2. In (2.2) we take

$$\eta(x) = \varphi(u(x)) \quad \text{for } x \in \Omega,$$

where φ is a Lipschitz function in $[0, T]$ with

$$\varphi(0) = 0.$$

The right-hand side of (2.2) becomes

$$\int_\Omega |\nabla u|^2 |\nabla \eta|^2 \, dx = \int_\Omega |\nabla u|^4 \varphi'(u)^2 \, dx = \int_0^T \Big(\int_{\Gamma_s} |\nabla u|^3 \, dV_s \Big) \varphi'(s)^2 \, ds$$

by the coarea formula. We have denoted by dV_s the volume element in Γ_s. The integral in ds is over the regular values of u, whose complement is of zero measure in $(0, T)$.

$d(x, S)$ to denote the Euclidean distance from a point $x \in \mathbf{R}^n$ to a set $S \subset \mathbf{R}^n$. We generally use C to denote any constant arising in an estimate which is independent of the small parameter ε.

We first establish two general identities. As was mentioned earlier, we have not encountered the first one before.

Lemma 2.1. *Let $U \subset \mathbf{R}^n$ be any open set. Then for any C^2 function $f : U \to \mathbf{R}$,*

$$\left(\sum_{j=1}^{n} |\nabla f_{x_j}|^2\right) - |\nabla|\nabla f||^2 = \begin{cases} |\nabla f|^2\left(\sum_{l=1}^{n-1}\kappa_l^2\right) + |\nabla_L|\nabla f||^2 & \text{for } x \in \{|\nabla f| > 0\} \cap U, \\ 0 & \text{for a.e. } x \in \{|\nabla f| = 0\} \cap U, \end{cases} \tag{2.1}$$

where κ_l are the principal curvatures of the level set of f at x and ∇_L denotes the orthogonal projection of the gradient along this level set.

Proof. Since f is of class C^2, we know that $|\nabla f|$ is Lipschitz continuous and therefore differentiable a.e. in U. Restricting our attention to the set of differentiability of $|\nabla f|$, we note that on the set $E \equiv \{x \in U : |\nabla f| = 0\}$, the co-area formula ([F]) yields

$$\int_E |\nabla|\nabla f|| + \sum_{j=1}^{n} |\nabla f_{x_j}|\, dx$$
$$= \int_{-\infty}^{\infty} H^{n-1}(\{|\nabla f| = s\} \cap E) + \sum_{j=1}^{n} H^{n-1}(\{f_{x_j} = s\} \cap E)\, ds = 0.$$

Hence, the left-hand side of (2.1) vanishes a.e. in E.

Now suppose that $|\nabla f(x_0)| > 0$ for some $x_0 \in U$. For all x in a neighborhood of x_0 we denote by τ_n the unit vector field $\tau_n(x) = \nabla f(x)/|\nabla f(x)|$ and then introduce $n-1$ vectors $\{\tau_i(x)\}$, $i = 1, \ldots, n-1$ forming an orthonormal basis for the tangent plane of the level set $\{y : f(y) = f(x)\}$ at x.

Near x_0 we have

$$(\nabla(|\nabla f|))_j = \nabla f_{x_j} \cdot \tau_n \tag{2.2}$$

for each j. By writing

$$\nabla f_{x_j} = \nabla f_{x_j} \cdot \tau_n + \sum_{i=1}^{n-1} (\nabla f_{x_j} \cdot \tau_i)\, \tau_i$$

for each j, $j = 1, \ldots, n$, we find using (2.2) that

$$\left(\sum_{j=1}^{n} |\nabla f_{x_j}|^2\right) - |\nabla|\nabla f||^2 = \sum_{j=1}^{n} |\nabla f_{x_j}|^2 - (\nabla f_{x_j} \cdot \tau_n)^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n} (\nabla f_{x_j} \cdot \tau_i)^2. \tag{2.3}$$

Observe that

$$\nabla f_{x_j} \cdot \tau_i = \frac{\partial}{\partial x_j}(|\nabla f|\, \tau_n) \cdot \tau_i = |\nabla f|\, (\tau_n)_{x_j} \cdot \tau_i. \tag{2.4}$$

Without loss of generality, we now assume that $\nabla f(x_0) = (0, 0, \ldots, f_{x_n}(x_0))$ and that $\tau_i(x_0)$ points in the direction of the coordinate x_i for $i = 1, \ldots, n-1$. Then (2.4) implies that at x_0 we have

$$\begin{aligned} \sum_{i,j=1}^{n-1} \left(\nabla f_{x_j} \cdot \tau_i\right)^2 &= |\nabla f|^2 \sum_{i,j=1}^{n-1} (D_{\tau_j} \tau_n \cdot \tau_i)^2 \\ &= |\nabla f|^2 \sum_{i,j=1}^{n-1} \left|B_{ij}\right|^2 = |\nabla f|^2 \sum_{l=1}^{n-1} \kappa_l^2 \end{aligned} \tag{2.5}$$

where $B = (B_{ij})$ denotes the second fundamental form at x_0 associated with the level set $\{x : f(x) = f(x_0)\}$.

We now evaluate (2.3) at $x = x_0$ and use (2.5) to obtain

$$\begin{aligned} \left(\sum_{j=1}^{n} \left|\nabla f_{x_j}\right|^2\right) - |\nabla|\nabla f||^2 &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\nabla f_{x_j} \cdot \tau_i\right)^2 + \sum_{i=1}^{n-1} \left(\nabla f_{x_n} \cdot \tau_i\right)^2 \\ &= |\nabla f|^2 \sum_{l=1}^{n-1} \kappa_l^2 + \sum_{i=1}^{n-1} (f_{x_i x_n})^2 \end{aligned} \tag{2.6}$$

Finally note that at x_0 we have

$$\nabla_L(|\nabla f|) = \sum_{i=1}^{n-1} (\nabla(|\nabla f|) \cdot \tau_i)\, \tau_i = \sum_{i=1}^{n-1} (\nabla(|\nabla f|))_i\, \tau_i,$$

so (2.2) gives

$$\begin{aligned} |\nabla_L(|\nabla f|)|^2 &= \sum_{i=1}^{n-1} (\nabla(|\nabla f|))_i^2 \\ &= \sum_{i=1}^{n-1} (\nabla f_{x_i} \cdot \tau_n)^2 = \sum_{i=1}^{n-1} (f_{x_i x_n})^2 \end{aligned}$$

Substituting this into (2.6) yields (2.1). □

Lemma 2.2. (cf. [CHo]) *Let $U \subset \mathbf{R}^n$ be any open set with C^2 boundary and outer unit normal ν. Then for any function $u \in C^2(\overline{U})$ satisfying a homogeneous Neumann condition $\nabla u \cdot \nu = 0$ on ∂U,*

$$\partial_\nu(|\nabla u|^2) = \begin{cases} -2B(\tau, \tau)\, |\nabla u|^2 & \text{on } \{|\nabla u| > 0\} \cap \partial U, \\ 0 & \text{on } \{|\nabla u| = 0\} \cap \partial U \end{cases} \tag{2.7}$$

where $\tau = \nabla u(x_0)/|\nabla u(x_0)|$ and $B(\cdot, \cdot)$ denotes the second fundamental form associated with ∂U at x_0.

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A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem

Juan Dávila[a,*], Louis Dupaigne[b], Kelei Wang[c], Juncheng Wei[d,e]

[a] *Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*
[b] *Institut Camille Jordan, UMR CNRS 5208, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France*
[c] *Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, China*
[d] *Department of Mathematics, University of British Columbia, Vancouver, B.C., V6T 1Z2, Canada*
[e] *Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong*

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ABSTRACT

We consider Liouville-type and partial regularity results for the nonlinear fourth-order problem

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

where $p > 1$ and $n \geqslant 1$. We give a complete classification of stable and finite Morse index solutions (whether positive or sign changing), in the full exponent range. We also compute an upper bound of the Hausdorff dimension of the singular set of extremal solutions. Our approach is motivated by Fleming's tangent cone analysis technique for minimal surfaces and Federer's dimension reduction principle in partial regularity theory. A key tool is the monotonicity formula for biharmonic equations.

* Corresponding author.
E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), dupaigne@math.univ-lyon1.fr (L. Dupaigne), wangkelei@wipm.ac.cn (K. Wang), wei@math.cuhk.edu.hk (J. Wei).

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1. Introduction

We study the following model biharmonic superlinear elliptic equation

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain or the entire space and $p > 1$ is a real number. Inspired by the tangent cone analysis in minimal surface theory, more precisely Fleming's key observation that the existence of an entire nonplanar minimal graph implies that of a singular area-minimizing cone (see his work on the Bernstein theorem [11]), we derive a monotonicity formula for solutions of (1.1) to reduce the non-existence of nontrivial entire solutions for the problem (1.1), to that of nontrivial homogeneous solutions. Through this approach we give a complete classification of stable solutions and those of finite Morse index, whether positive or sign changing, when $\Omega = \mathbb{R}^n$ is the whole Euclidean space. This in turn enables us to obtain partial regularity as well as an estimate of the Hausdorff dimension of the singular set of the extremal solutions in bounded domains.

Let us first describe the monotonicity formula. Eq. (1.1) has two important features. It is variational, with energy functional given by

$$\int \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}$$

and it is invariant under the scaling transformation

$$u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(\lambda x).$$

This suggests that the variations of the rescaled energy

$$r^{4\frac{p+1}{p-1}-n} \int_{B_r(x)} \left[\frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}\right]$$

with respect to the scaling parameter r are meaningful. Augmented by the appropriate boundary terms, the above quantity is in fact nonincreasing. More precisely, take $u \in W^{4,2}_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$, fix $x \in \Omega$, let $0 < r < R$ be such that $B_r(x) \subset B_R(x) \subset \Omega$, and define

$$E(r; x, u) := r^{4\frac{p+1}{p-1}-n} \int_{B_r(x)} \left[\frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}\right]$$
$$+ \frac{2}{p-1}\left(n - 2 - \frac{4}{p-1}\right) r^{\frac{8}{p-1}+1-n} \int_{\partial B_r(x)} u^2$$

$$+\frac{2}{p-1}\left(n-2-\frac{4}{p-1}\right)\frac{d}{dr}\left(r^{\frac{8}{p-1}+2-n}\int_{\partial B_r(x)} u^2\right)$$

$$+\frac{r^3}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+1-n}\int_{\partial B_r(x)}\left(\frac{4}{p-1}r^{-1}u+\frac{\partial u}{\partial r}\right)^2\right]$$

$$+\frac{1}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+4-n}\int_{\partial B_r(x)}\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right)\right]$$

$$+\frac{1}{2}r^{\frac{8}{p-1}+3-n}\int_{\partial B_r(x)}\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right),$$

where derivatives are taken in the sense of distributions. Then, we have the following monotonicity formula.

Theorem 1.1. *Assume that*

$$n \geqslant 5, \quad p > \frac{n+4}{n-4}. \tag{1.2}$$

Let $u \in W^{4,2}_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$ be a weak solution of (1.1). Then, $E(r; x, u)$ is non-decreasing in $r \in (0, R)$. Furthermore there is a constant $c(n,p) > 0$ such that

$$\frac{d}{dr}E(r;0,u) \geqslant c(n,p)r^{-n+2+\frac{8}{p-1}}\int_{\partial B_r}\left(\frac{4}{p-1}r^{-1}u+\frac{\partial u}{\partial r}\right)^2. \tag{1.3}$$

Remark 1.2. Monotonicity formulae have a long history that we will not describe here. Let us simply mention two earlier results that seem closest to our findings: the formula of Pacard [20] for the classical Lane–Emden equation and the one of Chang, Wang and Yang [2] for biharmonic maps.

Consider again Eq. (1.1) in the case where $\Omega = \mathbb{R}^n$, i.e.,

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

Let

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leqslant 4, \\ \frac{n+4}{n-4} & \text{if } n \geqslant 5, \end{cases}$$

denote the Sobolev exponent. When $1 < p \leqslant p_S(n)$, all positive solutions to (1.4) are classified: if $p < p_S(n)$, then $u \equiv 0$; if $p = p_S(n)$, then all solutions can be written in the form $u = c_n(\frac{\lambda}{\lambda^2+|x-x_0|^2})^{\frac{n-4}{2}}$ for some $c_n > 0, \lambda > 0, x_0 \in \mathbb{R}^n$, see the work of Xu and one

Solving the corresponding quartic equation, (1.7) holds if and only if $p \geqslant p_c(n)$ where $p_c(n) > p_S(n)$ is the fourth-order Joseph–Lundgren exponent computed by Gazzola and Grunau [12]:

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leqslant 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geqslant 13. \end{cases}$$

Equivalently, for fixed $p > p_S(n)$, define n_p to be the smallest dimension such that (1.7) holds. Then,

$$(1.7) \quad \Leftrightarrow \quad p \geqslant p_c(n) \quad \Leftrightarrow \quad n \geqslant n_p.$$

The existence, uniqueness and stability of regular radial positive solutions to (1.4) is by now well understood (see the works of Gazzola–Grunau, of Guo and one of the authors, and of Karageorgis [12,16,18]): for each $a > 0$ there exists a unique entire radial positive solution $u_a(|x|)$ to (1.4) with $u_a(0) = a$. This radial positive solution is stable if and only if (1.7) holds.

In our second result, which is a Liouville-type theorem, we give a complete characterization of all finite Morse index solutions (whether radial or not, whether positive or not).

Theorem 1.3. *Let u be a smooth solution of (1.4) with finite Morse index.*

- *If $p \in (1, p_c(n))$, $p \neq p_S(n)$, then $u \equiv 0$;*
- *If $p = p_S(n)$, then u has finite energy i.e.*

$$\int_{\mathbb{R}^n} (\Delta u)^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition u is stable, then in fact $u \equiv 0$.

Remark 1.4. According to the preceding discussions, Theorem 1.3 is sharp: on the one hand, in the critical case $p = p_S(n)$, Guo, Li and one of the authors [15] have constructed a large class of solutions to (1.1) with finite energy. Since in this case $\frac{(p-1)n}{4} = p + 1$, by a result of Rozenblum [26], such solutions have finite Morse index. On the other hand, for $p \geqslant p_c(n)$, all radial solutions are stable (see [16,18]).

Remark 1.5. The above theorem generalizes a similar result of Farina [10] for the classical Lane–Emden equation.

Now consider (1.1) when Ω is a smoothly bounded domain of $\mathbb{R}^n$ and supplement it with Navier boundary conditions:

of the authors [31]. However, there can be many sign-changing solutions to the equation (see the work by Guo, Li and one of the authors [15] for the critical case $p = p_S(n)$).

Here, we allow u to be sign-changing and p to be supercritical. Instead, we restrict the analysis to stable and finite Morse index solutions. A solution u to (1.4) is said to be stable if

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx, \quad \text{for all } \phi \in H^2(\mathbb{R}^n).$$

More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces E of $H^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx < p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx,$$

for any $\phi \in E \setminus \{0\}$. No assumption on the growth of u is needed in these definitions. Clearly, a solution is stable if and only if its Morse index is equal to zero. It is also standard knowledge that if a solution to (1.4) has finite Morse index, then there is a compact set $\mathcal{K} \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx, \quad \forall\phi \in H^2(\mathbb{R}^n\backslash\mathcal{K}).$$

Recall that if

$$\gamma = \frac{4}{p-1}, \quad K_0 = \gamma(\gamma+2)(\gamma-n+4)(\gamma-n+2), \tag{1.5}$$

then

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)} \tag{1.6}$$

is a singular solution to (1.4) in $\mathbb{R}^n \setminus \{0\}$. By the Hardy–Rellich inequality with best constant [25]

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{\phi^2}{|x|^4}\,dx, \quad \forall\phi \in H^2(\mathbb{R}^n),$$

the singular solution u_s is stable if and only if

$$pK_0 \leqslant \frac{n^2(n-4)^2}{16}. \tag{1.7}$$

$$\Delta u + \left(\frac{2}{p+1}\right)^{1/2} u^{\frac{p+1}{2}} \leqslant 0 \quad \text{in } \mathbb{R}^n. \tag{1.10}$$

As observed in [9] for a similar equation, the use of the above inequality can be completely avoided.

In this paper we take a completely new approach, which also avoids the use of (1.10) and requires minimal integrability. One of our motivations is Fleming's proof of the Bernstein theorem for minimal surfaces in dimension 3. Fleming used a monotonicity formula for minimal surfaces together with a compactness result to blow down the minimal surface. It turns out that the blow-down limit is a minimal cone. This is because the monotonic quantity is constant only for minimizing cones. Then, he proved that minimizing cones are flat, which implies in turn the flatness of the original minimal surface.

At last, let us sketch the proof of Theorem 1.3: we first derive a monotonicity formula for our equation (1.1). Then, we classify stable solutions: this is Theorem 4.1 in Section 4. To do this, we estimate solutions in the L^{p+1} norm, utilizing the aforementioned methods available in the literature, and then show that the *blow-down* limit $u^\infty(x) = \lim_{\lambda\to\infty} \lambda^{\frac{4}{p-1}} u(\lambda x)$ satisfies $E(r) \equiv \mathit{const}$. Then, Theorem 1.1 implies that u^∞ is a homogeneous stable solution, and we show in Theorem 3.1 that such solutions are trivial if $p < p_c(n)$. Then similar to Fleming's proof, the triviality of the *blow-down* limit implies that the original entire solution is also trivial. In Section 5, we extend our result to solutions of finite Morse index. Finally, in Section 6 we prove an ε-regularity result and use the Federer's dimension reduction principle to obtain the partial regularity of extremal solutions. This approach was used in [30] for (1.9), see also [6].

2. Proof of the monotonicity formula

In this section we derive a monotonicity formula for functions $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$ solving (1.1) in $B_R(0) \subset \Omega$. We assume that $p > \frac{n+4}{n-4}$.

Proof of Theorem 1.1. Since the boundary integrals in $E(r; x, u)$ only involve second order derivatives of u, the boundary integrals in $\frac{dE}{dr}(r; x, u)$ only involve third order derivatives of u. By our assumption $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$, for each $B_r(x) \subset B_R(0)$, $u \in W^{3,2}(\partial B_r(x))$. Thus, the following calculations can be rigorously verified. Assume that $x = 0$ and that the balls B_λ are all centered at 0. Take

$$\widetilde{E}(\lambda) := \lambda^{4\frac{p+1}{p-1} - n} \int_{B_\lambda} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}.$$

Define

$$v := \Delta u$$

and

$$u^\lambda(x) := \lambda^{\frac{4}{p-1}} u(\lambda x), \qquad v^\lambda(x) := \lambda^{\frac{4}{p-1}+2} v(\lambda x).$$

We still have $v^\lambda = \Delta u^\lambda$, $\Delta v^\lambda = |u^\lambda|^{p-1}u^\lambda$, and by differentiating in λ,

$$\Delta \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}.$$

Note that differentiation in λ commutes with differentiation and integration in x. A rescaling shows

$$\widetilde{E}(\lambda) = \int_{B_1} \frac{1}{2}\left(v^\lambda\right)^2 - \frac{1}{p+1}\left|u^\lambda\right|^{p+1}.$$

Hence

$$\begin{aligned}
\frac{d}{d\lambda}\widetilde{E}(\lambda) &= \int_{B_1} v^\lambda \frac{dv^\lambda}{d\lambda} - \left|u^\lambda\right|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{B_1} v^\lambda \Delta \frac{du^\lambda}{d\lambda} - \Delta v^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} v^\lambda \frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r}\frac{du^\lambda}{d\lambda}.
\end{aligned} \tag{2.1}$$

In what follows, we express all derivatives of u^λ in the $r = |x|$ variable in terms of derivatives in the λ variable. In the definition of u^λ and v^λ, directly differentiating in λ gives

$$\frac{du^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\left(\frac{4}{p-1}u^\lambda(x) + r\frac{\partial u^\lambda}{\partial r}(x)\right), \tag{2.2}$$

$$\frac{dv^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\left(\frac{2(p+1)}{p-1}v^\lambda(x) + r\frac{\partial v^\lambda}{\partial r}(x)\right). \tag{2.3}$$

In (2.2), taking derivatives in λ once again, we get

$$\lambda\frac{d^2u^\lambda}{d\lambda^2}(x) + \frac{du^\lambda}{d\lambda}(x) = \frac{4}{p-1}\frac{du^\lambda}{d\lambda}(x) + r\frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda}(x). \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1) we obtain

$$\frac{d\widetilde{E}}{d\lambda} = \int_{\partial B_1} v^\lambda\left(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{p-5}{p-1}\frac{du^\lambda}{d\lambda}\right) - \frac{du^\lambda}{d\lambda}\left(\lambda\frac{dv^\lambda}{d\lambda} - \frac{2(p+1)}{p-1}v^\lambda\right)$$

$$\Delta u + \left(\frac{2}{p+1}\right)^{1/2} u^{\frac{p+1}{2}} \leqslant 0 \quad \text{in } \mathbb{R}^n. \tag{1.10}$$

As observed in [9] for a similar equation, the use of the above inequality can be completely avoided.

In this paper we take a completely new approach, which also avoids the use of (1.10) and requires minimal integrability. One of our motivations is Fleming's proof of the Bernstein theorem for minimal surfaces in dimension 3. Fleming used a monotonicity formula for minimal surfaces together with a compactness result to blow down the minimal surface. It turns out that the blow-down limit is a minimal cone. This is because the monotonic quantity is constant only for minimizing cones. Then, he proved that minimizing cones are flat, which implies in turn the flatness of the original minimal surface.

At last, let us sketch the proof of Theorem 1.3: we first derive a monotonicity formula for our equation (1.1). Then, we classify stable solutions: this is Theorem 4.1 in Section 4. To do this, we estimate solutions in the L^{p+1} norm, utilizing the aforementioned methods available in the literature, and then show that the *blow-down* limit $u^\infty(x) = \lim_{\lambda\to\infty} \lambda^{\frac{4}{p-1}} u(\lambda x)$ satisfies $E(r) \equiv const$. Then, Theorem 1.1 implies that u^∞ is a homogeneous stable solution, and we show in Theorem 3.1 that such solutions are trivial if $p < p_c(n)$. Then similar to Fleming's proof, the triviality of the *blow-down* limit implies that the original entire solution is also trivial. In Section 5, we extend our result to solutions of finite Morse index. Finally, in Section 6 we prove an ε-regularity result and use the Federer's dimension reduction principle to obtain the partial regularity of extremal solutions. This approach was used in [30] for (1.9), see also [6].

2. Proof of the monotonicity formula

In this section we derive a monotonicity formula for functions $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$ solving (1.1) in $B_R(0) \subset \Omega$. We assume that $p > \frac{n+4}{n-4}$.

Proof of Theorem 1.1. Since the boundary integrals in $E(r; x, u)$ only involve second order derivatives of u, the boundary integrals in $\frac{dE}{dr}(r; x, u)$ only involve third order derivatives of u. By our assumption $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$, for each $B_r(x) \subset B_R(0)$, $u \in W^{3,2}(\partial B_r(x))$. Thus, the following calculations can be rigorously verified. Assume that $x = 0$ and that the balls B_λ are all centered at 0. Take

$$\widetilde{E}(\lambda) := \lambda^{4\frac{p+1}{p-1}-n} \int\limits_{B_\lambda} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}.$$

Define

$$v := \Delta u$$

$$= \int\limits_{\partial B_1} \lambda v^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 3v^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{du^\lambda}{d\lambda} \frac{dv^\lambda}{d\lambda}. \tag{2.5}$$

Observe that v^λ is expressed as a combination of x derivatives of u^λ. So we also transform v^λ into λ derivatives of u^λ. By taking derivatives in r in (2.2) and noting (2.4), we get on ∂B_1,

$$\begin{aligned}\frac{\partial^2 u^\lambda}{\partial r^2} &= \lambda \frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda} - \frac{p+3}{p-1}\frac{\partial u^\lambda}{\partial r} \\ &= \lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \frac{p-5}{p-1}\lambda\frac{du^\lambda}{d\lambda} - \frac{p+3}{p-1}\left(\lambda\frac{du^\lambda}{d\lambda} - \frac{4}{p-1}u^\lambda\right) \\ &= \lambda^2 \frac{d^2u^\lambda}{d\lambda^2} - \frac{8}{p-1}\lambda\frac{du^\lambda}{d\lambda} + \frac{4(p+3)}{(p-1)^2}u^\lambda.\end{aligned}$$

Then on ∂B_1,

$$\begin{aligned}v^\lambda &= \frac{\partial^2 u^\lambda}{\partial r^2} + \frac{n-1}{r}\frac{\partial u^\lambda}{\partial r} + \frac{1}{r^2}\Delta_\theta u^\lambda \\ &= \lambda^2 \frac{d^2u^\lambda}{d\lambda^2} - \frac{8}{p-1}\lambda\frac{du^\lambda}{d\lambda} + \frac{4(p+3)}{(p-1)^2}u^\lambda + (n-1)\left(\lambda\frac{du^\lambda}{d\lambda} - \frac{4}{p-1}u^\lambda\right) + \Delta_\theta u^\lambda \\ &= \lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \left(n-1-\frac{8}{p-1}\right)\lambda\frac{du^\lambda}{d\lambda} + \frac{4}{p-1}\left(\frac{4}{p-1} - n + 2\right)u^\lambda + \Delta_\theta u^\lambda.\end{aligned}$$

Here Δ_θ is the Beltrami–Laplace operator on ∂B_1 and below ∇_θ represents the tangential derivative on ∂B_1. For notational convenience, we also define the constants

$$\alpha = n-1-\frac{8}{p-1}, \qquad \beta = \frac{4}{p-1}\left(\frac{4}{p-1} - n + 2\right).$$

Now (2.5) reads

$$\begin{aligned}\frac{d}{d\lambda}\widetilde{E}(\lambda) &= \int\limits_{\partial B_1} \lambda\left(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda\frac{du^\lambda}{d\lambda} + \beta u^\lambda\right)\frac{d^2u^\lambda}{d\lambda^2} \\ &\quad + 3\left(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda\frac{du^\lambda}{d\lambda} + \beta u^\lambda\right)\frac{du^\lambda}{d\lambda} \\ &\quad - \lambda\frac{du^\lambda}{d\lambda}\frac{d}{d\lambda}\left(\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + \alpha\lambda\frac{du^\lambda}{d\lambda} + \beta u^\lambda\right) \\ &\quad + \int\limits_{\partial B_1} \lambda\Delta_\theta u^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 3\Delta_\theta u^\lambda \frac{du^\lambda}{d\lambda} - \lambda\frac{du^\lambda}{d\lambda}\Delta_\theta\frac{du^\lambda}{d\lambda} \\ &= R_1 + R_2.\end{aligned}$$

Integrating by parts on ∂B_1, we get

$$
\begin{aligned}
R_2 &= \int\limits_{\partial B_1} -\lambda \nabla_\theta u^\lambda \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2} - 3\nabla_\theta u^\lambda \nabla_\theta \frac{du^\lambda}{d\lambda} + \lambda \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&= -\frac{\lambda}{2} \frac{d^2}{d\lambda^2} \Bigl(\int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr) - \frac{3}{2} \frac{d}{d\lambda} \Bigl(\int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr) + 2\lambda \int\limits_{\partial B_1} \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&= -\frac{1}{2} \frac{d^2}{d\lambda^2} \Bigl(\lambda \int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr) - \frac{1}{2} \frac{d}{d\lambda} \Bigl(\int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr) + 2\lambda \int\limits_{\partial B_1} \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&\geqslant -\frac{1}{2} \frac{d^2}{d\lambda^2} \Bigl(\lambda \int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr) - \frac{1}{2} \frac{d}{d\lambda} \Bigl(\int\limits_{\partial B_1} |\nabla_\theta u^\lambda|^2 \Bigr).
\end{aligned}
$$

For R_1, after some simplifications we obtain

$$
\begin{aligned}
R_1 = \int\limits_{\partial B_1} & \lambda \Bigl(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda \Bigr) \frac{d^2 u^\lambda}{d\lambda^2} \\
& + 3 \Bigl(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda \Bigr) \frac{du^\lambda}{d\lambda} \\
& - \lambda \frac{du^\lambda}{d\lambda} \Bigl(\lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + (2+\alpha)\lambda \frac{d^2 u^\lambda}{d\lambda^2} + (\alpha+\beta) \frac{du^\lambda}{d\lambda} \Bigr) \\
= \int\limits_{\partial B_1} & \lambda^3 \Bigl(\frac{d^2 u^\lambda}{d\lambda^2} \Bigr)^2 + \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + \beta\lambda u^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 3\beta u^\lambda \frac{du^\lambda}{d\lambda} \\
& + (2\alpha - \beta)\lambda \Bigl(\frac{du^\lambda}{d\lambda} \Bigr)^2 - \lambda^3 \frac{du^\lambda}{d\lambda} \frac{d^3 u^\lambda}{d\lambda^3} \\
= \int\limits_{\partial B_1} & 2\lambda^3 \Bigl(\frac{d^2 u^\lambda}{d\lambda^2} \Bigr)^2 + 4\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda \Bigl(\frac{du^\lambda}{d\lambda} \Bigr)^2 \\
& + \frac{\beta}{2} \frac{d^2}{d\lambda^2} \bigl[\lambda (u^\lambda)^2 \bigr] - \frac{1}{2} \frac{d}{d\lambda} \Bigl[\lambda^3 \frac{d}{d\lambda} \Bigl(\frac{du^\lambda}{d\lambda} \Bigr)^2 \Bigr] + \frac{\beta}{2} \frac{d}{d\lambda} (u^\lambda)^2.
\end{aligned}
$$

Here we have used the relations (writing $f' = \frac{d}{d\lambda} f$ etc.)

$$
\lambda f f'' = \Bigl(\frac{\lambda}{2} f^2 \Bigr)'' - 2 f f' - \lambda (f')^2,
$$

and

$$
-\lambda^3 f' f''' = -\Bigl[\frac{\lambda^3}{2} \bigl((f')^2 \bigr)' \Bigr]' + 3\lambda^2 f' f'' + \lambda^3 (f'')^2.
$$

In particular, if $E(\lambda; 0, u) \equiv const.$ for all $\lambda \in (r, R)$, u is homogeneous in $B_R \setminus B_r$:

$$u(x) = |x|^{-\frac{4}{p-1}} u\left(\frac{x}{|x|}\right).$$

We end this section with the following observation: in the above computations we just need the inequality (2.6) to hold. In particular the formula can be easily extended to biharmonic equations with negative exponents. We state the following monotonicity formula for solutions of

$$\Delta^2 u = -\frac{1}{u^p}, \quad u > 0 \text{ in } \Omega \subset \mathbb{R}^n. \tag{2.8}$$

Lemma 2.2. *Assume that p satisfies*

$$n - 2 + \frac{8}{p+1} > \frac{4}{p+1}\left(\frac{4}{p+1} + n - 2\right). \tag{2.9}$$

Let u be a classical solution to (2.8) in $B_r(x) \subset B_R(x) \subset \Omega$. Then the following quantity

$$\begin{aligned}
\tilde{E}(r; x, u) :=\ & r^{4\frac{p-1}{p+1} - n} \int_{B_r(x)} \frac{1}{2}(\Delta u)^2 - \frac{1}{p-1} u^{1-p} \\
& - \frac{2}{p+1}\left(n - 2 + \frac{4}{p+1}\right) r^{-\frac{8}{p+1}+1-n} \int_{\partial B_r(x)} u^2 \\
& - \frac{2}{p+1}\left(n - 2 + \frac{4}{p+1}\right) \frac{d}{dr}\left(r^{-\frac{8}{p+1}+2-n} \int_{\partial B_r(x)} u^2\right) \\
& + \frac{r^3}{2}\frac{d}{dr}\left[r^{-\frac{8}{p+1}+1-n} \int_{\partial B_r(x)} \left(-\frac{4}{p+1} r^{-1} u + \frac{\partial u}{\partial r}\right)^2\right] \\
& + \frac{1}{2}\frac{d}{dr}\left[r^{-\frac{8}{p+1}+4-n} \int_{\partial B_r(x)} \left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right] \\
& + \frac{1}{2} r^{-\frac{8}{p+1}+3-n} \int_{\partial B_r(x)} \left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)
\end{aligned}$$

is increasing in r. Furthermore there exists $c_0 > 0$ such that

$$\frac{d}{dr} E(r; 0, u) \geqslant c_0 r^{-n+2-\frac{8}{p+1}} \int_{\partial B_r} \left(-\frac{4}{p+1} r^{-1} u + \frac{\partial u}{\partial r}\right)^2. \tag{2.10}$$

In the rest of the paper, sometimes we use $E(r; x)$ or $E(r)$ if no confusion occurs.

$$p\left(\int_{\mathbb{S}^{n-1}} |w|^{p+1}\,d\theta\right)\left(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\,dr\right)$$

$$\leqslant \left(\int_{\mathbb{S}^{n-1}} \left(|\Delta_\theta w|^2 + \frac{n(n-4)}{2}|\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16}w^2\right)d\theta\right)\left(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\,dr\right)$$

$$+ O\left[\left(\int_0^{+\infty} r\eta_\varepsilon'(r)^2 + r^3\eta_\varepsilon''(r)^2 + \left|\eta_\varepsilon'(r)\right|\eta_\varepsilon(r) + r\eta_\varepsilon(r)\left|\eta_\varepsilon''(r)\right|\,dr\right)\right.$$

$$\left.\times\left(\int_{\mathbb{S}^{n-1}} w(\theta)^2 + \left|\nabla_\theta w(\theta)\right|^2\,d\theta\right)\right].$$

Note that

$$\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\,dr \geqslant |\log\varepsilon|,$$

$$\int_0^{+\infty} r\eta_\varepsilon'(r)^2 + r^3\eta_\varepsilon''(r)^2 + \left|\eta_\varepsilon'(r)\right|\eta_\varepsilon(r) + r\eta_\varepsilon(r)\left|\eta_\varepsilon''(r)\right|\,dr \leqslant C,$$

for some constant C independent of ε. By letting $\varepsilon \to 0$, we obtain

$$p\int_{\mathbb{S}^{n-1}} |w|^{p+1}\,d\theta \leqslant \int_{\mathbb{S}^{n-1}} |\Delta_\theta w|^2 + \frac{n(n-4)}{2}|\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16}w^2.$$

Substituting (3.2) into this we get

$$\int_{\mathbb{S}^{n-1}} (p-1)|\Delta_\theta w|^2 + \left(pJ_1 - \frac{n(n-4)}{2}\right)|\nabla_\theta w|^2 + \left(pJ_2 - \frac{n^2(n-4)^2}{16}\right)w^2 \leqslant 0.$$

If $\frac{n+4}{n-4} < p < p_c(n)$, then $p-1>0$, $pJ_1 - \frac{n(n-4)}{2} > 0$ and $pJ_2 - \frac{n^2(n-4)^2}{16} > 0$ (cf. [13, p. 338]), so $w \equiv 0$ and then $u \equiv 0$. □

For applications in Section 6, we record the form of $E(R;0,u)$ for a homogeneous solution u.

Remark 3.2. Suppose $u(r,\theta) = r^{-\frac{4}{p-1}}w(\theta)$ is a homogeneous solution, where $p > \frac{n+4}{n-4}$ and $w \in W^{2,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$. In this case, for any $r > 0$,

$$\int_{B_r\setminus B_{r/2}} |\Delta u|^2 + |u|^{p+1} \leqslant cr^{n-4\frac{p+1}{p-1}}.$$

4. The blow down analysis

In this section we use the blow-down analysis to prove the Liouville theorem for stable solutions. Throughout this section u always denotes a smooth stable solution of (1.1) in $\mathbb{R}^n$.

Theorem 4.1. *Let u be a smooth stable solution of (1.1) on $\mathbb{R}^n$. If $1 < p < p_c(n)$, then $u \equiv 0$.*

The following lemma appears in [32] for positive solution. It remains valid for sign-changing solutions, see also [17].

Lemma 4.2. *Let u be a smooth stable solution of (1.1) and let $v = \Delta u$. Then for some C we have*

$$\int_{\mathbb{R}^n} (v^2 + |u|^{p+1})\eta^2 \leqslant C \int_{\mathbb{R}^n} u^2\big(|\nabla(\Delta\eta)\cdot\nabla\eta| + (\Delta\eta)^2 + \big|\Delta\big(|\nabla\eta|^2\big)\big|\big)\,dx + C\int_{\mathbb{R}^n} |uv||\nabla\eta|^2\,dx \tag{4.1}$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

Proof. For completeness we give the proof. We have the identity

$$\int_{\mathbb{R}^n} (\Delta^2\xi)\xi\eta^2\,dx = \int_{\mathbb{R}^n} \big(\Delta(\xi\eta)\big)^2 + \int_{\mathbb{R}^n} \big(-4(\nabla\xi\cdot\nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\big)\,dx + \int_{\mathbb{R}^n} \xi^2\big(2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\big)\,dx,$$

for $\xi \in C^4(\mathbb{R}^n)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$, see for example [32, Lemma 2.3].

Taking $\xi = u$ yields

$$\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\,dx = \int_{\mathbb{R}^n} \big(\Delta(u\eta)\big)^2 + \int_{\mathbb{R}^n} \big(-4(\nabla u\cdot\nabla\eta)^2 + 2uv|\nabla\eta|^2\big)\,dx + \int_{\mathbb{R}^n} u^2\big(2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\big)\,dx.$$

Using the stability inequality with $u\eta$ yields

$$p\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\,dx \leqslant \int_{\mathbb{R}^n} \big(\Delta(u\eta)\big)^2.$$

Since $p > \frac{n+4}{n-4}$, direct calculations show that

$$\alpha - \beta = \left(n - 1 - \frac{8}{p-1}\right) - \frac{4}{p-1}\left(\frac{4}{p-1} - n + 2\right) > 1. \tag{2.6}$$

Thus,

$$\begin{aligned} &2\lambda^3\left(\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + 4\lambda^2\frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\ &= 2\lambda\left(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{du^\lambda}{d\lambda}\right)^2 + (2\alpha - 2\beta - 2)\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\ &\geqslant 0. \end{aligned} \tag{2.7}$$

Then,

$$R_1 \geqslant \int_{\partial B_1} \frac{\beta}{2}\frac{d^2}{d\lambda^2}\left[\lambda(u^\lambda)^2\right] - \frac{1}{2}\frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\frac{du^\lambda}{d\lambda}\right)^2\right] + \frac{\beta}{2}\frac{d}{d\lambda}(u^\lambda)^2.$$

Now, rescaling back, we can write those λ derivatives in R_1 and R_2 as follows.

$$\int_{\partial B_1} \frac{d}{d\lambda}(u^\lambda)^2 = \frac{d}{d\lambda}\left(\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda} u^2\right),$$

$$\int_{\partial B_1} \frac{d^2}{d\lambda^2}\left[\lambda(u^\lambda)^2\right] = \frac{d^2}{d\lambda^2}\left(\lambda^{\frac{8}{p-1}+2-n}\int_{\partial B_\lambda} u^2\right),$$

$$\int_{\partial B_1} \frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\frac{du^\lambda}{d\lambda}\right)^2\right] = \frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u + \frac{\partial u}{\partial r}\right)^2\right)\right],$$

$$\frac{d^2}{d\lambda^2}\left(\lambda\int_{\partial B_1}|\nabla_\theta u^\lambda|^2\right) = \frac{d^2}{d\lambda^2}\left[\lambda^{1+\frac{8}{p-1}+2+1-n}\int_{\partial B_\lambda}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right],$$

$$\frac{d}{d\lambda}\left(\int_{\partial B_1}|\nabla_\theta u^\lambda|^2\right) = \frac{d}{d\lambda}\left[\lambda^{\frac{8}{p-1}+2+1-n}\int_{\partial B_\lambda}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right].$$

Substituting these into $\frac{d}{d\lambda}E(\lambda; 0, u)$ we finish the proof. □

Denote $c(n,p) = 2\alpha - 2\beta - 2 > 0$. By (2.7), we have

Corollary 2.1.

$$\frac{d}{dr}E(r; 0, u) \geqslant c(n,p)r^{-n+2+\frac{8}{p-1}}\int_{\partial B_r}\left(\frac{4}{p-1}r^{-1}u + \frac{\partial u}{\partial r}\right)^2.$$

Therefore

$$\int_{\mathbb{R}^n} \left(|u|^{p+1}\eta^2 + \left(\Delta(u\eta)\right)^2\right) dx \leqslant C \int_{\mathbb{R}^n} \left(|\nabla u|^2|\nabla\eta|^2 + |uv||\nabla\eta|^2\right) dx + C \int_{\mathbb{R}^n} u^2\left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2\right) dx.$$

Using $\Delta(\eta u) = v\eta + 2\nabla\eta\cdot\nabla u + u\Delta\eta$ we obtain

$$\int_{\mathbb{R}^n} \left(|u|^{p+1} + v^2\right)\eta^2\, dx \leqslant C \int_{\mathbb{R}^n} \left(|\nabla u|^2|\nabla\eta|^2 + |uv||\nabla\eta|^2\right) dx + C \int_{\mathbb{R}^n} u^2\left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2\right) dx.$$

But

$$2\int_{\mathbb{R}^n} |\nabla u|^2|\nabla\eta|^2\, dx = \int_{\mathbb{R}^n} \Delta\left(u^2\right)|\nabla\eta|^2\, dx - 2\int_{\mathbb{R}^n} uv|\nabla\eta|^2\, dx = \int_{\mathbb{R}^n} u^2\Delta\left(|\nabla\eta|^2\right) dx - 2\int_{\mathbb{R}^n} uv|\nabla\eta|^2\, dx,$$

and hence

$$\int_{\mathbb{R}^n} \left(|u|^{p+1} + v^2\right)\eta^2\, dx \leqslant C \int_{\mathbb{R}^n} u^2\left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2 + \left|\Delta\left(|\nabla\eta|^2\right)\right|\right) dx + C\int_{\mathbb{R}^n} |uv||\nabla\eta|^2\, dx.$$

This proves (4.1) □

Corollary 4.3. *There exists a constant C such that*

$$\int_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{-4} \int_{B_{2R}(x)\setminus B_R(x)} u^2 + CR^{-2} \int_{B_{2R}(x)\setminus B_R(x)} |uv|, \tag{4.2}$$

and

$$\int_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{n-4\frac{p+1}{p-1}}, \tag{4.3}$$

for all $B_R(x)$.

Proof. The first inequality is a direct consequence of (4.1), by choosing a cut-off function $\eta \in C_0^\infty(B_{2R}(x))$, such that $\eta \equiv 1$ in $B_R(x)$, and for $k \leqslant 3$, $|\nabla^k \eta| \leqslant \frac{1000}{R^k}$.

Exactly the same argument as in [32] or [17] provides the second estimate. For completeness, we record the proof here. Replace η in (4.1) by η^m, where m is a large integer and η is a cut-off function as before. Then

$$\int |uv||\nabla \eta^m|^2 = m^2 \int_{B_{2R}(x)\setminus B_R(x)} |uv|\eta^{2m-2}|\nabla \eta|^2$$
$$\leqslant \frac{1}{2C}\int v^2\eta^{2m} + C\int u^2\eta^{2m-4}|\nabla \eta|^4.$$

Substituting this into (4.1), we obtain

$$\int \left(v^2 + |u|^{p+1}\right)\eta^{2m} \leqslant CR^{-4}\int_{B_{2R}(x)} u^2\eta^{2m-4}$$
$$\leqslant CR^{-4}\Bigg(\int_{B_{2R}(x)} |u|^{p+1}\eta^{(m-2)(p+1)}\Bigg)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})}.$$

This gives (4.3). Here we have used the fact $\eta^{2m} \geqslant \eta^{(m-2)(p+1)}$ because $0 \leqslant \eta \leqslant 1$, m is large, and $p > 1$. $\square$

Proof of Theorem 4.1 for $1 < p \leqslant \frac{n+4}{n-4}$. For $p < \frac{n+4}{n-4}$, we can let $R \to +\infty$ in (4.3) to get $u \equiv 0$ directly. If $p = \frac{n+4}{n-4}$, this gives

$$\int_{\mathbb{R}^n} v^2 + |u|^{p+1} < +\infty.$$

So

$$\lim_{R\to+\infty} \int_{B_{2R}(x)\setminus B_R(x)} v^2 + |u|^{p+1} = 0.$$

Then by (4.2), and noting that now $n = 4\frac{p+1}{p-1}$,

$$\int_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{-4}\int_{B_{2R}(x)\setminus B_R(x)} u^2 + C\int_{B_{2R}(x)\setminus B_R(x)} |v|^2$$
$$\leqslant CR^{-4}\Bigg(\int_{B_{2R}(x)\setminus B_R(x)} |u|^{p+1}\Bigg)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})} + C\int_{B_{2R}(x)\setminus B_R(x)} |v|^2$$

$$\leqslant C\Big(\int_{B_{2R}(x)\setminus B_R(x)} |u|^{p+1}\Big)^{\frac{2}{p+1}} + C\int_{B_{2R}(x)\setminus B_R(x)} |v|^2.$$

This goes to 0 as $R \to +\infty$, and we still get $u \equiv 0$. □

Next we concentrate on the case $p > \frac{n+4}{n-4}$. We first use (4.3) to show

Lemma 4.4. $\lim_{r\to+\infty} E(r;0,u) < +\infty$.

Proof. Let us write $E(r) = E(r;0,u)$. Since $E(r)$ is non-decreasing in r, we have

$$E(r) \leqslant \frac{1}{r}\int_r^{2r} E(t)\,dt \leqslant \frac{1}{r^2}\int_r^{2r}\int_t^{t+r} E(\lambda)\,d\lambda\,dt.$$

By (4.3),

$$\frac{1}{r^2}\int_r^{2r}\int_t^{t+r}\Big(\lambda^{4\frac{p+1}{p-1}-n}\int_{B_\lambda}\frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}\Big)\,d\lambda\,dt \leqslant C.$$

Next

$$\begin{aligned}
&\frac{1}{r^2}\int_r^{2r}\int_t^{t+r}\Big(\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda} u^2\Big)\,d\lambda\,dt\\
&= \frac{1}{r^2}\int_r^{2r}\int_{B_{t+r}\setminus B_t} |x|^{\frac{8}{p-1}+1-n} u(x)^2\,dx\,dt\\
&\leqslant \frac{1}{r^2}\int_r^{2r}\Big(\int_{B_{3r}\setminus B_r} |x|^{(\frac{8}{p-1}+1-n)\frac{p+1}{p-1}}\Big)^{\frac{p-1}{p+1}}\Big(\int_{B_{3r}} |u(x)|^{p+1}\Big)^{\frac{2}{p+1}}\,dt\\
&\leqslant C.
\end{aligned}$$

The same estimate holds for the term in $E(r)$ containing

$$\int_{\partial B_\lambda}\Big(|\nabla u|^2 - \Big|\frac{\partial u}{\partial r}\Big|^2\Big).$$

For this we need to note the following estimate

$$\int_{B_r} |\nabla u|^2 \leqslant Cr^2\int_{B_{2r}} (\Delta u)^2 + Cr^{-2+n\frac{p-1}{p+1}}\Big(\int_{B_{2r}} |u|^{p+1}\Big)^{\frac{2}{p+1}} \leqslant Cr^{n-\frac{8}{p-1}-2}. \qquad (4.4)$$

Now consider

$$
\begin{aligned}
&\frac{1}{r^2}\int_r^{2r}\int_t^{t+r}\frac{\lambda^3}{2}\frac{d}{d\lambda}\left[\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u+\frac{\partial u}{\partial r}\right)^2\right]d\lambda\,dt\\
&=\frac{1}{2r^2}\int_r^{2r}\left\{(t+r)^{\frac{8}{p-1}+4-n}\int_{\partial B_{t+r}}\left(\frac{4}{p-1}(t+r)^{-1}u+\frac{\partial u}{\partial r}\right)^2\right.\\
&\quad\left.-t^{\frac{8}{p-1}+4-n}\int_{\partial B_t}\left(\frac{4}{p-1}t^{-1}u+\frac{\partial u}{\partial r}\right)^2\right\}dt\\
&\quad-\frac{3}{2r^2}\int_r^{2r}\int_t^{t+r}\lambda^{\frac{8}{p-1}+3-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u+\frac{\partial u}{\partial r}\right)^2 d\lambda\,dt\\
&\leqslant\frac{C}{r^2}\int_{B_{3r}\setminus B_r}|x|^{\frac{8}{p-1}+4-n}\left(\frac{4}{p-1}|x|^{-1}u+\frac{\partial u}{\partial r}\right)^2\\
&\leqslant C.
\end{aligned}
$$

The remaining terms in $E(r)$ can be treated similarly. □

For any $\lambda>0$, define

$$
u^\lambda(x):=\lambda^{\frac{4}{p-1}}u(\lambda x),\qquad v^\lambda(x):=\lambda^{\frac{4}{p-1}+2}v(\lambda x).
$$

u^λ is also a smooth stable solution of (1.1) on $\mathbb{R}^n$.

By rescaling (4.3), for all $\lambda>0$ and balls $B_r(x)\subset\mathbb{R}^n$,

$$
\int_{B_r(x)}(v^\lambda)^2+|u^\lambda|^{p+1}\leqslant Cr^{n-4\frac{p+1}{p-1}}.
$$

In particular, u^λ are uniformly bounded in $L^{p+1}_{loc}(\mathbb{R}^n)$ and $v^\lambda=\Delta u^\lambda$ are uniformly bounded in $L^2_{loc}(\mathbb{R}^n)$. By elliptic estimates, u^λ are also uniformly bounded in $W^{2,2}_{loc}(\mathbb{R}^n)$. Hence, up to a subsequence of $\lambda\to+\infty$, we can assume that $u^\lambda\to u^\infty$ weakly in $W^{2,2}_{loc}(\mathbb{R}^n)\cap L^{p+1}_{loc}(\mathbb{R}^n)$. By compactness embedding for Sobolev functions, $u^\lambda\to u^\infty$ strongly in $W^{1,2}_{loc}(\mathbb{R}^n)$. Then for any ball $B_R(0)$, by interpolation between L^q spaces and noting (4.3), for any $q\in[1,p+1)$, as $\lambda\to+\infty$,

$$
\|u^\lambda-u^\infty\|_{L^q(B_R(0))}\leqslant\|u^\lambda-u^\infty\|^t_{L^1(B_R(0))}\|u^\lambda-u^\infty\|^{1-t}_{L^{p+1}(B_R(0))}\to 0,\tag{4.5}
$$

where $t\in(0,1]$ satisfies $\frac{1}{q}=t+\frac{1-t}{p+1}$. That is, $u^\lambda\to u^\infty$ in $L^q_{loc}(\mathbb{R}^n)$ for any $q\in[1,p+1)$.

For any function $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Delta u^\infty \Delta\varphi - (u^\infty)^p \varphi = \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} \Delta u^\lambda \Delta\varphi - (u^\lambda)^p \varphi = 0,$$

$$\int_{\mathbb{R}^n} (\Delta\varphi)^2 - p(u^\infty)^{p-1}\varphi^2 = \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} (\Delta\varphi)^2 - p(u^\lambda)^{p-1}\varphi^2 \geqslant 0.$$

Thus $u^\infty \in W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$ is a stable solution of (1.1) in $\mathbb{R}^n$.

Lemma 4.5. *u^∞ is homogeneous.*

Proof. For any $0 < r < R < +\infty$, by the monotonicity of $E(r;0,u)$ and Lemma 4.4,

$$\lim_{\lambda\to+\infty} E(\lambda R;0,u) - E(\lambda r;0,u) = 0.$$

Therefore, by the scaling invariance of E

$$\lim_{\lambda\to+\infty} E\big(R;0,u^\lambda\big) - E\big(r;0,u^\lambda\big) = 0.$$

We note that $E(r;0,u^\lambda)$ is absolutely continuous with respect to r, since we assume u^λ smooth. This still holds if we assume $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$, since boundary integrals only involve second order derivatives of u and so for each $B_r(0) \subset B_R(0)$, $u \in W^{3,2}(\partial B_r(0))$. Then by Corollary 2.1 we see that

$$\begin{aligned} 0 &= \lim_{\lambda\to+\infty} E\big(R;0,u^\lambda\big) - E\big(r;0,u^\lambda\big) \\ &\geqslant c(n,p) \lim_{\lambda\to+\infty} \int_{B_R\setminus B_r} \frac{(\frac{4}{p-1}|x|^{-1}u^\lambda(x) + \frac{\partial u^\lambda}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx \\ &\geqslant c(n,p) \int_{B_R\setminus B_r} \frac{(\frac{4}{p-1}|x|^{-1}u^\infty(x) + \frac{\partial u^\infty}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx. \end{aligned}$$

Note that in the last inequality we only used the weak convergence of u^λ to u^∞ in $W^{1,2}_{loc}(\mathbb{R}^n)$. Now

$$\frac{4}{p-1} r^{-1} u^\infty + \frac{\partial u^\infty}{\partial r} = 0, \quad \text{a.e. in } \mathbb{R}^n.$$

Integrating in r shows that

$$u^\infty(x) = |x|^{-\frac{4}{p-1}} u^\infty\left(\frac{x}{|x|}\right).$$

That is, u^∞ is homogeneous. □

By Theorem 3.1, $u^\infty \equiv 0$. Since this holds for the limit of any sequence $\lambda \to +\infty$, by (4.5) we get

$$\lim_{\lambda\to+\infty} u^\lambda = 0 \quad \text{strongly in } L^2\big(B_4(0)\big).$$

Now we show

Lemma 4.6. $\lim_{r\to+\infty} E(r;0,u) = 0$.

Proof. For all $\lambda \to +\infty$,

$$\lim_{\lambda\to+\infty} \int\limits_{B_4(0)} \left(u^\lambda\right)^2 = 0.$$

Because v^λ are uniformly bounded in $L^2(B_4(0))$, by the Cauchy inequality we also have

$$\lim_{\lambda\to+\infty} \int\limits_{B_4(0)} \left|u^\lambda v^\lambda\right| \leqslant \lim_{\lambda\to+\infty} \Big(\int\limits_{B_4(0)} \left(u^\lambda\right)^2\Big)^{\frac{1}{2}} \Big(\int\limits_{B_4(0)} \left(v^\lambda\right)^2\Big)^{\frac{1}{2}} = 0.$$

By (4.2),

$$\begin{aligned}\lim_{\lambda\to+\infty} \int\limits_{B_3(0)} \left(v^\lambda\right)^2 + \left|u^\lambda\right|^{p+1} &\leqslant C \lim_{\lambda\to+\infty} \Big(\int\limits_{B_4(0)} \left(u^\lambda\right)^2 + \int\limits_{B_4(0)} \left|u^\lambda v^\lambda\right|\Big) \\ &= 0. \end{aligned} \tag{4.6}$$

By the interior L^2 estimate, we get

$$\lim_{\lambda\to+\infty} \int\limits_{B_2(0)} \sum_{k\leqslant 2} \left|\nabla^k u^\lambda\right|^2 = 0.$$

In particular, we can choose a sequence $\lambda_i \to +\infty$ such that

$$\int\limits_{B_2(0)} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 \leqslant 2^{-i}.$$

By this choice we have

$$\int_1^2 \sum_{i=1}^{+\infty} \int\limits_{\partial B_r} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 dr \leqslant \sum_{i=1}^{+\infty} \int_1^2 \int\limits_{\partial B_r} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 dr \leqslant 1.$$

That is, the function

$$f(r) := \sum_{i=1}^{+\infty} \int_{\partial B_r} \sum_{k \leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 \in L^1((1,2)).$$

There exists an $r_0 \in (1,2)$ such that $f(r_0) < +\infty$. From this we get

$$\lim_{i \to +\infty} \left\|u^{\lambda_i}\right\|_{W^{2,2}(\partial B_{r_0})} = 0.$$

Combining this with (4.6) and the scaling invariance of $E(r)$, we get

$$\lim_{i \to +\infty} E(\lambda_i r_0; 0, u) = \lim_{i \to +\infty} E\left(r_0; 0, u^{\lambda_i}\right) = 0.$$

Since $\lambda_i r_0 \to +\infty$ and $E(r; 0, u)$ is non-decreasing in r, we get

$$\lim_{r \to +\infty} E(r; 0, u) = 0. \qquad \square$$

By the smoothness of u, $\lim_{r \to 0} E(r; 0, u) = 0$. Then again by the monotonicity of $E(r; 0, u)$ and the previous lemma, we obtain

$$E(r; 0, u) = 0 \quad \text{for all } r > 0.$$

Then again by Corollary 2.1, u is homogeneous, and then $u \equiv 0$ by Theorem 3.1 (or by the smoothness of u). This finishes the proof of Theorem 4.1.

5. Finite Morse index solutions

In this section we prove Theorem 1.3 and we always assume that u is a smooth solution. First, by the doubling lemma [22] and our Liouville theorem for stable solutions, Theorem 4.1, we have

Lemma 5.1. *Let u be a smooth, finite Morse index (positive or sign changing) solution of (1.1). There exist a constant C and R_0 such that for all $x \in B_{R_0}(0)^c$,*

$$\left|u(x)\right| \leqslant C|x|^{-\frac{4}{p-1}}.$$

Proof. Assume that u is stable outside B_{R_0}. For $x \in B^c_{R_0}$, let $M(x) = |u(x)|^{\frac{p-1}{4}}$ and $d(x) = |x| - R_0$, the distance to B_{R_0}. Assume that there exists a sequence of $x_k \in B^c_{R_0}$ such that

$$M(x_k) d(x_k) \geqslant 2k. \tag{5.1}$$

Since u is bounded on any compact set of $\mathbb{R}^n$, $d(x_k) \to +\infty$.

Consider $\eta_k := sgn(\theta_k)\chi$ for $1 \leq k \leq m$, where again $sgn(x)$ is the Sign function. The geometric Poincaré inequality (32) yields

$$\int_{B_R \setminus B_{\sqrt{R}}} \sum_i |\nabla u_i|^2 |\nabla \chi|^2 \geq \sum_i \int_{|\nabla u_i| \neq 0} \left(|\nabla u_i|^2 \kappa_i^2 + |\nabla_T |\nabla u_i||^2\right) \chi^2$$
$$+ \sum_{i \neq j} \int_{\mathbf{R}^N} \left(\nabla u_i \cdot \nabla u_j - sgn(\theta_i) sgn(\theta_j) |\nabla u_i||\nabla u_j|\right) H_{u_i u_j} \chi^2$$
$$= I_1 + I_2. \tag{38}$$

Note that I_1 is clearly nonnegative. Moreover, (37) yields that $H_{u_i u_j} sgn(\theta_i) sgn(\theta_j) \leq 0$ for all $i < j$, and therefore, I_2 can be written as

$$I_2 = \sum_{i \neq j} \int_{\mathbf{R}^N} \left(sgn\left(H_{u_i u_j}\right) \nabla u_i \cdot \nabla u_j + |\nabla u_i||\nabla u_j|\right) H_{u_i u_j} sgn\left(H_{u_i u_j}\right) \chi^2,$$

which is also nonnegative.

On the other hand, since

$$\int_{B_R \setminus B_{\sqrt{R}}} \sum_i |\nabla u_i|^2 |\nabla \chi|^2 \leq C \begin{cases} \dfrac{1}{\log R}, & \text{if } N = 2, \\ \dfrac{R^{N-2} + R^{(N-2)/2}}{|N-2||\log R|^2}, & \text{if } N \neq 2, \end{cases}$$

one can see that in dimension two the left hand side of (38) goes to zero as $R \to \infty$. Since $I_1 = 0$, one concludes that all u_i for $i = 1, \ldots, m$ are one-dimensional and from the fact that $I_2 = 0$, provided $H_{u_i u_j}$ is not identically zero, we obtain that for all $x \in \mathbf{R}^2$,

$$-sgn(H_{u_i u_j}) \nabla u_i \cdot \nabla u_j = |\nabla u_i||\nabla u_j|,$$

which completes the proof of the theorem. □

Now, we are ready to state and prove the main result of this paper.

Theorem 5 *Conjecture (2) holds for $N \leq 3$.*

Proof Let again $\phi_i := \partial_N u_i$ and $\psi_i := \nabla u_i \cdot \eta$ for any fixed $\eta = (\eta', 0) \in \mathbf{R}^{N-1} \times \{0\}$ in such a way that $\sigma_i := \frac{\psi_i}{\phi_i}$ is a solution of system (13) for $h_{i,j}(x) = H_{u_i u_j} \phi_i(x) \phi_j(x)$ and f to be the identity. Since $|\nabla u_i| \in L^\infty(\mathbf{R}^N)$, we have $||\phi_i \sigma_i||_{L^\infty(\mathbf{R}^N)} < \infty$.

In dimension $N = 2$, assumption (12) holds and Proposition 1 then yields that σ_i is constant, which finishes the proof as argued before.

In dimension $N = 3$, we shall follow ideas used by Ambrosio and Cabré [4] and Alberti et al. [2] in the case of a single equation. We first note that u being H-monotone means that u is a stable solution of (6). Moreover, the function $v(x_1, x_2) := \lim_{x_3 \to \infty} u(x_1, x_2, x_3)$ is also a bounded stable solution for (6) in $\mathbf{R}^2$. Indeed, it suffices to test (10) on $\zeta_k(x) = \eta_k(x')\chi_R(x_N)$ where $\eta_k \in C_c^1(\mathbf{R}^{N-1})$ and $\chi_R \in C_c^1(\mathbf{R})$ is defined as

$$\chi_R(t) := \begin{cases} 1, & \text{if } R + 1 < t < 2R + 1, \\ 0, & \text{if } t < R \text{ or } t > 2R + 2, \end{cases}$$

for $R > 1$, $0 \leq \chi_R \leq 1$ and $0 \leq \chi_R' \leq 2$. Note also that since u is an H-monotone solution, the system (6) is then orientable. It follows from Theorem 4 that v is one dimensional and consequently the energy of v in a two-dimensional ball of radius R is bounded by a multiple of R, which yields that

$$\limsup_{t\to\infty} E(u^t) \le CR^2, \tag{39}$$

where here $u^t(x') := u(x', x_n + t)$ for $t \in \mathbf{R}$ and $E_R(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + H(u) - c_u d\mathbf{x}$ for $c_u := \inf H(u)$.

To finish the proof, we shall show that

$$\int_{B_R} |\nabla u|^2 \le CR^2. \tag{40}$$

Note that shifted function u^t is also a bounded solution of (6) with $|\nabla u_i^t| \in L^\infty(\mathbf{R}^N)$, i.e.,

$$\Delta u^t = \nabla H(u^t) \text{ in } \mathbf{R}^N, \tag{41}$$

and also

$$\partial_t u_i^t > 0 > \partial_t u_j^t \quad \text{for all } i \in I \text{ and } j \in J \text{ and in } \mathbf{R}^N. \tag{42}$$

Since u_i^t converges to v_i in $C^1_{loc}(\mathbf{R}^N)$ for all $i = 1, \dots, m$, we have

$$\lim_{t\to\infty} E(u^t) = E(v).$$

Now, we claim that the following upper bound for the energy holds.

$$E_R(u) \le E_R\left(u^t\right) + M \int_{\partial B_R} \left(\sum_{i\in I} \left(u_i^t - u_i\right) + \sum_{j\in J} \left(u_j - u_j^t\right) \right) dS \quad \text{for all } t \in \mathbf{R}^+, \tag{43}$$

where $M = \max_i ||\nabla u_i||_{L^\infty(\mathbf{R}^N)}$. Indeed, by differentiating the energy functional along the path u^t, one gets

$$\partial_t E_R\left(u^t\right) = \int_{B_R} \nabla u^t \cdot \nabla\left(\partial_t u^t\right) + \int_{B_R} \nabla H\left(u^t\right) \partial_t u^t, \tag{44}$$

where $\nabla H(u^t)\partial_t u^t = \sum_i H_{u_i}(u^t)\partial_t u_i^t$. Now, multiply (41) with $\partial_t u^t$, to obtain

$$-\int_{B_R} \nabla u^t \cdot \nabla\left(\partial_t u^t\right) + \int_{\partial B_R} \partial_\nu u^t \partial_t u^t = \int_{B_R} \nabla H\left(u^t\right) \partial_t u^t. \tag{45}$$

From (45) and (44) we obtain

$$\partial_t E_R\left(u^t\right) = \int_{\partial B_R} \partial_\nu u^t \partial_t u^t = \sum_i \int_{\partial B_R} \partial_\nu u_i^t \partial_t u_i^t. \tag{46}$$

Note that $-M \le \partial_\nu u^t \le M$ and $\partial_t u_i^t > 0 > \partial_t u_j^t$ for $i \in I$ and $j \in J$. Therefore,

$$\partial_t E_R\left(u^t\right) \ge M \int_{\partial B_R} \left(\sum_j \partial_t u_j^t - \sum_i \partial_t u_i^t \right) dS. \tag{47}$$

On the other hand,

$$
\begin{aligned}
E_R(u) &= E_R\left(u^t\right) - \int_0^t \partial_t E_R\left(u^s\right) ds, \\
&\leq E_R\left(u^t\right) + M \int_0^t \int_{\partial B_R} \left(\sum_i \partial_s u_i^s - \sum_j \partial_s u_j^s\right) dS ds \\
&= E_R\left(u^t\right) + M \int_{\partial B_R} \left(\sum_i \left(u_i^t - u_i\right) + \sum_j \left(u_j - u_j^t\right)\right) dS. \qquad (48)
\end{aligned}
$$

To finish the proof of the theorem just note that $u_i < u_i^t$ and $u_j^t < u_j$ for all $i \in I$, $j \in J$ and $t \in \mathbf{R}^+$. Moreover, from (39) we have $\lim_{t\to\infty} E_R(u^t) \leq CR^2$. Therefore, (48) yields

$$E_R(u) \leq C|\partial B_R| \leq CR^2,$$

and we are done. □

The above proof suggests that—just as in the case of a single equation—any H-monotone solution u of (6) must satisfy the following estimate

$$\int_{B_R} |\nabla u|^2 \leq CR^{N-1} \quad \text{for any } R > 1, \qquad (49)$$

for some constant $C > 0$. This can be done in the following particular case.

Theorem 6 *If u is a bounded H-monotone solution of (6) such that for $i = 1, \ldots, m$,*

$$\lim_{x_N \to \infty} u_i\left(\mathbf{x}', x_N\right) = a_i, \quad \forall \mathbf{x} = \left(\mathbf{x}', x_N\right) \in \mathbf{R}^{N-1} \times \mathbf{R}$$

where a_i are constants, then

$$E_R(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + H(u) - H(a) d\mathbf{x} \leq CR^{N-1}, \qquad (50)$$

where $a = \{a_i\}_{i=1}^{i=m}$ and C is a positive constant independent of R.

Proof We first note the following decay on the energy of the shifted function u^t as defined above,

$$\lim_{t\to\infty} E_R\left(u^t\right) = 0. \qquad (51)$$

Indeed, since u^t is convergent to a pointwise, one can see that

$$\lim_{t\to\infty} \int_{B_R} \left(H\left(u^t\right) - H(a)\right) d\mathbf{x} \to 0.$$

Therefore, we need to prove that

$$\lim_{t\to\infty} \int_{B_R} |\nabla u_i^t|^2 d\mathbf{x} \to 0.$$

To do so, multiply both sides of (41) with $u_i^t - a_i$ and integrate by parts to get

$$-\int_{B_R} |\nabla u_i^t|^2 + \int_{\partial B_R} \partial_\nu u_i^t \left(u_i^t - a_i\right) = \int_{B_R} \nabla H\left(u^t\right)\left(u_i^t - a_i\right),$$

which yields (51).

To get the energy bound in (50), one can follow the proof of the previous theorem to end up with

$$E_R(u) \leq E_R\left(u^t\right) + C|\partial B_R| \quad \text{for all } t \in \mathbf{R}^+.$$

To conclude, it suffices to send $t \to \infty$ and to use the fact that $\lim_{t\to\infty} E_R(u^t) = 0$ to finally obtain that

$$E_R(u) \leq C|\partial B_R| \leq CR^{N-1}.$$

Remark 2 Using Pohozaev type arguments one can see that

$$\Gamma_R = \frac{E_R(u)}{R^{N-1}} \text{ is increasing} \tag{52}$$

provided the following pointwise estimate holds:

$$|\nabla u|^2 \leq 2H(u). \tag{53}$$

Note that this is an extension of the pointwise estimate that Modica [14] proved in the case of a single equation. It is still not known for systems, though Caffarelli and Lin in [10] and later, Alikakos in [3] have shown, in the case where $H \geq 0$, the following weaker monotonicity formula, namely that

$$\Lambda_R = \frac{E_R(u)}{R^{N-2}} \text{ is increasing in } R. \tag{54}$$

Remark 3 The H-monotonicity assumption seems to be crucial for concluding that the solutions are one-dimensional. Indeed, it was shown in [1] that when H is a multiple-well potential on $\mathbf{R}^2$, the system has entire heteroclinic solutions (u, v), meaning that for each fixed $x_2 \in \mathbf{R}$, they connect (when $x_1 \to \pm\infty$) a pair of constant global minima of W, while if $x_2 \to \pm\infty$, they connect a pair of distinct one dimensional stationary wave solutions $z_1(x_1)$ and $z_2(x_1)$. Note that these convergence are even uniform, which means that the corresponding Gibbons conjecture for systems of equations is not valid in general, without the assumption of H-monotonicity.

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