MATH 517 (2017-2018) Homework Three

Due Date: March 9, 2018

1. Prove the following one-dimensional version of maximum principle by comparison function: Let u and w satisfy

$$u^{''}+V_1(x)u=0, \quad w^{''}+V_2(x)w=0$$

where $V_2 \geq V_1$. Show that if $w(x) \neq 0$ in the open interval (a,b), then u can have at most one zero on the closed interval $[a,b]$ unless $w(a)=w(b)=0, V_1 \equiv V_2$ and u is a constant multiple of w.

2. (a) State the narrow domain principle; (b) Prove a general version of Gidas-Ni-Nirenberg symmetry theorem: any positive solution to

$$\Delta u + f(|x|,u)=0 \text{ in } B_1, u=0 \text{ on } \partial B_1$$

must be radially symmetric, where $f(|x|,u)$ is monotone decreasing in $|x|$. Show that the last condition is also necessary.

3. Consider the following problem

$$\Delta u + f(u)=0, u>0 \text{ in } \Omega, u=0 \text{ on } \partial\Omega$$

(a) If Ω is a convex domain, show that there is a fixed neighborhood of $\partial\Omega$ (depending only on the geometry of the domain) such that $\nabla u \neq 0$. (b) When $n=2$, show that the same result holds for any domain. (3) When $n \geq 3$, show that the same result holds for any domain if $f(u)=u^p, p \leq \frac{n+2}{n-2}$.

Hint: when Ω is not convex at a boundary, one can use Kelvin transform to make it convex near the point.

4. Use the method of moving planes to show that any positive solutions to

$$\Delta u - u + u^p = 0, u>0 \text{ in } \mathbb{R}^n, \ u(x) \to 0 \text{ as } |x| \to +\infty$$

must be radially symmetric around some point. (No fast-decaying principle needed.) Here $p>1$. Hint: first prove that $u \lesssim e^{-|x|}$.

5.(a) State the fast-decaying principle. (b) Consider the following problem

$$\Delta u + f(u) = 0, u>0 \text{ in } \mathbb{R}^n$$

where $f(u) \geq 0$ satisfies the following condition

$$\frac{f(t)}{t^{\frac{n+2}{n-2}}} \text{ is nonincreasing for } \ t>0$$

Show that u is radially symmetric around point.