MATH 517 (2017-2018) Homework One
Due Date: Jan. 31, 2018

1. Use the following De Giorgi's iteration to prove the following simple estimate: Let u satisfy

$$-\Delta u = f \text{ in } \Omega$$

then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\|f\|_{L^\infty(\Omega)}$$

Hint: Step 1: Multiplying the equation by $(u-k)_+$ and showing that for $h > k \geq k_0$

$$|A(h)| \leq (\frac{C\|f\|_{L^\infty}}{h-k})^p |A(k)|^{p-1}$$

where $A(h) = |\{u > h\}$. Here $p > 2$. Step 2: use iteration to show that $|A(h)| = 0$ for $h \geq k_0 + d$.

2. Consider the following energy functional

$$E[u] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega (1-u^2)^2$$

in the space

$$M = \{u \in H^1(\Omega);\ \frac{1}{|\Omega|}\int_\Omega u = m\}$$

where $-1 < m < 1$. Use direct method to show the existence of minimizers of E and find the Euler-Lagrange equation.

3. (a) Let $\mathcal{D}^{1,2}(R^N)$ be the completion of C_0^∞ under the norm $\|u\| = (\int |\nabla u|^2)^{1/2}$. Show that for $N \geq 3$ and $u = u(r) \in \mathcal{D}^{1,2}$
Then $|u(r)| \leq Cr^{-\frac{N-2}{2}}$.

(b) Consider the following minimization problem

$$S_N = \inf\{\int |\nabla u|^2 \mid \int |u|^{\frac{2N}{N-2}} = 1\}$$

Show that S_N is attained. Hint: for compactness, use (a) and the scaling invariance of $\int |\nabla u|^2, \int u^{\frac{2N}{N-2}}$.

(c) Compute the minimizers in (b) and the value S_N.

4. (a) State and prove Brezis-Lieb Lemma.

(b) Consider the following problem

$$\Delta u - a(x)u + b(x)u^p = 0 \quad u \in H^1(R^N)$$

Assume that

$$1 < p < \frac{N+2}{N-2},$$

$$a(x) \to a_\infty, b(x) \to b_\infty \text{ as } |x| \to +\infty$$

State a condition for existence and prove it.

5. Prove the monotone iteration scheme for the following elliptic system

$$\begin{cases} \Delta u + f(u,v) = 0 \text{ in } \Omega; \\ \Delta v + g(u,v) = 0 \text{ in } \Omega; \\ B[u] = B[v] = 0 \text{ on } \partial\Omega \end{cases}$$

where $B[u] = \frac{\partial u}{\partial \nu} + b(x)u, b(x) \geq 0$. Here $\frac{\partial f}{\partial v} \leq 0, \frac{\partial g}{\partial u} \leq 0$.

Hint: Consider two pairs of sub-super solution $(\underline{u}, \underline{v})$:

$$\Delta\underline{u} + f(\underline{u}, \underline{v}) > 0, \Delta\underline{v} + g(\underline{u}, \underline{v}) < 0$$

and super-sub solution $(\bar{u}, \bar{v})$:

$$\Delta\bar{u} + f(\bar{u}, \bar{v}) < 0, \Delta\bar{v} + g(\bar{u}, \bar{v}) > 0$$

so that

$$\underline{u} < \bar{u}, \underline{v} > \bar{v}$$

From sub-super solution, one constructs an increasing sequence of u_n, and a decreasing sequence of v_n.