MATH 517 (2017-2018) Homework Four
Due Date: March 23, 2018

1. Show that if $p \leq \frac{n}{n-2}$ (Serrin's exponent) and u satisfies the following differential inequality

$$\Delta u + u^p \leq 0, u \geq 0 \text{ in } R^n$$

then $u \equiv 0$.

2. Prove the following Hardy's inequality: If $n \geq 3$ and $\phi \in C_0^\infty(R^n \backslash \{0\})$, then

$$\frac{(n-2)^2}{4} \int_{R^n} \frac{\phi^2}{|x|^2} dx \leq \int_{R^n} |\nabla \phi|^2$$

Show that $\frac{(n-2)^2}{4}$ is the best constant.

3. Consider the following equation:

$$\Delta u + e^u = 0 \text{ in } R^n, n \geq 2$$

Show that for $n \leq 9$ there is no stable solution.
Hint: test the function $e^{\gamma u} \eta^2$.

4. (a) Derive a monotonicity formula for solutions of

$$\Delta u + |x|^l u^p = 0 \quad \text{in } R^n$$

where $l > -2$. (b) Compute the corresponding Joseph-Lundgren exponent $p_{JL}(l,n)$. (c) Show that for $p < p_{JL}(l,n)$ the homogeneous stable solution must be zero.
What can you say about the stable solutions?

5. Consider the Allen-Cahn equation

$$\Delta u + u - u^3 = 0, \ |u| \leq 1 \text{ in } R^n$$

(a) Prove the Modica's estimate

$$\frac{1}{2} |\nabla u|^2 \leq \frac{1}{4} (1 - u^2)^2$$

(b) Using Modica' estimate to prove the following monotonicity formula:

$$E(r) = \frac{1}{r^{n-2}} \int_{B_r} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - u^2)^2\right)$$

$$\frac{dE}{dr} \geq 0$$