MATH 517 (2017-2018) Homework Five (Last Written Homework)
Due Date: April 6, 2018

1. Consider the following problem

$$\epsilon^2 \Delta u - V(x)u + K(x)u^p = 0, u \in H^1, u > 0 \text{ in } \mathbb{R}^n$$

where $1 < p < \frac{n+2}{n-2}$ and

$$0 < C_1 \leq V(x) \leq C_2, 0 < C_1 \leq K(x) \leq C_2$$

Use the reduction method to construct a single spike solution.

2. Let $U_0 = c_n (\frac{1}{1+|y|^2})^{\frac{n-2}{2}}$ be the radial solution of

$$\Delta U + U^{\frac{n+2}{n-2}} = 0$$

Show that it is nondegenerate, i.t., all bounded solutions to

$$\Delta \phi + \frac{n+2}{n-2} U_0^{\frac{4}{n-2}} \phi = 0$$

are given by $Z_0, Z_j, j = 1, ..., n$. Here $Z_0 = \frac{n-2}{2} U_0 + x \cdot \nabla U_0$.

3. Use reduction method to construct a solution to

$$\Delta u + u^{\frac{n+2}{n-2} - \epsilon} = 0, u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega$$

concentrating on a minimum of the Robin function $R(\xi, \xi)$, as $\epsilon \to 0+$.

4. (a) Let $w(t)$ be the unique homoclinic solution to

$$w^{''} - w + w^p = 0, -\infty < x < +\infty, w^{'}(0) = 0, w(\pm\infty) = 0$$

Show that the first eigenvalue is

$$\lambda_1 = \frac{(p-1)(p+3)}{4}, Z_1(x) = w^{\frac{p+1}{2}}$$

(b) Prove that all bounded solutions to

$$\Delta \phi - \phi + p w^{p-1}(x) \phi = 0 \text{ in } \mathbb{R}^2$$

are

$$\phi = c w^{'}(x) + d_1 Z_1(x) \cos(\sqrt{\lambda_1} y) + d_2 Z_1(x) \sin(\sqrt{\lambda_1} y)$$

5. Consider the following problem

$$\epsilon^2 \Delta u + a(x)(u - u^3) = 0 \quad x \in \mathbb{R}^2$$

Let Γ be a closed and compact curve in $\mathbb{R}^2$. We want to construct a solution whose zero level set lying close to Γ.

Derive the necessary condition for Γ. Set-up the inner-outer gluing procedure. Derive the reduced Jacobi equation.