MATH 517 (2017-2018) Homework Two
Due Date: Feb. 14, 2018

1. Use Mountain-Pass Lemma to prove the existence of one positive solution to

$$-\Delta u = -\lambda u + u^p \ \text{ in } \Omega, u > 0 \text{ in } \Omega, \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

where $p < \frac{n+2}{n-2}$ and $\lambda > 0$. Show that for λ large the Mountain-Pass solution is nontrivial, i.e., $u \not\equiv \lambda^{\frac{1}{p-1}}$.
2. Consider the critical exponent problem in an annulus

$$-\Delta u = u^{\frac{n+2}{n-2}} \text{ in } B_{r_2} \backslash B_{r_1}, u = 0 \text{ on } \partial(B_{r_2} \backslash B_{r_1})$$

where $r_2 > r_1, n \geq 3$. Show that there always exists a solution.
3. Consider the following Henon's equation:

$$-\Delta u = |x|^\alpha u^p \text{ in } B_1(0), u > 0 \text{ in } B_1, u = 0 \text{ on } \partial B_1$$

where $\alpha > 0, p > 1$, and $B_1(0)$ denotes the unit ball in $R^n, n \geq 3$.
(a) For $p \geq \frac{n+2+2\alpha}{n-2}$, show that $u \equiv 0$.
(b) Show that for $p < \frac{n+2+2\alpha}{n-2}$ there exists a solution.

Hint: Reference: Wei-Ming Ni, A nonlinear Dirichlet problem on the unit ball and its applications. Indiana Univ. Ma 31 (1982), no. 6, 801807.

(c) Show that the exterior problem

$$-\Delta u = u^p \text{ in } R^n \backslash B_1, u = 0 \text{ on } \partial B_1$$

admits a radially symmetric solution when $p > \frac{n+2}{n-2}$.

Hint: use Kelvin transformation $|x|^{n-2} u(\frac{x}{|x|^2})$ to transform it into Henon's equation.
4. Consider the following problem in R^n:

$$-\Delta u = -u + u^p, \text{ in } R^n, u > 0, u \in H^1(R^n)$$

where $n \geq 3$.
(a) Show that for $p \geq \frac{n+2}{n-2}$, there are no solutions
(b) Show that the following problem

$$-\Delta u = -u + u^p + u^{\frac{n+2}{n-2}}, \text{ in } R^n, u > 0, u \in H^1(R^n)$$

admits a solution when $n \geq 5$.

Hint: Show that the mountain-pass value for the energy fucntional

$$J[u] = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \int u^2 - \frac{n-2}{2n} \int u^{\frac{2n}{n-2}} - \frac{1}{p+1} \int u^{p+1}$$

is strictly below $\frac{1}{n} S_n^{\frac{n}{2}}$. Here the functional space is $H_r^1 = H^1 \cap \{u = u(r)\}$.
5. Derive a Pohozaev identity for bi-harmonic equations

$$\Delta^2 u = f(u) \text{ in } \Omega$$

with either Navier boundary condition

$$u = \Delta u = 0 \text{ on } \partial\Omega$$

or Dirichlet boundary condition

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

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6. Consider the following inhomogeneous critical problem

$$-\Delta u = u^{\frac{n+2}{n-2}} + \mu f(x) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

where $\Omega \subset R^n$ and $n \geq 3$. Here $f(x) > 0$ is a smooth function.
(a) Use sub-super solution method to the existence of one solution u_μ for small μ.
(b) Let $u = u_\mu + v$. Use Mountain-Pass-Lemma to show the existence of the second solution for $n \geq 5$.
Hint: For example let $n = 6$. Then v satisfies

$$-\Delta v = 2u_\lambda v + v^2$$

For the new energy functional

$$J[v] = \frac{1}{2}\int |\nabla v|^2 - \int u_\mu v^2 - \frac{1}{3}\int v^3$$

Check the mountain-pass condition, and the energy condition.