Existence and Blow-up Profiles of Ground States in Second Order Multi-population Mean-field Games Systems

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Abstract

In this paper, we utilize the variational structure to study the existence and asymptotic profiles of ground states in multi-population ergodic Mean-field Games systems subject to some local couplings with mass critical exponents. Of concern the attractive and repulsive interactions, we impose some mild conditions on trapping potentials and firstly classify the existence of ground states in terms of intra-population and interaction coefficients. Next, as the intra-population and inter-population coefficients approach some critical values, we show the ground states blow up at one of global minima of potential functions and the corresponding profiles are captured by ground states to potential-free Mean-field Games systems for single population up to translations and rescalings. Moreover, under certain types of potential functions, we establish the refined blow-up profiles of corresponding ground states. In particular, we show that the ground states concentrate at the flattest global minima of potentials.

Keywords: Multi-population Mean-field Games Systems; Variational Approaches; Constrained Minimization; Blow-up Solutions

1 Introduction

Mean-field Games systems are proposed to describe decision-making among a huge number of indistinguishable rational agents. In real world, various problems involve numerous interacting players, which causes theoretical analysis and even numerical study become impractical. To overcome this issue, Huang et al. [8] and Lasry et al. [9] borrowed the ideas arising from particle physics and introduced Mean-field Games theories and systems independently. For their rich applications in economics, finance, management, etc, we refer the readers to [7].

Focusing on the derivation of Mean-field Games systems, we assume that the i-th agent with $i = 1, \cdots, n$ satisfies the following controlled stochastic differential equation (SDE):

$$dX_t^i = -\gamma_t^i dt + \sqrt{2} dB_t^i, \ \ X_0^i = x^i \in \mathbb{R}^N,$$

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where x^i is the initial state, γ_t^i denotes the controlled velocity and B_t^i represent the independent Brownian motion. Suppose all agents are indistinguishable and minimize the following average cost:

$$J(\gamma_t) := \mathbb{E}\int_0^T [L(\gamma_t) + V(X_t) + f(m(X_t))]dt + u_T(X_T), \tag{1.1}$$
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where L is the Lagrangian, V describes the spatial preference and function f depends on the population density. By applying the standard dynamic programming principle, the coupled PDE system consisting of Hamilton–Jacobi–Bellman equation and Fokker-Planck equation is formulated, in which the second equation characterize the distribution of the population. The crucial assumption here is all agents are homogeneous and minimize the same cost (1.1). Whereas, in some scenarios, the game processes involve several classes of players with distinct objectives and constraints. Correspondingly, the distributions of games can not be modelled by classical Mean-field Games systems. Motivated by this, multi-population Mean-field Games systems were proposed and the derivations of multi-population stationary problems used to describe Nash equilibria are shown in [6]. For some relevant results of the study of multi-population Mean-field Games systems, we refer the readers to [3].

The objective of this paper is to study the following stationary two-population second order Mean-field Games system:

$$\begin{cases} -\Delta u_1 + H(\nabla u_1) + \lambda_1 = V_1(x) + f_1(m_1, m_2), & x \in \mathbb{R}^N, \\ \Delta m_1 + \nabla\cdot(m_1 \nabla H(\nabla u_1)) = 0, & x \in \mathbb{R}^N, \\ -\Delta u_2 + H(\nabla u_2) + \lambda_2 = V_2(x) + f_2(m_1, m_2), & x \in \mathbb{R}^N, \\ \Delta m_2 + \nabla\cdot(m_2 \nabla H(\nabla u_2)) = 0, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m_1\, dx = \int_{\mathbb{R}^N} m_2\, dx = 1, \end{cases} \tag{1.2}$$
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where $H : \mathbb{R}^N \to \mathbb{R}$ is a Hamiltonian, (m_1, m_2) represents the population density, (u_1, u_2) denotes the value function and (f_1, f_2) is the coupling. Here $V_i(x)$, $i = 1, 2$ are potential functions and $(\lambda_1, \lambda_2) \in \mathbb{R}\times\mathbb{R}$ denotes the Lagrange multiplier. In particular, Hamiltonian H is in general chosen as

$$H(p) = C_H|p|^\gamma \text{ with } C_H > 0 \text{ and } \gamma > 1. \tag{1.3}$$
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In light of the definition, the corresponding Lagrangian is given by

$$L = C_L|\gamma|^{\gamma'},\ \ \gamma' = \frac{\gamma}{\gamma-1} > 1,\ C_L = \frac{1}{\gamma'}(\gamma C_H)^{\frac{1}{1-\gamma}} > 0.$$

From the viewpoint of variational methods, the single population counterpart of (1.2) has been studied intensively when the coupling f is local and satisfies $f = -em^\alpha$ with constant $e > 0$, see [2, 5, 10]. In detail, there exists a mass critical exponent $\alpha = \alpha^* := \frac{\gamma'}{N}$ such that only when $\alpha < \alpha^*$, the stationary problem admits ground states for any $e > 0$. Moreover, when $\alpha = \alpha^*$, one can find $e^* > 0$ such that the stationary Mean-field Games system has ground states only for $e < e^*$ [5]. In this paper, we shall extend the above results into two-species stationary Mean-field Games system (1.2). Similarly as in [5], we consider the mass critical exponent case and define

$$f_1 = -\alpha_1 m_1^{\frac{\gamma'}{N}} - \beta m_1^{\frac{\gamma'}{2N}-\frac{1}{2}} m_2^{\frac{1}{2}+\frac{\gamma'}{N}},\ \ f_2 = -\alpha_2 m_2^{\frac{\gamma'}{N}} - \beta m_2^{\frac{\gamma'}{2N}-\frac{1}{2}} m_1^{\frac{1}{2}+\frac{\gamma'}{N}}, \tag{1.4}$$
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where $\alpha_i > 0$, $i = 1, 2$ and β measure the strengths of intra-population and inter-population interactions, respectively. We shall employ the variational approach to classify the existence of ground states and analyze their asymptotic profiles to (1.2) in terms of α_i, $i = 1, 2$ and β. Noting the forms of nonlinearities shown in (1.4), we assume $\gamma' > N$ here and in the sequel for our analysis; otherwise the strong singularities might cause difficulties for finding ground states to (1.2) while taking limits. It is an intriguing but challenging problem to explore the existence of global minimizers in the case of $1 < \gamma' \leq N$.

By employing the variational methods, the existence of ground states to (1.2) is associated with the following constrained minimization problem:

$$e_{\alpha_1,\alpha_2,\beta} = \inf_{(m_1,w_1,m_2,w_2)\in\mathcal{K}} \mathcal{E}(m_1, w_1, m_2, w_2), \tag{1.5}$$

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where

$$\begin{aligned}\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1, w_1, m_2, w_2) := \sum_{i=1,2}\Big(C_L \int_{\mathbb{R}^N} \Big|\frac{w_i}{m_i}\Big|^{\gamma'} m_i\,dx + \int_{\mathbb{R}^N} V_i m_i\,dx - \frac{N}{N+\gamma'}\alpha_i \int_{\mathbb{R}^N} m_i^{1+\frac{\gamma'}{N}}\,dx\Big)\\ - \frac{2\beta N}{N+\gamma'}\int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx,\end{aligned} \tag{1.6}$$

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and $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ with

$$\begin{aligned}\mathcal{K}_i = \Big\{(m_i, w_i)\Big| - \int_{\mathbb{R}^N} \nabla m_i \cdot \nabla\varphi\,dx + \int_{\mathbb{R}^N} w_i \cdot \nabla\varphi\,dx = 0,\ \forall\varphi \in C_c^\infty(\mathbb{R}^N),\\ m_i \in W^{1,\gamma'}(\mathbb{R}^N),\ w_i \in L^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} m_i\,dx = 1,\ \int_{\mathbb{R}^N} V_i m_i\,dx < +\infty,\ m_i \geq 0 \text{ a.e.}\Big\}\end{aligned} \tag{1.7}$$

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for $i = 1, 2$. Due to the technical restriction of our analysis, we impose the following assumptions on potential functions $V_i(x)$ with $i = 1, 2$:

(H1).

$$\inf_{x\in\mathbb{R}^N} V_i(x) = 0,\ V_i \in C^1(\mathbb{R}^N) \text{ and } \lim_{|x|\to+\infty} V_i(x) = +\infty; \tag{1.8}$$

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(H2).

$$\liminf_{|x|\to+\infty} \frac{V_i(x)}{|x|^b} > 0, \quad \limsup_{|x|\to+\infty} \frac{V_i(x)}{e^{\delta|x|}} < +\infty \text{ with constants } b > 0,\ \delta > 0. \tag{1.9}$$

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Similarly as shown in [5], the existence of ground states to (1.2) has a strong connection with the following minimization problem for the single species potential-free Mean-field Games System:

$$(M^*)^{\frac{\gamma'}{N}} = \inf_{(m,w)\in\mathcal{A}} \frac{\Big(\int_{\mathbb{R}^N} C_L\big|\frac{w}{m}\big|^{\gamma'} m\,dx\Big)\Big(\int_{\mathbb{R}^N} dx\Big)^{\frac{\gamma'}{N}}}{\frac{1}{1+\frac{\gamma'}{N}}\int_{\mathbb{R}^N} m^{1+\frac{\gamma'}{N}}\,dx}, \tag{1.10}$$

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where

$$\begin{aligned}\mathcal{A} := \Big\{(m, w) \in W^{1,\gamma'}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\Big| - \int_{\mathbb{R}^N} \nabla m \cdot \nabla\varphi\,dx + \int_{\mathbb{R}^N} w \cdot \nabla\varphi\,dx = 0,\ \forall\varphi \in C_c^\infty(\mathbb{R}^N),\\ 0 \leq m \not\equiv 0,\ \int_{\mathbb{R}^N} m|x|^b\,dx < +\infty \text{ with } b > 0 \text{ given by (1.9)}\Big\}.\end{aligned}$$

We would like to point out that it was shown in Theorem 1.2 [5] that problem (1.10) is attainable and admits at least a minimizer satisfying

$$\begin{cases} -\Delta u + C_H|\nabla u|^\gamma - \frac{\gamma'}{NM^*} = -m^{\frac{\gamma'}{N}}, \\ \Delta m + C_H\gamma\nabla\cdot(m|\nabla u|^{\gamma-2}\nabla u) = 0,\ w = -C_H\gamma m|\nabla u|^{\gamma-2}\nabla u, \\ \int_{\mathbb{R}^N} m\,dx = M^*,\ 0 < m < Ce^{-\delta_0|x|}, \end{cases} \tag{1.11}$$

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where $\delta_0 > 0$ is some constant. As a consequence, the following Gagliardo-Nirenberg type's inequality holds:

$$\frac{N}{N+\gamma'}\int_{\mathbb{R}^N} m^{1+\frac{\gamma'}{N}}\,dx \le \frac{1}{a^*}\Big(C_L\int_{\mathbb{R}^N}\Big|\frac{w}{m}\Big|^{\gamma'} m\,dx\Big)\Big(\int_{\mathbb{R}^N} m\,dx\Big)^{\frac{\gamma'}{N}}\ \forall\,(m,w)\in\mathcal{A}, \tag{1.12}$$

GNinequal

where $a^* := (M^*)^{\frac{\gamma'}{N}}$. With the aid of (1.12), we shall establish several results for the existence and non-existence of global minimizers to (1.2) and further study the blow-up behaviors of ground states in terms of α_i, $i = 1,2$ and β defined in (1.4). We emphasize that $\alpha_i > 0$, $i = 1,2$ represent the self-focusing of the i-th component and $\beta > 0$ denotes the attractive interaction, while $\beta < 0$ represents the repulsive interaction.

In the next subsection, we shall first state our existence results for attractive and repulsive interactions then discuss the corresponding blow-up profiles results.

1.1 Main Results

⟨thm11multi⟩ **Theorem 1.1.** *Suppose that $V_i(x)$ with $i = 1,2$ satisfy (H1) and (H2) given by (1.8) and (1.9), respectively. Define $a^* := (M^*)^{\frac{\gamma'}{N}}$ with M^* given in (1.11), then we have*

(i). if $0 < \alpha_1,\alpha_2 < a^*$ and $-\infty < \beta < \beta_* := \sqrt{(a^*-\alpha_1)(a^*-\alpha_2)}$, problem (1.5) has at least one global minimizer $(m_{1,\boldsymbol{a}}, w_{1,\boldsymbol{a}}, m_{2,\boldsymbol{a}}, w_{2,\boldsymbol{a}}) \in \mathcal{K}$. Correspondingly, there exists a solution $(m_{1,\boldsymbol{a}}, m_{2,\boldsymbol{a}}, u_{1,\boldsymbol{a}}, u_{2,\boldsymbol{a}}) \in W^{1,p}(\mathbb{R}^N)\times W^{1,p}(\mathbb{R}^N)\times C^2(\mathbb{R}^N)\times C^2(\mathbb{R}^N)$ with any $p > 1$ and $(\lambda_{1,\boldsymbol{a}},\lambda_{2,\boldsymbol{a}}) \in \mathbb{R}\times\mathbb{R}$ such that

$$\begin{cases} -\Delta u_1 + C_H|\nabla u_1|^\gamma + \lambda_1 = V_1(x) - \alpha_1 m_1^{\frac{\gamma'}{N}} - \beta m_1^{\frac{\gamma'}{2N}-\frac12} m_2^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N, \\ \Delta m_1 + C_H\gamma\nabla\cdot(m_1|\nabla u_1|^{\gamma-2}\nabla u_1) = 0, & x\in\mathbb{R}^N, \\ -\Delta u_2 + C_H|\nabla u_2|^\gamma + \lambda_2 = V_2(x) - \alpha_2 m_2^{\frac{\gamma'}{N}} - \beta m_2^{\frac{\gamma'}{2N}-\frac12} m_1^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N, \\ \Delta m_2 + C_H\gamma\nabla\cdot(m_2|\nabla u_2|^{\gamma-2}\nabla u_2) = 0, & x\in\mathbb{R}^N, \\ \int_{\mathbb{R}^N} m_1\,dx = \int_{\mathbb{R}^N} m_2\,dx = 1; \end{cases} \tag{1.13}$$

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(ii). either $\alpha_1 > a^*$ or $\alpha_2 > a^*$ or $\beta > \beta^* := \frac{2a^*-\alpha_1-\alpha_2}{2}$, problem (1.5) has no minimizer.

Theorem 1.1 indicates that when the self-focusing cofficients α_i, $i = 1,2$ are small and the interaction is repulsive, or attractive but with the weak effect, problem (1.5) admits minimizers and correspondingly, there exist classical solutions to (1.13). Whereas, if the self-focusing effects and the attractive interaction are strong, problem (1.5) does not have any minimizer. In fact, there are some gap regions for the existence results shown in Theorem 1.1 since we have $\beta^* \ge \beta_*$ and the equality holds only when $\alpha_1 = \alpha_2$. It is also an interesting problem to explore the case of $\alpha_i < a^*$, $i = 1,2$ and $\beta_* < \beta < \beta^*$.

Of concern one borderline case $\beta = \beta_* = \beta^*$ with $\alpha_1 = \alpha_2 < a^*$ shown in Theorem 1.1, we further obtain

⟨thm12⟩ **Theorem 1.2.** *Assume all conditions in Theorem 1.1 hold and suppose $V_i(x)$, $i = 1, 2$ satisfy*

$$\inf_{x\in\mathbb{R}^N} (V_1(x) + V_2(x)) = 0. \tag{1.14}$$ moreassum

Then if $\alpha := \alpha_1 = \alpha_2 < a^*$ and $0 < \beta = \beta^* = \beta_* = a^* - \alpha < a^*$, we have problem (1.5) has no minimizer.

Theorem 1.2 demonstrates that when the self-focusing effects are subcritical but the attractive interaction is strong and under critical case, there is no minimizer to problem (1.5). Besides the borderline case discussed in Theorem 1.2, we also study the case of $\alpha_i = a^*$ for $i = 1$ or 2 and obtain

finalexistence⟩ **Theorem 1.3.** *Assume all conditions in Theorem 1.1 hold. If one of the following conditions holds:*

(i). $\alpha_1 = \alpha_2 = a^*$ *and* $-\infty < \beta \le 0$;

(ii). $\alpha_1 = a^*$, $0 < \alpha_2 < a^*$ *and* $0 \le \beta \le \beta^* = \frac{a^*-\alpha_2}{2}$,

then we have problem (1.5) does not admit any minimizer.

Remark 1.1. *We remark that when $\alpha_2 = a^*$, $0 < \alpha_1 < a^*$ and $0 \le \beta \le \beta^*$, (1.5) also does not have any minimizer since m_1-population and m_2-population are symmetric in (1.2),*

Theorem 1.3 shows that if one of self-focusing coefficients are critical, system (1.2) does not admit the ground state. We next summarize results for the study of blow-up profiles of ground states in some singular limits, in which two cases are concerned: attractive interactions with $\beta > 0$ and repulsive ones with $\beta < 0$. Before stating our results, we give some preliminary notations. Define

$$Z_i := \{x | V_i(x) = 0\}, i = 1, 2. \tag{1.15}$$ zerosdefi

For any $p > 0$, we denote

$$H_{\bar m,p}(y) := \int_{\mathbb{R}^N} |x + y|^p \bar m(x)\, dx, \text{ and } \bar\nu_p := \inf_{(\bar m,\bar w)\in\mathcal{M}} \inf_{y\in\mathbb{R}^N} H_{\bar m,p}(y), \tag{1.16}$$ Hmoibarnu

with

$$\mathcal{M} := \{(\bar m, \bar w) | \exists u \text{ such that } (\bar m, \bar w, u) \text{ satisfies (1.11) and } (\bar m, \bar w) \text{ is a minimizer of (1.10)}\}. \tag{1.17}$$ mathcalMd

The following two theorems address the attractive case with $(\alpha_1, \alpha_2) \nearrow (a^* - \beta, a^* - \beta)$ and $Z_1 \cap Z_2 \neq \emptyset$, which are

hm13attractive⟩ **Theorem 1.4.** *Assume that $V_i(x)$ satisfies (1.8), (1.9) and $Z_1 \cap Z_2 \neq \emptyset$. Let $0 < \beta < a^*$, $0 < \alpha_1, \alpha_2 < a^* - \beta := \alpha^*_\beta$, $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ be a minimizer of $e_{\alpha_1,\alpha_2,\beta}$ with $\mathbf{a} := (\alpha_1, \alpha_2)$ and $(m_{1,\mathbf{a}}, u_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{2,\mathbf{a}})$ be a solution of (1.13). Define $\mathbf{a}^*_\beta := (\alpha^*_\beta, \alpha^*_\beta) = (a^* - \beta, a^* - \beta)$, then as $\mathbf{a} \nearrow \mathbf{a}^*_\beta$, we have for $i = 1, 2$,*

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} \Big(\int_{\mathbb{R}^N} C_L \Big| \frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}} \Big|^{\gamma'} m_{i,\mathbf{a}}\, dx - \frac{N(\alpha_i + \beta)}{N + \gamma'} \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx \Big) = 0, \tag{1.18}$$ thm13conc

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} \int_{\mathbb{R}^N} V_1(x) m_{1,\mathbf{a}} + V_2(x) m_{2,\mathbf{a}}\, dx = 0, \ \lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} \int_{\mathbb{R}^N} \Big(m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}} - m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}} \Big)^2 dx = 0, \tag{1.19}$$ thm13conc

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\,dx \to +\infty \text{ for both } i = 1,2 \tag{1.20}$$

and

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} \frac{\int_{\mathbb{R}^N} \left|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\right|^{\gamma'} m_{1,\mathbf{a}}\,dx}{\int_{\mathbb{R}^N} \left|\frac{w_{2,\mathbf{a}}}{m_{2,\mathbf{a}}}\right|^{\gamma'} m_{2,\mathbf{a}}\,dx} = 1, \quad \lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} \frac{\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx}{\int_{\mathbb{R}^N} m_{2,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx} = 1. \tag{1.21}$$

Moreover, define

$$\varepsilon := \varepsilon_{\boldsymbol{a}} := \left(C_L \int_{\mathbb{R}^N} \left|\frac{w_{1,\boldsymbol{a}}}{m_{1,\boldsymbol{a}}}\right|^{\gamma'} m_{1,\boldsymbol{a}}\right)^{-\frac{1}{\gamma'}} \to 0. \tag{1.22}$$

Let $x_{i,\varepsilon}$, $i = 1,2$ be one global minimal point of $u_{i,\boldsymbol{a}}$ and $y_{i,\varepsilon}$, $i = 1,2$ be one global maximal point of $m_{i,\boldsymbol{a}}$. Then we have up to a subsequence $\exists x_0$ s.t. $V_1(x_0) = V_2(x_0) = 0$, and

$$x_{i,\varepsilon}, y_{i,\varepsilon} \to x_0, \text{ as } \boldsymbol{a} \nearrow \boldsymbol{a}^*_\beta;$$

moreover, we find

$$\limsup_{\varepsilon\to0^+} \frac{|x_{1,\varepsilon} - x_{2,\varepsilon}|}{\varepsilon} < +\infty, \tag{1.23}$$

and

$$\limsup_{\varepsilon\to0^+} \frac{|x_{i,\varepsilon} - y_{j,\varepsilon}|}{\varepsilon} < +\infty, \quad i,j = 1,2. \tag{1.24}$$

In addition, let

$$u_{i,\varepsilon} := \varepsilon^{\frac{2-\gamma}{\gamma-1}} u_{i,\boldsymbol{a}}(\varepsilon x + x_{1,\varepsilon}),\ m_{i,\varepsilon} := \varepsilon^N m_{i,\boldsymbol{a}}(\varepsilon x + x_{1,\varepsilon}),\ w_{i,\varepsilon} := \varepsilon^{N+1} w_{i,\boldsymbol{a}}(\varepsilon x + x_{1,\varepsilon}), \tag{1.25}$$

then there exist $u \in C^2(\mathbb{R}^N)$, $0 \le m \in W^{1,\gamma'}(\mathbb{R}^N)$, and $w \in L^{\gamma'}(\mathbb{R}^N)$ such that

$$u_{i,\varepsilon} \to u \text{ in } C^2_{loc}(\mathbb{R}^N), \quad m_{i,\varepsilon} \to m \text{ in } L^p(\mathbb{R}^N),\ \forall p \ge 1, \quad w_{i,\varepsilon} \rightharpoonup w \text{ in } L^{\gamma'}(\mathbb{R}^N),\ i = 1,2. \tag{1.26}$$

In particular, (m,w) is a minimizer of problem (1.10) and (u,m,w) solves

$$\begin{cases} -\Delta u + C_H|\nabla u|^\gamma - \frac{\gamma'}{N} = -a^* m^{\frac{\gamma'}{N}}, & x \in \mathbb{R}^N, \\ \Delta m + C_H\gamma\nabla\cdot(m|\nabla u|^{\gamma-2}\nabla u) = 0,\ w = -C_H\gamma|\nabla u|^{\gamma-2}\nabla u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} m\,dx = 1. \end{cases} \tag{1.27}$$

Theorem 1.4 implies that as $(\alpha_1,\alpha_2) \nearrow (a^*-\beta, a^*-\beta)$, there are concentration phenomena in the multi-population Mean-field Games system (1.2) with attractive interactions under the mass critical exponent case. In addition, the basic blow-up profiles of ground states are given in Theorem 1.4. Moreover, by imposing the local polynomial expansion on potential functions, we obtain the following results of refined blow-up profiles:

Theorem 1.5. *Assume all conditions in Theorem 1.4 hold. Suppose that $V_1(x)$ and $V_2(x)$ have l common global minimum points, i.e., $Z_1 \cap Z_2 = \{x_1, \cdots, x_l \in \mathbb{R}^N\}$, and there exist $d > 0$, $a_{ij} > 0$, $p_{ij} > 0$ with $i = 1,2$, $j = 1,\cdots,l$ such that*

$$V_i = a_{ij}|x - x_j|^{p_{ij}} + O(|x - x_j|^{p_{ij}+1}) \text{ for } 0 < |x - x_j| < d. \tag{1.28}$$

Let $p_j := \min\{p_{1j}, p_{2j}\}$, $p_0 := \max_{1\le j\le l} p_j$ *and*

$$\mu_j = \lim_{x\to x_j} \frac{V_1(x)+V_2(x)}{|x-x_j|^{p_j}} = \begin{cases} a_{1j}, \ \text{if } p_{1j} < p_{2j}, \\ a_{1j}+a_{2j}, \ \text{if } p_{1j} = p_{2j}, \\ a_{2j}, \ \text{if } p_{1j} > p_{2j}. \end{cases} \tag{1.29}$$

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Define $\bar Z := \{x_j | p_j = p_0, \ j = 1,\cdots,l\}$, $\mu = \min\{\mu_j | x_j \in \bar Z\}$ *and* $Z_0 = \{x_j | x_j \in \bar Z \text{ and } \mu_j = \mu\}$. *Let* $(u_{i,\varepsilon}, m_{i,\varepsilon}, w_{i,\varepsilon})$, $i = 1,2$ *be given as (1.25). Then we have*

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\left(\frac{2\gamma'}{p_0\mu\bar v_{p_0} a^*}\right)^{\frac{1}{\gamma'+p_0}} \left(a^* - \frac{\alpha_1+\alpha_2+2\beta}{2}\right)^{\frac{1}{\gamma'+p_0}}} = 1, \tag{1.30}$$

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and

$$\frac{x_{1,\varepsilon}-x_0}{\varepsilon} \to y_0 \text{ with } x_0 \in Z_0 \text{ and } y_0 \in \mathbb{R}^N satisfying H_{m,p_0}(y_0) = \bar v_{p_0}, \tag{1.31}$$

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where m *and* $\bar v_{p_0}$ *are given in (1.26) and (1.16), respectively.*

Next, we discuss the blow-up profiles of ground states to (1.2) under repulsive interactions. We remark that on one hand, one has shown in Theorem 1.1 that (1.2) admits ground states when $0 < \alpha_1, \alpha_2 < a^*$ and $\beta \le 0$; on the other hand, Theorem 1.3 indicates that (1.5) does not have any minimizer when $\alpha_1 = \alpha_2 = a^*$ and $\beta \le 0$. Similarly as discussed in the proof of Theorem 1.4, we investigate the concentration phenomena in (1.2) with repulsive interactions and obtain

blowupnegative⟩ **Theorem 1.6.** *Assume that* $V_i(x)$ *with* $i = 1,2$ *satisfy (H1) and (H2) given by (1.8) and (1.9), respectively. Suppose*

$$Z_1 \cap Z_2 = \emptyset, \tag{1.32}$$

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where Z_1 *and* Z_2 *are given by (1.15). Let* $\beta < 0$, $0 < \alpha_1, \alpha_2 < a^*$, $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ *be a minimizer of* $e_{\alpha_1,\alpha_2,\beta}$ *with* $\mathbf{a} := (\alpha_1, \alpha_2)$ *and* $(m_{1,\mathbf{a}}, u_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{2,\mathbf{a}})$ *be a solution of (1.13). Define* $\mathbf{a}^* := (a^*, a^*)$, *then we have as* $\mathbf{a} \nearrow \mathbf{a}^*$,

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*} \left(\int_{\mathbb{R}^N} C_L \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx + \int_{\mathbb{R}^N} V_i m_{i,\mathbf{a}}\, dx - \frac{N\alpha_i}{N+\gamma'} \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx \right) = 0; \tag{1.33}$$

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$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*} \int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\, dx = 0, \tag{1.34}$$

C8notesne

and

$$\int_{\mathbb{R}^N} V_1 m_{1,\mathbf{a}} + V_2 m_{2,\mathbf{a}}\, dx \to 0; \tag{1.35}$$

C9notesne

$$C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx \to +\infty, \quad \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx \to +\infty, \ \ i = 1,2. \tag{1.36}$$

C10betane

Moreover, define

$$\hat\varepsilon_i := \left(C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx \right)^{-\frac{1}{\gamma'}} \to 0 \text{ as } \mathbf{a} \nearrow \mathbf{a}^*, \ i = 1,2.$$

Let $x_{i,\hat\varepsilon}$, $i=1,2$ be a global minimum point of $u_{i,\mathbf{a}}$ and

$$m_{i,\hat\varepsilon}=\hat\varepsilon_i^N m_{i,\boldsymbol{a}}(\hat\varepsilon_i x+x_{i,\hat\varepsilon}),\ \ w_{i,\hat\varepsilon}=\hat\varepsilon_i^{N+1}w_{i,\boldsymbol{a}}(\hat\varepsilon_i x+x_{i,\hat\varepsilon}),\ u_{i,\hat\varepsilon}=\hat\varepsilon_i^{\frac{2-\gamma}{\gamma-1}}u_{i,\boldsymbol{a}}(\hat\varepsilon_i x+x_{i,\hat\varepsilon}), \tag{1.37}$$

then there exist $(u_i,m_i,w_i)\in C^2(\mathbb{R}^N)\times W^{1,\gamma'}(\mathbb{R}^N)\times L^{\gamma'}(\mathbb{R}^N)$ with $i=1,2$ such that

$$u_{i,\varepsilon}\to u_i \text{ in } C^2_{loc}(\mathbb{R}^N),\ \ m_{i,\varepsilon}\to m_i \text{ in } L^p(\mathbb{R}^N),\ \forall p\geq 1,\ \ w_{i,\varepsilon}\rightharpoonup w_i \text{ in } L^{\gamma'}(\mathbb{R}^N),\ i=1,2. \tag{1.38}$$

In particular, (m_i,u_i,w_i), $i=1,2$ both solve system (1.27).

Remark 1.2. *We point out that unlike the attractive case discussed in Theorem 1.4 and Theorem 1.5, $\hat\varepsilon_1$ and $\hat\varepsilon_2$ given in Theorem 1.6 both converge to zero but might not be in the same order since $\beta<0$ and the behaviors of V_1 and V_2 might be distinct around global minimum points locally.*

Theorem 1.6 indicates that when the interaction is repulsive, there are concentration phenomena within system (1.2) in some singular limit of parameters α_1, α_2 and β. Moreover, similarly as the conclusion shown in Theorem 1.5, we explore the refined blow-up profiles and obtain

Theorem 1.7. *Assume all conditions in Theorem 1.6 hold. Suppose that each V_i, $i=1,2$ has only one global minimum point x_i with $x_1\neq x_2$ and there exist $d>0$, $b_i>0$ and $p_i>0$ such that*

$$V_i(x)=b_i|x-x_i|^{p_i}+O(|x-x_i|^{p_i+1}) \text{ for } 0<|x-x_i|<d. \tag{1.39}$$

Define for $i=1,2$,

$$\tilde\epsilon_i:=(a^*-\alpha_i)^{\frac{1}{\gamma'+p_i}} \text{ and assume } \exists\, s\in(0,1] \text{ such that } \tilde\epsilon_1=O(\tilde\epsilon_2^s). \tag{1.40}$$

Let $(m_{1,\boldsymbol{a}},w_{1,\boldsymbol{a}},m_{2,\boldsymbol{a}},w_{2,\boldsymbol{a}})$ be a minimizer of (1.5) and $(m_{i,\hat\varepsilon},w_{i,\hat\varepsilon},u_{i,\hat\varepsilon})$ be defined as (1.37). Then we have

$$\frac{x_{i,\hat\varepsilon}-x_i}{\hat\varepsilon_i}\to y_{i0} \text{ such that } H_{m_i,p_i}(y_{i0})=\bar\nu_{p_i}, \tag{1.41}$$

where m_i and $\bar\nu_{p_i}$, $i=1,2$ are given by (1.38) and (1.16), respectively. Moreover, the following asymptotics hold as $\mathbf{a}\nearrow\mathbf{a}^*$,

$$\hat\varepsilon_i^{\gamma'}=(1+o(1))\Big(\frac{\gamma'(a^*-\alpha_i)}{a^*b_i\bar\nu_{p_i}p_i}\Big)^{\frac{1}{\gamma'+p_i}},\ i=1,2.$$

Remark 1.3. *In Theorem 1.7, we discuss the refined blow-up profiles of ground states when the interaction coefficient is non-positive under some technical assumption (1.40). We would like to remark that this condition is technical and could be improved if the refined decay estimate of population density m is given. In fact, the improved condition will be exhibited in Section 5.*

The rest of this paper is organized as follows: In Section 2, we give some preliminary results for the existence and properties of the solutions to Hamilton-Jacobi equations and Fokker-Planck equations, which are used to investigate the existence and blow-up behaviors of minimizers to problem (1.5) . Section 3 is devoted to the exploration of the effect of the potentials $V_i(x)$, $i=1,2$ and coefficients α_1,α_2,β on the existence of minimizers. Correspondingly, the proof of Theorems 1.1-1.3 will be finished. In Section 4, we perform the blow-up analysis of minimizers under the case of attractive interactions $\beta>0$, and show the conclusions of Theorem 1.4 and Theorem 1.5. Finally, in Section 5, we focus on the asymptotic profiles of ground states with $\beta<0$ and complete the proof of Theorem 1.6 and Theorem 1.7.

2 Preliminary Results

⟨preliminary⟩ In this section, we collect some preliminaries for the existence and regularities of solutions to Hamilton-Jacobi equations and Fokker-Planck equations, respectively. Furthermore, some useful equalities and estimates satisfied by the solution to the single population Mean-field Games system will be listed.

2.1 Hamilton-Jacobi Equations

?⟨subsection1⟩? Consider the following second order Hamilton-Jacobi equations:

$$-\Delta u_k + C_H|\nabla u_k|^\gamma + \lambda_k = V_k(x) + f_k(x), \quad x \in \mathbb{R}^N, \tag{2.1}$$

HJB-regul

where $\gamma > 1$ is fixed, C_H is a given positive constant independent of k and (u_k, λ_k) denote the solutions to (2.1). For the gradient estimates of u_k, we find

ma21-gradientu⟩ **Lemma 2.1.** *Suppose that $f_k \in L^\infty(\mathbb{R}^N)$ satisfies $\|f_k\|_{L^\infty} \le C_f$, $|\lambda_k| \le \lambda$, and the potential functions $V_k(x) \in C^{0,\theta}_{\rm loc}(\mathbb{R}^N)$ with $\theta \in (0,1)$ satisfy $0 \le V_k(x) \to +\infty$ as $|x| \to +\infty$, and $\exists\, R > 0$ sufficiently large such that*

$$0 < C_1 \le \frac{V_k(x+y)}{V_k(x)} \le C_2, \text{ for all } k \text{ and all } |x| \ge R \text{ with } |y| < 2,$$

where the positive constants C_f, λ, R, C_1 and C_2 are independent of k. Let $(u_k, \lambda_k) \in C^2(\mathbb{R}^N) \times \mathbb{R}$ be a sequence of solutions to (2.1). Then, for all k,

$$|\nabla u_k(x)| \le C(1 + V_k(x))^{\frac{1}{\gamma}}, \text{ for all } x \in \mathbb{R}^N,$$

where constant C depends on C_H, C_1, C_2, λ, γ, N and C_f.

In particular, if there exist $b \ge 0$ and $C_F > 0$ independent of k, such that following conditions hold on V_k

$$C_F^{-1}(\max\{|x| - C_F, 0\})^b \le V_k(x) \le C_F(1 + |x|)^b, \quad \text{for all } k \text{ and } x \in \mathbb{R}^N, \tag{2.2}$$

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then we have

$$|\nabla u_k| \le C(1 + |x|)^{\frac{b}{\gamma}}, \text{ for all } k \text{ and } x \in \mathbb{R}^N,$$

where constant C depends on C_H, C_F, b, λ, γ, N and C_f.

Proof. See Lemma 3.1 in [5] and the argument is the slight modification of the proof of Theorem 2.5 in [2]. □

For the lower bound of u_k, we have

generallemma22⟩ **Lemma 2.2** (C.f. Lemma 3.2 in [5])**.** *Suppose all conditions in Lemma 2.1 hold. Let u_k be a family of C^2 solutions and assume that $u_k(x)$ are bounded from below uniformly. Then there exist positive constants C_3 and C_4 independent of k such that*

$$u_k(x) \ge C_3 V_k^{\frac{1}{\gamma'}}(x) - C_4, \ \forall x \in \mathbb{R}^n, \text{ for all } k. \tag{2.3}$$

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In particular, if the following conditions hold on V_k

$$C_F^{-1}(\max\{|x| - C_F, 0\})^b \le V_k(x) \le C_F(1 + |x|)^b, \quad \text{for all } k \text{ and } x \in \mathbb{R}^n, \tag{2.4}$$

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where constants $b > 0$ and C_F are independent of k, then we have

$$u_k(x) \geq C_3|x|^{1+\frac{b}{r'}} - C_4, \text{ for all } k, x \in \mathbb{R}^n. \tag{2.5}$$ usolution

If $b = 0$ in (2.4) and there exist $R > 0$ and $\hat{\delta} > 0$ independent of k such that

$$f_k + V_k - \lambda_k > \hat{\delta} > 0 \text{ for all } |x| > R, \tag{2.6}$$?lemma22hc

then (2.5) also holds.

The existence result of the classical solution to (2.1) is summarized as

a22preliminary⟩ **Lemma 2.3** (C.f. Lemma 3.3 in [5]). *Suppose $V_k + f_k$ are locally Hölder continuous and bounded from below uniformly in k. Define*

$$\bar{\lambda}_k := \sup\{\lambda \in \mathbb{R} \mid (2.1) \text{ has a solution } u_k \in C^2(\mathbb{R}^n)\}.$$

Then

(i). $\bar{\lambda}_k$ are finite for every k and (2.1) admits a solution $(u_k, \lambda_k) \in C^2(\mathbb{R}^n) \times \mathbb{R}$ with $\lambda_k = \bar{\lambda}_k$ and $u_k(x)$ being bounded from below (may not uniform in k). Moreover,

$$\bar{\lambda}_k = \sup\{\lambda \in \mathbb{R} \mid (2.1) \text{ has a subsolution } u_k \in C^2(\mathbb{R}^n)\}.$$

(ii). If V_k satisfies (2.2) with $b > 0$, then u_k is unique up to constants for fixed k and there exists a positive constant C independent of k such that

$$u_k(x) \geq C|x|^{\frac{b}{r'}+1} - C, \forall x \in \mathbb{R}^n. \tag{2.7}$$ lowerboun

In particular, if $V_k \equiv 0$ in (2.1) and there exists $\sigma > 0$ independent of k such that

$$f_k - \lambda_k \geq \sigma > 0, \quad \text{for } |x| > K_2,$$

where $K_2 > 0$ is a large constant independent of k, then (2.7) also holds.

(iii). If V_k satisfies (1.9), then there exist uniformly bounded from below classical solutions u_k to problem (2.1) satisfying estimate (2.3).

2.2 Fokker-Planck Equations

?⟨subsection2⟩? Of concern the second order Fokker-Planck equation

$$-\Delta m + \nabla \cdot w = 0, \quad x \in \mathbb{R}^N, \tag{2.8}$$ sect2-FP-

where w is given and m denotes the solution, we have the following results for the regularity:

21-crucial-cor⟩ **Lemma 2.4.** *Let $(m, w) \in \left(L^1(\mathbb{R}^N) \cap W^{1,\hat{q}}(\mathbb{R}^N)\right) \times L^1(\mathbb{R}^N)$ be a solution to (2.8) with*

$$\hat{q} := \begin{cases} \frac{N}{N-\gamma'+1} & \text{if } \gamma' < N, \\ \in \left(\frac{2N}{N+2}, N\right) & \text{if } \gamma' = N, \\ \gamma' & \text{if } \gamma' > N. \end{cases}$$

Assume that

$$\Lambda_{\gamma'} := \int_{\mathbb{R}^n} |m| \left|\frac{w}{m}\right|^{\gamma'} dx < \infty,$$

then we have $w \in L^1(\mathbb{R}^N) \cap L^{\hat{q}}(\mathbb{R}^N)$ and there exists $C = C(\Lambda_{\gamma'}, \|m\|_{L^1(\mathbb{R}^N)}) > 0$ such that

$$\|m\|_{W^{1,\hat{q}}(\mathbb{R}^N)}, \|w\|_{L^1(\mathbb{R}^N)}, \|w\|_{L^{\hat{q}}(\mathbb{R}^N)} \leq C.$$

Proof. See the proof of Lemma 3.5 in [5]. □

Next, we state some useful identities satisfied by the single population Mean-field Games system. First of all, we have the exponential decay estimates of m when some condition is imposed on the Lagrange multiplier, which is

⟨mdecaylemma⟩ **Lemma 2.5** (C.f. Proposition 5.3 in [2]). *Assume $\gamma' > N$. Let $(u,\lambda,m)\in C^2(\mathbb{R}^n)\times\mathbb{R}\times(W^{1,\gamma'}(\mathbb{R}^n)\cap L^1(\mathbb{R}^n))$ with u bounded from below, and $\lambda<0$ be the solution of the following Mean-field Games system*

$$\begin{cases} -\Delta u + C_H|\nabla u|^\gamma + \lambda = -m^\nu, & x\in\mathbb{R}^N,\\ \Delta m + C_H\gamma\nabla\cdot(m|\nabla u|^{\gamma-2}\nabla u) = 0, & x\in\mathbb{R}^N, \end{cases} \tag{2.9}$$ 26prelimi

where $\nu\in(0,\frac{\gamma'}{N}]$. Then, we have there exist $\kappa_1,\kappa_2>0$ such that

$$m(x)\leq \kappa_1 e^{-\kappa_2|x|}\ \text{ for all } x\in\mathbb{R}^N.$$

With the aid of Lemma 2.5, we have the following results for the Pohozaev identities satisfied by the solution to system (2.9):

⟨poholemma⟩ **Lemma 2.6** (C.f. Proposition 3.1 in [4]). *Assume all conditions in Lemma 2.5 hold and denote $w=-C_H\gamma m|\nabla u|^{\gamma-2}\nabla u$. Then we have the following Pohozaev type identities hold:*

$$\begin{cases} \lambda\int_{\mathbb{R}^N} m\,dx = -\frac{(\nu+1)\gamma'-N\nu}{(\alpha+1)\gamma'}\int_{\mathbb{R}^N} m^{\nu+1}\,dx,\\ C_L\int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'}dx = \frac{N\nu}{(\nu+1)\gamma'}\int_{\mathbb{R}^N} m^{\nu+1}\,dx = (\gamma-1)C_H\int_{\mathbb{R}^N} m|\nabla u|^\gamma\,dx. \end{cases}$$

3 Existence of ground states

⟨sec-existence⟩ In this section, we shall discuss the existence of ground states to system (1.2) under some conditions of coefficients α_i with $i=1,2$ and β. To this end, we first estimate the energy $\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1,w_1,m_2,w_2)$ from below. Then, if the energy is shown to have some finite lower bound and the minimizers is proved to exist, we will find the existence of ground states to (1.2) by the standard duality argument. Before stating our main results for the existence of minimizers, we give some preliminary definitions, which are

$$e^i_{\alpha_i} := \inf_{(m,w)\in\mathcal{K}_i}\mathcal{E}^i_{\alpha_i}(m,w),\ \ i=1,2, \tag{3.1}$$ problem51

where $\mathcal{K}_i$ is given by (1.7) and

$$\mathcal{E}^i_{\alpha_i}(m,w) = C_L\int_{\mathbb{R}^N}\left|\frac{w}{m}\right|^{\gamma'} m\,dx + \int_{\mathbb{R}^N} V_i m\,dx - \frac{\alpha_i}{1+\frac{\gamma'}{N}}\int_{\mathbb{R}^N} m^{1+\frac{\gamma'}{N}}\,dx. \tag{3.2}$$ mathcalea

Concerning the existence of ground states in (1.2), we have

nceleastenergy⟩ **Lemma 3.1.** *Assume all conditions in Theorem 1.1 hold, then we have*

(i). if $0<\alpha_1<a^*$, $0<\alpha_2<a^*$ and $-\infty<\beta<\beta_*:=\sqrt{(a^*-\alpha_1)(a^*-\alpha_2)}$, then problem (1.5) has a global minimizer $(m_{1,\boldsymbol{a}},w_{1,\boldsymbol{a}},m_{2,\boldsymbol{a}},w_{2,\boldsymbol{a}})\in\mathcal{K}$;

(ii). either $\alpha_1>a^*$ or $\alpha_2>a^*$ or $\beta>\beta^*:=\frac{2a^*-\alpha_1-\alpha_2}{2}$, then problem (1.5) has no minimizer.

Proof. (i). Invoking inequality (1.12) and condition (1.8) satisfied by V_i with $i = 1, 2$, we have for any $(m_1, w_1, m_2, w_2) \in \mathcal{K}$,

$$\begin{aligned}
&\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1, w_1, m_2, w_2) \\
\geq &\sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i m_i \, dx + \frac{N}{N+\gamma'}\Big[\sum_{i=1}^{2}(a^* - \alpha_i)\int_{\mathbb{R}^N} m_i^{1+\frac{\gamma'}{N}} \, dx - 2\beta \int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}} \, dx\Big] \\
\geq &\frac{2(\beta_* - \beta)N}{N+\gamma'} \int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}} \, dx,
\end{aligned} \tag{3.3}$$ citeinequ

where $\mathcal{E}_{\alpha_1,\alpha_2,\beta}$ is given by (1.6). Then, letting $\{(m_{1,k}, w_{1,k}, m_{2,k}, w_{2,k}\} \subset \mathcal{K}$ with $k \in \mathbb{Z}^+$ being a minimizing sequence of $e_{\alpha_1,\alpha_2,\beta}$, one has from (3.3) and $-\infty < \beta < \beta_*$ that

$$\sup_k \sum_{i=1}^{2} \int_{\mathbb{R}^N} V_i m_{i,k} \, dx < +\infty, \quad \sup_k \int_{\mathbb{R}^N} m_{1,k}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,k}^{\frac{1}{2}+\frac{\gamma'}{2N}} \, dx < +\infty, \tag{3.4}$$ between31

and then

$$\sup_k \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left|\frac{w_{i,k}}{m_{i,k}}\right|^{\gamma'} m_{i,k} \, dx < +\infty. \tag{3.5}$$ moreimpor

Thanks to Lemma 2.4 and (3.5), one obtains as $k \to +\infty$, for $i = 1, 2$,

$$(m_{i,k}, w_{i,k}) \rightharpoonup (m_{i,\mathbf{a}}, w_{i,\mathbf{a}}) \text{ in } W^{1,\gamma'}(\mathbb{R}^N) \times L^{\gamma'}(\mathbb{R}^N).$$

Moreover, by the compactly Sobolev embedding (C.f. Lemma 5.1 in [5]) and Fatou's lemma, we find from (3.4) that

$$m_{i,k} \to m_{i,\mathbf{a}} \text{ in } L^{1+\frac{\gamma'}{N}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N).$$

Then it follows that $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) \in \mathcal{K}$ is a minimizer.

(ii). Let $\mathcal{M}$ be given by (1.17). Since $\gamma' > N$, by using Morrey's embedding, the standard elliptic regularity and the maximum principle, one follows the idea shown in [1] then obtain for any $(m, w) \in \mathcal{M}$, $m(x) > 0$ for all $x \in \mathbb{R}^N$. Next, we utilize some rescaled pair of $(m_0, w_0) \in \mathcal{M}$ to analyze the bound of $\mathcal{E}_{\alpha_1,\alpha_2,\beta}$ from below.

Let $(m_0, w_0) \in \mathcal{M}$ and define

$$(m_t, w_t) = \left(\frac{t^N}{M^*} m_0(t(x - x_0)), \frac{t^{N+1}}{M^*} w_0(t(x - x_0))\right), \text{ for } t > 0 \text{ and } x_0 \in \mathbb{R}^N. \tag{3.6}$$ byusingin

From Lemma 2.5 and Lemma 2.6, we have that

$$C_L \int_{\mathbb{R}^N} \left|\frac{w_0}{m_0}\right|^{\gamma'} m_0 \, dx = 1, \quad \int_{\mathbb{R}^N} m_0^{1+\frac{\gamma'}{N}} \, dx = \frac{N+\gamma'}{N}, \quad \int_{\mathbb{R}^N} m_0 \, dx = M^*. \tag{3.7}$$ combine20

Combining (3.6) with (3.7), one finds

$$C_L \int_{\mathbb{R}^N} \left|\frac{w_t}{m_t}\right|^{\gamma'} m_t \, dx = C_L \frac{t^{\gamma'}}{M^*} \int_{\mathbb{R}^N} \left|\frac{w_0}{m_0}\right|^{\gamma'} m_0 \, dx = \frac{t^{\gamma'}}{M^*}, \tag{3.8}$$ onefindsi

and

$$\int_{\mathbb{R}^N} m_t^{1+\frac{\gamma'}{N}}\,dx = \frac{t^{N(1+\frac{\gamma'}{N})}}{(M^*)^{1+\frac{\gamma'}{N}}}\int_{\mathbb{R}^N} m_0^{1+\frac{\gamma'}{N}}(tx)\,dx = \frac{N+\gamma'}{N}\frac{t^{\gamma'}}{(M^*)^{1+\frac{\gamma'}{N}}}. \tag{3.9}$$

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Then it follows from (3.2), (3.6), (3.8) and (3.9) that

$$\begin{aligned}\mathcal{E}^1_{\alpha_1}(m_t,w_t) =& C_L\int_{\mathbb{R}^N}\left|\frac{w_t}{m_t}\right|^{\gamma'} m_t\,dx - \frac{N\alpha_1}{N+\gamma'}\int_{\mathbb{R}^N} m_t^{1+\frac{\gamma'}{N}}\,dx + \int_{\mathbb{R}^N} V_1 m_t\,dx\\ =& \frac{t^{\gamma'}}{M^*}\left(1-\frac{\alpha_1}{a^*}\right) + \frac{1}{M^*}\int_{\mathbb{R}^N} V\left(\frac{x}{t}+x_0\right)m_0\,dx.\end{aligned} \tag{3.10}$$

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On the other hand, we choose

$$\bar m = \frac{e^{-\delta_1|x|}}{\|e^{-\delta_1|x|}\|_{L^1}},\quad \bar w = \nabla\bar m \text{ with } (\bar m,\bar w)\in\mathcal{K}_2,$$

and apply Hölder's inequality to get

$$\int_{\mathbb{R}^N} m_t^{\frac12+\frac{\gamma'}{2N}}\bar m^{\frac12+\frac{\gamma'}{2N}}\,dx \le \left(\int_{\mathbb{R}^N} m_t^{1+\frac{\gamma'}{N}}\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\bar m^{1+\frac{\gamma'}{N}}\,dx\right)^{\frac12} \le Ct^{\frac{\gamma'}{2}}, \tag{3.11}$$

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where $C>0$ is some constant. Upon collecting (3.10) and (3.11), we obtain if $\alpha_1 > a^*$,

$$\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_t,w_t,\bar m,\bar w) \ge \frac{t^{\gamma'}}{M^*}\left(1-\frac{\alpha_1}{a^*}\right) - Ct^{\frac{\gamma'}{2}} - C \to -\infty,\ \text{ as } t\to+\infty.$$

Thus, $e_{\alpha_1,\alpha_2,\beta} = -\infty$ when $\alpha_1 > a^*$. Similarly, we find if $\alpha_2 > a^*$, then $e_{\alpha_1,\alpha_2,\beta} = -\infty$. Consequently, we have if any $\alpha_i > a^*$ or $\alpha_2 > a^*$, problem (1.5) does not have a minimizer.

It is left to study the case of $\beta > \beta^*$. To this end, we compute and obtain

$$\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_t,w_t,m_t,w_t) = \frac{t^{\gamma'}}{M^*}\left(2-\frac{\alpha_1}{a^*}-\frac{\alpha_2}{a^*}-\frac{2\beta}{a^*}\right) + O(1) \to -\infty,\ \text{ as } t\to+\infty,$$

when $\beta > \beta^* := \frac{2a^*-\alpha_1-\alpha_2}{2}$. This completes the proof.

□

Lemma 3.1 states some existence results for the global minimizers (m_1,w_1,m_2,w_2) to (1.5) under some conditions of α_1, α_2 and β. In particular, when intra-population and inter-population coefficients are all small, Lemma 3.1 implies there exists a minimizer to (1.5). Whereas, the existence of ground states to (1.2) can not be shown unless (u_1,u_2) and (λ_1,λ_2) are obtained. Hence, to finish the proof of Theorem 1.1, we establish the following lemma for the existence of the value function pair (u_1,u_2) and Lagrange multipliers (λ_1,λ_2):

emma32multimfg⟩ **Lemma 3.2.** *Let $(m_{1,a},w_{1,a},m_{2,a},w_{2,a})\in\mathcal{K}$ be a minimizer of $e_{\alpha_1,\alpha_2,\beta}$ with $\mathcal{K}=\mathcal{K}_1\times\mathcal{K}_2$ defined by (1.7), then there exist $(u_{1,a},u_{2,a})\in(C^2(\mathbb{R}^N))^2$ and $(\lambda_{1,a},\lambda_{2,a})\in\mathbb{R}^2$ such that $(m_{1,a},u_{1,a},m_{2,a},u_{2,a},\lambda_{1,a},\lambda_{2,a})$ solves*

$$\begin{cases} -\Delta u_1 + C_H|\nabla u_1|^\gamma + \lambda_1 = V_1(x) - \alpha_1 m_1^{\frac{\gamma'}{N}} - \beta m_1^{\frac{\gamma'}{2N}-\frac12} m_2^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N,\\ \Delta m_1 + \nabla\cdot w_1 = 0,\ w_1 = -\gamma C_H m_1|\nabla u_1|^{\gamma-2}\nabla u_1 & x\in\mathbb{R}^N,\\ -\Delta u_2 + C_H|\nabla u_2|^\gamma + \lambda_2 = V_2(x) - \alpha_2 m_2^{\frac{\gamma'}{N}} - \beta m_2^{\frac{\gamma'}{2N}-\frac12} m_1^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N,\\ \Delta m_2 + \nabla\cdot w_2 = 0,\ w_2 = -\gamma C_H m_2|\nabla u_2|^{\gamma-2}\nabla u_2, & x\in\mathbb{R}^N.\end{cases} \tag{3.12}$$

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Moreover, we have the following identities and estimates hold:

$$\lambda_{i,a} = C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,a}}{m_{i,a}}\right|^{\gamma'} m_{i,a}\,dx + \int_{\mathbb{R}^N} V_i m_{i,a}\,dx - \alpha_i \int_{\mathbb{R}^N} m_{i,a}^{1+\frac{\gamma'}{N}}\,dx - \beta \int_{\mathbb{R}^N} m_{1,a}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,a}^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx,\ i = 1,2, \tag{3.13}$$
lambda177

and there exists a constant $C > 0$ such that

$$|\nabla u_{i,a}(x)| \le C\Big(1 + V_i^{\frac{1}{\gamma}}(x)\Big),\ \ u_{i,a}(x) \ge C V_i^{\frac{1}{\gamma}} - C, \textit{ for all } x \in \mathbb{R}^N,\ i = 1,2. \tag{3.14}$$
gradientu

Proof. To prove this lemma, we follow the approaches employed to show Proposition 3.4 in [2] and make slight modifications. Define admissible sets $\mathcal{A}_i$ as

$$\mathcal{A}_i = \left\{\psi \in C^2(\mathbb{R}^N) \,\middle|\, \limsup_{|x|\to\infty} \frac{|\nabla\psi|}{V_i^{\frac{1}{\gamma}}} < +\infty,\ \limsup_{|x|\to\infty} \frac{|\Delta\psi|}{V_i} < +\infty \right\},\ \ i = 1,2, \tag{3.15}$$
?definitio

then we proceed the similar argument shown in the proof of Proposition 5.1 in [5] and obtain

$$-\int_{\mathbb{R}^N} m_{i,\mathbf{a}} \Delta\psi\,dx = \int_{\mathbb{R}^N} w_{i,\mathbf{a}} \cdot \nabla\psi\,dx,\ \ \forall \psi \in \mathcal{A}_i,\ i = 1,2. \tag{3.16}$$
80innotes

Next, we define

$$\tilde{J}_1(m,w) := \int_{\mathbb{R}^N} \left[C_L \left|\frac{w}{m}\right|^{\gamma'} m + [V_1(x) + f_1(m_{1,\mathbf{a}}, m_{2,\mathbf{a}})]m\right] dx, \tag{3.17}$$
definitio

where

$$f_1(m_{1,\mathbf{a}}, m_{2,\mathbf{a}}) := -\alpha_1 m_{1,\mathbf{a}}^{\frac{\gamma'}{N}} - \beta m_{2,\mathbf{a}}^{\frac{\gamma'}{2N}+\frac{1}{2}} m_{1,\mathbf{a}}^{\frac{\gamma'}{2N}-\frac{1}{2}},$$

and set

$$\begin{aligned}\mathcal{B}_i := \Big\{(m,w) \in (L^1(\mathbb{R}^N) \cap W^{1,\gamma'}(\mathbb{R}^N)) \times L^{\gamma'}(\mathbb{R}^N) \Big| -\int_{\mathbb{R}^N} m\Delta\psi\,dx = \int_{\mathbb{R}^N} w \cdot \nabla\psi\,dx\,, \forall \psi \in \mathcal{A}_i,\\ m \ge 0 \text{ a.e. in } \mathbb{R}^N,\ \int_{\mathbb{R}^N} m\,dx = 1,\ \int_{\mathbb{R}^N} V_i m\,dx < +\infty,\ \int_{\mathbb{R}^N} |w| V_i^{\frac{1}{\gamma'}}\,dx < +\infty\Big\},\ i = 1,2.\end{aligned}$$

We have the fact that $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ is a minimizer of $\mathcal{E}_{\alpha_1,\alpha_2,\beta}$ in $\mathcal{B}_1 \times \mathcal{B}_2$, i.e.

$$e_{\alpha_1,\alpha_2,\beta} := \inf_{(m_1,w_1,m_2,w_2)\in\mathcal{K}_1\times\mathcal{K}_2} \mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1,w_1,m_2,w_2) = \inf_{(m_1,w_1,m_2,w_2)\in\mathcal{B}_1\times\mathcal{B}_2} \mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1,w_1,m_2,w_2). \tag{3.18}$$
minimumB1

Now, we claim

$$\tilde{J}_1(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) = \min_{(m,w)\in\mathcal{B}_1} \tilde{J}_1(m,w), \tag{3.19}$$
tildeJ1m1

where $\tilde{J}_1$ is defined by (3.17). Indeed, we set

$$J_1(m,w) := \mathcal{E}_{\alpha_1,\alpha_2,\beta}(m,w,m_{2,\mathbf{a}},w_{2,\mathbf{a}}) := \varphi(m,w) + \Lambda(m) + \tilde{G}, \tag{3.20}$$
definitio

where

$$\varphi(m,w) := C_L \int_{\mathbb{R}^N} m\left|\frac{w}{m}\right|^{\gamma'} dx,$$

$$\Lambda(m) := -\frac{N\alpha_1}{N+\gamma'}\int_{\mathbb{R}^N} m^{1+\frac{\gamma'}{N}}\,dx + \int_{\mathbb{R}^N} V_1 m\,dx - \frac{2\beta N}{N+\gamma'}\int_{\mathbb{R}^N} m^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx,$$

and

$$\tilde{G} := C_L \int_{\mathbb{R}^N} \left|\frac{w_{2,\mathbf{a}}}{m_{2,\mathbf{a}}}\right|^{\gamma'} m_{2,\mathbf{a}}\,dx - \frac{N\alpha_1}{\gamma'+N}\int_{\mathbb{R}^N} m_{2,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx + \int_{\mathbb{R}^N} V_2 m_{2,\mathbf{a}}\,dx.$$

For any $(m, w) \in \mathcal{B}_1$, we define

$$m_\lambda = \lambda m + (1-\lambda)m_{1,\mathbf{a}},\ \ w_\lambda = \lambda w + (1-\lambda)w_{1,\mathbf{a}},\ \ 0 < \lambda < 1,$$

and have the fact that $(m_\lambda, w_\lambda) \in \mathcal{B}_1$. Thus, by using (3.18) and (3.20), we obtain

$$J_1(m_\lambda, w_\lambda) = \mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_\lambda, w_\lambda, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) \geq \mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) = J_1(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}),$$

which implies

$$\varphi(m_\lambda, w_\lambda) + \Lambda(m_\lambda) \geq \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) + \Lambda(m_{1,\mathbf{a}}),$$

i.e.

$$\varphi(m_\lambda, w_\lambda) - \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) \geq \Lambda(m_{1,\mathbf{a}}) - \Lambda(m_\lambda). \tag{3.21}$$ 89innotes

Next, we simplitfy (3.21). On one hand, by the convexity of φ in (m, w), we have

$$\varphi(m_\lambda, w_\lambda) \leq \lambda\varphi(m, w) + (1-\lambda)\varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}),$$

i.e.

$$\varphi(m_\lambda, w_\lambda) - \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) \leq \lambda[\varphi(m, w) - \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}})]. \tag{3.22}$$ 90innotes

On the other hand, for $\lambda > 0$ sufficiently small, we have

$$\Lambda(m_\lambda) = \Lambda(m_{1,\mathbf{a}}) + \lambda\langle\nabla\Lambda(m_{1,\mathbf{a}}), (m - m_{1,\mathbf{a}})\rangle + O(\lambda). \tag{3.23}$$ 91innotes

In addition, invoking (3.17) and (3.20), one can obtain

$$\nabla\Lambda(m_{1,\mathbf{a}}) = V_1 + f_1(m_{1,\mathbf{a}}, m_{2,\mathbf{a}}).$$

Upon substituting (3.22) and (3.23) into (3.21), we get

$$\varphi(m, w) - \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) \geq -\langle\nabla\Lambda(m_{1,\mathbf{a}}), m - m_{1,\mathbf{a}}\rangle.$$

Hence,

$$\tilde{J}_1(m, w) = \varphi(m, w) + \langle\nabla\Lambda(m_{1,\mathbf{a}}), m\rangle \geq \varphi(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) + \langle\nabla\Lambda(m_{1,\mathbf{a}}), m_{1,\mathbf{a}}\rangle = \tilde{J}_1(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}),$$

which indicates that claim (3.19) holds.

Now, we prove

$$\sup\{\lambda : -\Delta\psi + C_H|\nabla\psi|^\gamma + \lambda \leq V_1 + f_1(m_{1,\mathbf{a}}, m_{2,\mathbf{a}}) \text{ in } \mathbb{R}^N \text{ for some } \psi \in \mathcal{B}_1\} = \min_{(m,w)\in\mathcal{B}_1} \tilde{J}_1(m, w). \tag{3.24}$$ ourclaimh

In fact, by following the similar argument shown in the proof of Proposition 3.4 in [2], we define

$$\mathcal{L}_1(m,w,\lambda,\psi) := \tilde{J}_1(m,w) + \int_{\mathbb{R}^N} (m\Delta\psi + w\cdot\nabla\psi - \lambda m)\,dx + \lambda,$$

and obtain

$$\min_{(m,w)\in\mathcal{B}_1} \tilde{J}_1(m,w) = \min_{(m,w)\in\Gamma} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}_1} \mathcal{L}_1(m,w,\lambda,\psi),$$

where $\Gamma := (L^1(\mathbb{R}^N)\cap W^{1,\gamma'}(\mathbb{R}^N))\times L^{\gamma'}(\mathbb{R}^N)$. Invoking the convexity of $\mathcal{L}_1(\cdot,\cdot,\lambda,\psi)$ and the linearity of $\mathcal{L}_1(m,w,\cdot,\cdot)$, one has

$$\begin{aligned}
&\min_{(m,w)\in\Gamma} \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}_1} \mathcal{L}_1(m,w,\lambda,\psi) = \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}_1} \min_{(m,w)\in\Gamma} \mathcal{L}_1(m,w,\lambda,\psi)\\
&= \sup_{(\lambda,\psi)\in\mathbb{R}\times\mathcal{A}_1} \int_{\mathbb{R}^N} \min_{(m,w)\in\mathbb{R}\times\mathbb{R}^N} \left[C_L\left|\frac{w}{m}\right|^{\gamma'} m + [V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}})]m + m\Delta\psi + w\cdot\nabla\psi - \lambda m\right]dx + \lambda\\
&= \begin{cases} 0, & V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) - [-\Delta\psi + C_H|\nabla\psi|^\gamma] \geq 0,\\ -\infty, & V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) - [-\Delta\psi + C_H|\nabla\psi|^\gamma] < 0 \end{cases}\\
&= \sup\{\lambda | V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) - [-\Delta\psi + C_H|\nabla\psi|^\gamma] \geq 0 \text{ for some } \psi\in\mathcal{A}_1\},
\end{aligned}$$

which shows (3.24). Moreover, with the aid of Lemma 2.3, we have

$$\begin{aligned}
\lambda_{1,\mathbf{a}} &:= \sup\{\lambda | V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) - [-\Delta\psi + C_H|\nabla\psi|^\gamma] \geq 0 \text{ for some } \psi\in\mathcal{A}_1\}\\
&= \min_{(m,w)\in\mathcal{B}_1} \tilde{J}_1(m,w) < +\infty,
\end{aligned}\tag{3.25}$$ lambda1su

and there exists $u_{1,\mathbf{a}}\in C^2(\mathbb{R}^N)$ such that

$$-\Delta u_{1,\mathbf{a}} + C_H|\nabla u_{1,\mathbf{a}}|^\gamma + \lambda_{1,\mathbf{a}} = V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) \text{ in } \mathbb{R}^N. \tag{3.26}$$ u1eqf1m1m

In particular, we have from Lemma 2.1 and Lemma 2.2 that (3.14) holds for $u_{1,\mathbf{a}}$.

Since $m_{1,\mathbf{a}}$, $m_{2,\mathbf{a}}\in L^\infty(\mathbb{R}^N)$ by Sobolev embedding, one obtains $f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}})\in L^\infty(\mathbb{R}^N)$. Then it follows from (3.14) and (3.26) that

$$|-\Delta u_{1,\mathbf{a}}(x)| \leq C(1+V_1(x)).$$

Thus, $u_{1,\mathbf{a}}\in\mathcal{A}_1$. Combining (3.19) with (3.25), one finds (3.13) holds for $i=1$, i.e.

$$\lambda_{1,\mathbf{a}} = \tilde{J}_1(m_{1,\mathbf{a}},w_{1,\mathbf{a}}) = \int_{\mathbb{R}^N}\left[C_L\left|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\right|^{\gamma'} m_{1,\mathbf{a}} + [V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}})]m_{1,\mathbf{a}}\right]dx,$$

where we have used (3.17). Next, we shall show

$$w_{1,\mathbf{a}} = -C_H\gamma m_{1,\mathbf{a}}|\nabla u_{1,\mathbf{a}}|^{\gamma-2}\nabla u_{1,\mathbf{a}}.$$

First of all, (3.13) and (3.26) imply that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N}\left[C_L\left|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\right|^{\gamma'} + V_1 + f_1(m_{1,\mathbf{a}},m_{2,\mathbf{a}}) - \lambda_{1,\mathbf{a}}\right]m_{1,\mathbf{a}}\,dx\\
&= \int_{\mathbb{R}^N}\left[C_L\left|\frac{w_1}{m_{1,\mathbf{a}}}\right|^{\gamma'} - \Delta u_{1,\mathbf{a}} + C_H|\nabla u_{1,\mathbf{a}}|^{\gamma'}\right]m_{1,\mathbf{a}}\,dx.
\end{aligned}$$

Then we take $\psi = u_{1,\mathbf{a}}$ in (3.16) to get

$$0 = \int_{\{x|m_{1,\mathbf{a}}>0\}} \left[C_L \left| \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \right|^{\gamma'} + C_H |\nabla u_{1,\mathbf{a}}|^{\gamma'} + \nabla u_{1,\mathbf{a}} \cdot \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \right] m_{1,\mathbf{a}} \, dx. \tag{3.27}$$

therefore

By using the definition of H that

$$L\Big(- \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \Big) = C_L \left| \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \right|^{\gamma'} = \sup_{p \in \mathbb{R}^N} \Big(- p \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} - H(p) \Big) \geq -C_H |\nabla u_{1,\mathbf{a}}|^{\gamma} - \nabla u_{1,\mathbf{a}} \cdot \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}},$$

where $H(p) = C_H |p|^{\gamma}$. Therefore, (3.27) indicates that

$$C_L \left| \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \right|^{\gamma'} + C_H |\nabla u_{1,\mathbf{a}}|^{\gamma} + \nabla u_{1,\mathbf{a}} \cdot \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} \geq 0 \text{ a.e. in } \{x \in \mathbb{R}^N | m_{1,\mathbf{a}} > 0\}. \tag{3.28}$$

indicates

Since $\sup_{p \in \mathbb{R}^N} (-p \frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} - H(p))$ is attained by $p = \nabla u_{1,\mathbf{a}}$ when $m_{1,\mathbf{a}} > 0$, one has from (3.28) that

$$\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}} = -\nabla H(\nabla u_{1,\mathbf{a}}) \text{ in } \{x \in \mathbb{R}^N | m_{1,\mathbf{a}} > 0\}.$$

Thus, we obtain

$$-\Delta m_{1,\mathbf{a}} - C_H \nabla \cdot (m_{1,\mathbf{a}} |\nabla u_{1,\mathbf{a}}|^{\gamma - 2} \nabla u_{1,\mathbf{a}}) = 0 \text{ in a weak sense.}$$

Proceeding the similar argument shown above, we have (3.13) holds for $i = 2$ and there exists $u_2 \in C^2(\mathbb{R}^N)$ such that

$$w_{2,\mathbf{a}} = -C_H \gamma m_{2,\mathbf{a}} |\nabla u_{2,\mathbf{a}}|^{\gamma - 2} \nabla u_{2,\mathbf{a}}, \ x \in \mathbb{R}^N \text{ in a weak sense.}$$

Finally, by the standard elliptic regularity, we find (3.12) holds in a classical sense. This completes the proof of this lemma. □

By summarizing Lemma 3.1 and Lemma 3.2, we are able to show conclusions stated in Theorem 1.1, which are

Proof of Theorem 1.1:

Proof. For Conclusion (i), we invoke Lemma 3.1 to get there exists a minimizer $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, m_{2,\mathbf{a}}) \in \mathcal{K}$ to (1.5). Moreover, Lemma 3.2 implies there exist $(u_{1,\mathbf{a}}, u_{2,\mathbf{a}}) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ and $(\lambda_{1,\mathbf{a}}, \lambda_{2,\mathbf{a}}) \in \mathbb{R}^2$ such that $(m_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{1,\mathbf{a}}, u_{2,\mathbf{a}}, \lambda_{1,\mathbf{a}}, \lambda_{2,\mathbf{a}})$ solves (1.13). By standard regularity arguments, we have from Lemma 2.4 that

$$(m_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{1,\mathbf{a}}, u_{2,\mathbf{a}}) \in W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N),$$

which completes the proof of this conclusion. Conclusion (ii) is the straightforward corollary of Lemma 3.1. □

We next focus on the borderline case when $\alpha_1 = \alpha_2$ shown in Theorem 1.1. In detail, we impose the extra assumption (1.14) on the potentials and investigate the conclusions shown in Theorem 1.2, which are

Proof of Theorem 1.2:

Proof. In light of the assumption (1.14), we let (m_t, w_t) be (3.6) with $x_0 \in \mathbb{R}^N$ satisfying

$$V_1(x_0) = V_2(x_0) = 0.$$

Then for $i = 1, 2$, we compute to get

$$\int_{\mathbb{R}^N} V_i(x) m_t \, dx = \frac{1}{M^*} \int_{\mathbb{R}^N} V_i(x) t^N m_0(t(x - x_0)) \, dx = \frac{1}{M^*} \int_{\mathbb{R}^N} V_i\Big(\frac{y}{t} + x_0\Big) m_0(y) \, dy.$$

By invoking Lebesgue Convergence Dominated Theorem, we further obtain as $t \to +\infty$,

$$\int_{\mathbb{R}^N} V_i(x) m_t \, dx \to V_i(x_0) = 0, \text{ for } i = 1, 2.$$

Proceeding the similar argument shown in the proof of Lemma 3.1, we get

$$\mathcal{E}_{a^*-\beta, a^*-\beta, \beta}(m_t, w_t, m_t, w_t) = [V_1(x_0) + V_2(x_0)] + o_t(1), \tag{3.29}$$

takeinfty

where $o_t(1) \to 0$ as $t \to +\infty$. We take $t \to +\infty$ in (3.29) to obtain

$$e_{\alpha^*-\beta, \alpha^*-\beta, \beta} \leq 0. \tag{3.30}$$

criticald

On the other hand, we rewrite (1.6) as

$$\begin{aligned} \mathcal{E}_{\alpha_1, \alpha_2, \beta}(m_1, w_1, m_2, w_2) = & \sum_{i=1}^{2} \Big(\int_{\mathbb{R}^N} C_L \Big|\frac{w_i}{m_i}\Big|^{\gamma'} m_i + V_i m_i - \frac{N(\alpha_i + \beta)}{N + \gamma'} \int_{\mathbb{R}^N} m_i^{1+\frac{\gamma'}{N}} \, dx \Big) \\ & + \frac{N\beta}{N + \gamma'} \int_{\mathbb{R}^N} \Big(m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} - m_2^{\frac{1}{2}+\frac{\gamma'}{2N}} \Big)^2 dx. \end{aligned} \tag{3.31}$$

uponsubst

Upon substituting $\alpha_1 = \alpha_2 = a^* - \beta$ and $\beta = a^* - \alpha$ into (3.31), we deduce that

$$e_{a^*-\beta, a^*-\beta, \beta} \geq 0. \tag{3.32}$$

criticald

Combining (3.30) with (3.32), one has

$$e_{a^*-\beta, a^*-\beta, \beta} = 0. \tag{3.33}$$

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Now, we argue by contradiction and assume that (m_1, w_1, m_2, w_2) is a minimizer of (1.5) with $\alpha_1 = \alpha_2 = a^* - \beta$ and $\beta = a^* - \alpha$. Then we have

$$\begin{aligned} \mathcal{E}_{a^*-\beta, a^*-\beta, \beta}(m_1, w_1, m_2, w_2) = & \sum_{i=1}^{2} C_L \int_{\mathbb{R}^N} \Big|\frac{w_i}{m_i}\Big|^{\gamma'} m_i \, dx - \frac{Na^*}{N + \gamma'} \int_{\mathbb{R}^N} m_i^{1+\frac{\gamma'}{N}} \, dx \\ & + \frac{N\beta}{N + \gamma'} \int_{\mathbb{R}^N} \Big(m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} - m_2^{\frac{1}{2}+\frac{\gamma'}{2N}} \Big)^2 dx \\ & + \int_{\mathbb{R}^N} V_1(x) m_1 + V_2(x) m_2 \, dx \\ := & I_1 + I_2 + I_3. \end{aligned} \tag{3.34}$$

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In light of (3.33), one finds from (3.34) that $I_1 = I_2 = I_3 = 0$, in which $I_1 = 0$ implies each (m_i, w_i), $i = 1, 2$ is a minimizer of problem (1.10). In addition, $I_2 = 0$ indicates that $m_1 = m_2$ in $\mathbb{R}^N$. Morever, one gets from $I_3 = 0$ that

$$\int_{\mathbb{R}^N} V_1(x) m_1 + V_2(x) m_2 \, dx = 0,$$

which leads to a contradiction since $m_i > 0$ for $i = 1, 2$ by using the compactly Sobolev embedding and the maximum principle as shown in [1]. □

For the existence of minimizers, we next consider the case of $\alpha_1 = a^*$ and show Theorem 1.3, which is

Proof of Theorem 1.3:

Proof. We define the test solution-pair as

$$m_{i,\tau}(x) = \frac{\tau^N}{M^*} m_0\Big(\tau\Big(x - \bar{x}_i + (-1)^i \iota \frac{\ln\tau}{\tau}\nu\Big)\Big),\ w_{i,\tau} = \frac{\tau^{N+1}}{M^*} w_0\Big(\tau\Big(x - \bar{x}_i + (-1)^i \iota \frac{\ln\tau}{\tau}\nu\Big)\Big), \tag{3.35}$$

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where (m_0, w_0) denotes a minimizer of (1.10) satisfying (1.11), $\nu \in \mathbb{S}^{N-1}$, $\bar{x}_i \in \mathbb{R}^N$ and constant ι will be determined later.

By using Lemma 2.6, we have

$$C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\tau}}{m_{i,\tau}}\right|^{\gamma'} m_{i,\tau}\, dx = \frac{\tau^{\gamma'}}{M^*}, \quad \int_{\mathbb{R}^N} m_{i,\tau}^{1+\frac{\gamma'}{N}}\, dx = \frac{N+\gamma'}{N} \frac{\tau^{\gamma'}}{(M^*)^{1+\frac{\gamma'}{N}}}, \tag{3.36}$$

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and

$$\int_{\mathbb{R}^N} m_{1,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx = \frac{\tau^{\gamma'}}{(M^*)^{1+\frac{\gamma'}{N}}} \int_{\mathbb{R}^N} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x) m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x + \tau(\bar{x}_1 - \bar{x}_2) + 2\iota \ln\tau\nu)\, dx. \tag{3.37}$$

B6innotes

We have the fact that

$$|\tau(\bar{x}_1 - \bar{x}_2) + 2\iota\ln\tau\nu| \ge 2\iota\ln\tau \ \text{ when } \tau \gg 1.$$

Hence, for τ large, if $x \in B_{\iota\ln\tau} = \{x \| x| < \iota\ln\tau\}$, one gets from Lemma 2.5 that

$$m_0(x + \tau(\bar{x}_1 - \bar{x}_2) + 2\iota\ln\tau\nu) \le Ce^{-\delta_0\iota\ln\tau}, \tag{3.38}$$

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where $C > 0$ is a constant. And if $x \in B^c_{\iota\ln\tau}$, then

$$m_0 \le Ce^{-\delta_0\iota\ln\tau}, \tag{3.39}$$

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where C is a positive constant, $\delta_0 > 0$ and we have used the exponential decay property of m_0.

Combining (3.38) and (3.39), one finds from (3.37) that as $\tau \to +\infty$,

$$\begin{aligned}\int_{\mathbb{R}^N} m_{1,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx = \frac{\tau^{\gamma'}}{(M^*)^{1+\frac{\gamma'}{N}}} \Big[&\int_{B_{\iota\ln\tau}} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x) m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x + \tau(\bar{x}_1 - \bar{x}_2) + 2\iota\ln\tau\nu)\, dx \\ &+ \int_{B^c_{\iota\ln\tau}} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x) m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}(x + \tau(\bar{x}_1 - \bar{x}_2) + 2\iota\ln\tau\nu)\, dx\Big] \\ \le\, & C_{N,\gamma'} \tau^{\gamma'} e^{-(\frac{1}{2}+\frac{\gamma'}{2N})\delta_0\iota\ln\tau} = C_{N,\gamma'} \tau^{\gamma' - (\frac{1}{2}+\frac{\gamma'}{2N})\delta_0\iota} \to 0,\end{aligned} \tag{3.40}$$

B8innotes

where constant ι is chosen as $\iota > \frac{2\gamma' N}{(N+\gamma')\delta_0}$. In addition,

$$\int_{\mathbb{R}^N} V_i m_{i,\tau}\, dx = \frac{1}{M^*} \int_{\mathbb{R}^N} V_i\Big(\frac{x}{\tau} + \bar{x}_i - (-1)^i \iota \frac{\ln\tau}{\tau}\nu\Big) m_0\, dx := \frac{1}{M^*} \int_{\mathbb{R}^N} g_\tau(x)\, dx.$$

Noting that $g_\tau(x) \to V_i(\bar{x}_i) m_0(x)$ a.e. in $\mathbb{R}^N$, we obtain from (1.9), (3.38) and (3.39) that when τ is large,

$$|g_\tau(x)| \le Ce^{\delta|\frac{x}{\tau} + \bar{x}_i - (-1)^i \iota \frac{\ln\tau}{\tau}\nu|} e^{-\delta_0|x|} \le Ce^{-\frac{\delta_0}{2}|x|} \in L^1(\mathbb{R}^N).$$

Thus, by Lebesgue dominated theorem, we further get

$$\int_{\mathbb{R}^N} V_i m_{i,\tau}\, dx \to V_i(\bar{x}_i) \text{ as } \tau \to +\infty. \tag{3.41}$$ b10innote

Collecting (3.37), (3.40) and (3.41), one finds if $\alpha_1 = \alpha_2 = a^*$ and $\beta \le 0$, then

$$\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_{1,\tau}, w_{1,\tau}, m_{2,\tau}, w_{2,\tau}) = V_1(\bar{x}_1) + V_2(\bar{x}_2) + o_\tau(1),$$

where $o_\tau(1) \to 0$ as $\tau \to +\infty$. It follows that

$$e_{\alpha_1,\alpha_2,\beta} \le V_1(\bar{x}_1) + V_2(\bar{x}_2) = 0,$$

where $\bar{x}_i \in \mathbb{R}^N$ with $i = 1, 2$. If we choose $\bar{x}_i$ such that $V_i(\bar{x}_i) = 0$ for $i = 1, 2$, then by using (1.12) and $\beta \le 0$, one has $e_{\alpha_1,\alpha_2,\beta} \ge 0$. Therefore, we summarize to get $e_{\alpha_1,\alpha_2,\beta} = 0$. Proceeding the same argument as shown in the proof of Theorem 1.2, we show there is no minimizer in case (i).

For case (ii), if $\beta = 0$, one finds

$$e_{\alpha_1,\alpha_2,0} = e^1_{a^*} + e^2_{\alpha_2},$$

where $e^1_{a^*}$ and $e^2_{\alpha_2}$ are given by (3.1). Noting that this is the decoupled case, we have the fact that there is no minimizer as shown in [5].

If $0 < \beta < \frac{a^*-\alpha_2}{2}$, taking $\iota = 0$ in (3.35), we compute to get

$$\int_{\mathbb{R}^N} m_{1,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx = \frac{\tau^{\frac{1}{2}(\gamma'-N)}}{(M^*)^{\frac{1}{2}+\frac{\gamma'}{2N}}} \int_{\mathbb{R}^N} m_0\Big(\frac{x}{\tau} + \bar{x}_1\Big) m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx := \frac{\tau^{\frac{1}{2}(\gamma'-N)}}{(M^*)^{\frac{1}{2}+\frac{\gamma'}{2N}}} I_\tau. \tag{3.42}$$ b12innote

We choose $\bar{x}_1 \in \mathbb{R}^N$ such that $m_0(\bar{x}_1) > C_0 > 0$ then obtain

$$\lim_{\tau\to+\infty} I_\tau \ge C_0 \int_{\mathbb{R}^N} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx \ge C_1 > 0 \text{ as } \tau \to +\infty.$$

Thus, (3.42) implies

$$\int_{\mathbb{R}^N} m_{1,\tau}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_0^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx \ge C_1 \tau^{\frac{1}{2}(\gamma'-N)} \to +\infty,$$

It follows that

$$\mathcal{E}_{a^*.\alpha_2,\beta}(m_{1,\tau}, w_{1,\tau}, m_0, w_0) \le o_\tau(1) + C - C_{\gamma'}\beta\tau^{\frac{1}{2}(\gamma'-N)} \to -\infty \text{ for } \beta > 0.$$

Hence $e_{a^*,\alpha_2,\beta} = -\infty$ if $\beta > 0$, which indicates (1.5) has no minimizer.

□

As shown in Theorem 1.1 and Theorem 1.2, we have obtained when all coefficients α_1, α_2 and β are subcritical, (1.2) admits classical ground states; whereas, if $\alpha_1 = \alpha_2$ are subcritical and β is critical, then (1.5) has no minimizer. A natural question is the behaviors of ground states as $(\alpha_1, \alpha_2) \nearrow (a^* - \beta, a^* - \beta)$. In fact, we can show there are concentration phenomena as coefficients approach critical ones. In the next section, we shall discuss the asymptotic profiles of ground states in the singular limits mentioned above.

4 Asymptotic Profiles of Ground States with $\beta > 0$

ultipopulation⟩ This section is devoted to the blow-up behaviors of ground states to (1.2) in some singular limits. We focus on the attractive interaction case and obtain

Proof of Theorem 1.4:

Proof. FIrst of all, we have from (3.31) that

$$\begin{aligned}
\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) =& \sum_{i=1}^{2}\Big(\int_{\mathbb{R}^N} C_L\Big|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\Big|^{\gamma'} m_{i,\mathbf{a}} - \frac{N(\alpha_i+\beta)}{N+\gamma'}\int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\Big)\\
&+\frac{N\beta}{N+\gamma'}\int_{\mathbb{R}^N}\Big(m_{1,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}} - m_{2,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\Big)^2\,dx\\
&+\int_{\mathbb{R}^N} V_1(x)m_{1,\mathbf{a}} + V_2(x)m_{2,\mathbf{a}}\,dx\\
:=& II_1 + II_2 + II_3. \qquad (4.1)
\end{aligned}$$

thm13scal

In light of (1.8) and (1.12), one finds $II_j \geq 0$, $j = 1, 2, 3$. Moreover, assumption (1.8) implies $II_3 \geq 0$. Proceeding the same argument shown in the proof of Lemma 3.1, we use the test pair (3.6) and compute from (4.1) that

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} e_{\alpha_1,\alpha_2,\beta} = e_{a^*-\beta,a^*-\beta,\beta} = 0. \qquad (4.2)$$

thm13scal

Combining (4.1) with (4.2), we obtain (1.18) and (1.19).

We next prove (1.20) and argue by contradiction. Without loss of generality, we assume that

$$\limsup_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} C_L\int_{\mathbb{R}^N}\Big|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\Big|^{\gamma'} m_{1,\mathbf{a}}\,dx < +\infty.$$

Then, it follows from (1.18), (1.19) and Lemma 2.4 that $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ is uniformly bounded in $(W^{1,\gamma'}(\mathbb{R}^N)\times L^{\gamma'}(\mathbb{R}^N))^2$. Moreover, by compactly Sobolev embedding (C.f. Lemma 5.1 in [5]), one finds $m_{i,\mathbf{a}} \to m_{i,0}$ strongly in $L^1(\mathbb{R}^N)\cap L^{1+\frac{\gamma'}{N}}(\mathbb{R}^N)$ for $i = 1, 2$. By using the convexity of $\int_{\mathbb{R}^N}\big|\frac{w}{m}\big|^{\gamma'} m\,dx$, we have

$$\begin{aligned}
e_{a^*-\beta,a^*-\beta,\beta} = \lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta} e_{\alpha_1,\alpha_2,\beta} =& \lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta}\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})\\
\geq& \mathcal{E}_{a^*-\beta,a^*-\beta,\beta}(m_0, w_0, m_0, w_0) \geq e_{a^*-\beta,a^*-\beta,\beta},
\end{aligned}$$

which implies (m_0, w_0, m_0, w_0) is a minimizer of $e_{a^*-\beta,a^*-\beta,\beta}$ and it is a contradiction since we have showed that $e_{a^*-\beta,a^*-\beta,\beta}$ has no minimizer in Theorem 1.2.

Now, we find (1.20) holds and further obtain from (1.18) that for $i = 1, 2$

$$\int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx \to +\infty \text{ and } \lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta}\frac{C_L\int_{\mathbb{R}^N}\big|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\big|^{\gamma'} m_{i,\mathbf{a}}\,dx}{\int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx} = \frac{N+\gamma'}{N}a^*.$$

Noting that as $\mathbf{a}\nearrow\mathbf{a}^*_\beta$,

$$\Big[\Big(\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\Big)^{\frac{1}{2}} - \Big(\int_{\mathbb{R}^N} m_{2,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\Big)^{\frac{1}{2}}\Big]^2 \leq \int_{\mathbb{R}^N}\Big(m_{1,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}} - m_{2,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\Big)^2\,dx \to 0,$$

one gets (1.21) holds.

Noting that $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ satisfy (3.12), we have from the integration by parts that

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla u_{1,\mathbf{a}}\cdot\nabla m_{1,\mathbf{a}}\,dx + C_H\int_{\mathbb{R}^N}|\nabla u_{1,\mathbf{a}}|^{\gamma} m_{1,\mathbf{a}}\,dx+\lambda_1\\ =&\int_{\mathbb{R}^N} V_1 m_{1,\mathbf{a}}\,dx-\alpha_1\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx-\beta\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\,dx,\end{aligned}\tag{4.3}$$

ibp120240

and

$$\int_{\mathbb{R}^N} \nabla u_{1,\mathbf{a}}\cdot\nabla m_{1,\mathbf{a}}\,dx=-C_H\gamma\int_{\mathbb{R}^N} m_{1,\mathbf{a}}|\nabla u_{1,\mathbf{a}}|^{\gamma}\,dx.\tag{4.4}$$

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Combining (4.3) with (4.4), one finds

$$\begin{aligned}\lambda_1=&C_L\int_{\mathbb{R}^N}\left|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\right|^{\gamma'} m_{1,\mathbf{a}}\,dx+\int_{\mathbb{R}^N} V_1 m_{1,\mathbf{a}}\,dx-\alpha_1\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx-\beta\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\,dx\\ =&\left(C_L\int_{\mathbb{R}^N}\left|\frac{w_{1,\mathbf{a}}}{m_{1,\mathbf{a}}}\right|^{\gamma'} m_{1,\mathbf{a}}\,dx-\frac{N(\alpha_1+\beta)}{N+\gamma'}\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\right)-\frac{\gamma'(\alpha_1+\beta)}{\gamma'+N}\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\\ &+\beta\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\left(m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}-m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\right)dx+\int_{\mathbb{R}^N} V_1 m_{1,\mathbf{a}}\,dx\\ =&o_\varepsilon(1)-\frac{\gamma'(\alpha_1+\beta)}{\gamma'+N}\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx+\beta\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\left(m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}-m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\right)dx,\end{aligned}\tag{4.5}$$

lambda1fo

where we have used (1.18) and (1.19) as $\mathbf{a}\nearrow\mathbf{a}^*_\beta$. To further simplify (4.5), we use (1.19) to get

$$\begin{aligned}\left|\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\left(m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}-m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\right)dx\right|\leq&\left(\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\right)^{\frac12}\left[\int_{\mathbb{R}^N}\left(m_{1,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}-m_{2,\mathbf{a}}^{\frac12+\frac{\gamma'}{2N}}\right)^2dx\right]^{\frac12}\\ =&o_\varepsilon(1)\left(\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx\right)^{\frac12}.\end{aligned}\tag{4.6}$$

lambda1fo

By utilizing (1.18) and (1.22), one finds

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*_\beta}\frac{N\varepsilon_{\mathbf{a}}^{\gamma'}(\alpha_1+\beta)}{N+\gamma'}\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{2N}}\,dx=1.\tag{4.7}$$

lambda1fo

Collecting (4.5), (4.6) and (4.7), we have

$$\lambda_1=-\frac{(\alpha_1+\beta)\gamma'}{N+\gamma'}\int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{1+\frac{\gamma'}{N}}\,dx+o_\varepsilon(1)=\frac{\gamma'}{N}\varepsilon^{-\gamma'}+o_\varepsilon(1),\tag{4.8}$$

lambda1in

where $\varepsilon\to0$ given by (1.22). Proceeding the similar argument shown above, one obtains from (1.21) that

$$\lim_{\varepsilon\to0}\lambda_2\varepsilon^{\gamma'}=-\frac{\gamma'}{N}.$$

Now, we substitute (1.25) into (3.12) and obtain

$$\begin{cases}-\Delta u_{1,\varepsilon}+C_H|\nabla u_{1,\varepsilon}|^{\gamma}+\lambda_1\varepsilon^{\gamma'}=\varepsilon^{\gamma'}V_1(\varepsilon x+x_{1,\varepsilon})-\alpha_1 m_{1,\varepsilon}^{\frac{\gamma'}{N}}-\beta m_{1,\varepsilon}^{\frac{\gamma'}{2N}-\frac12}m_{2,\varepsilon}^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N,\\ \Delta m_{1,\varepsilon}+\nabla\cdot w_{1,\varepsilon}=0,\ w_{1,\varepsilon}=-\gamma C_H m_{1,\varepsilon}|\nabla u_{1,\varepsilon}|^{\gamma-2}\nabla u_{1,\varepsilon} & x\in\mathbb{R}^N,\\ -\Delta u_{2,\varepsilon}+C_H|\nabla u_{2,\varepsilon}|^{\gamma}+\lambda_2\varepsilon^{\gamma'}=\varepsilon^{\gamma'}V_2(\varepsilon x+x_{1,\varepsilon})-\alpha_2 m_{2,\varepsilon}^{\frac{\gamma'}{N}}-\beta m_{2,\varepsilon}^{\frac{\gamma'}{2N}-\frac12}m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{N}}, & x\in\mathbb{R}^N,\\ \Delta m_{2,\varepsilon}+\nabla\cdot w_{2,\varepsilon}=0,\ w_{2,\varepsilon}=-\gamma C_H m_{2,\varepsilon}|\nabla u_{2,\varepsilon}|^{\gamma-2}\nabla u_{2,\varepsilon}, & x\in\mathbb{R}^N,\\ \int_{\mathbb{R}^N} m_{1,\varepsilon}\,dx=\int_{\mathbb{R}^N} m_{2,\varepsilon}\,dx=1.\end{cases}\tag{4.9}$$

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Without loss of the generality, we assume

$$\inf_{x\in\mathbb{R}^N} u_{1,\mathbf{a}} = \inf_{x\in\mathbb{R}^N} u_{2,\mathbf{a}} = 0.$$

In light of (1.21), (1.22) and (1.25), one finds

$$\sup_{\varepsilon\to 0^+} C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\varepsilon}}{m_{i,\varepsilon}}\right|^{\gamma'} m_{i,\varepsilon}\,dx < +\infty,\ \ i=1,2.$$

Then it follows from Lemma 2.4 that for $i=1,2$

$$\sup_{\varepsilon\to 0^+}\|m_{i,\varepsilon}\|_{W^{1,\gamma'}(\mathbb{R}^N)} < +\infty,\quad \sup_{\varepsilon\to 0^+}\|w_{i,\varepsilon}\|_{L^1(\mathbb{R}^N)} < +\infty,\quad \sup_{\varepsilon\to 0^+}\|w_{i,\varepsilon}\|_{L^{\gamma'}(\mathbb{R}^N)} < +\infty. \tag{4.10}$$

fromregul

Invoking (1.19) and (1.22) , one finds for $i=1,2$, as $\varepsilon\to 0^+$,

$$\int_{\mathbb{R}^N} V_i(\varepsilon x + x_{1,\varepsilon}) m_{i,\varepsilon}\,dx \to 0, \tag{4.11}$$

convergen

and

$$\int_{\mathbb{R}^N} \left(m_{1,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}} - m_{2,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}}\right)^2 dx \to 0. \tag{4.12}$$

convergen

By using the standard Sobolev embedding, we have from (4.10) and (4.12) that

$$m_{i,\varepsilon} \rightharpoonup m \text{ in } W^{1,\gamma'}(\mathbb{R}^N),\ \ m_{i,\varepsilon}\to m\geq 0,\ \text{a.e. in } \mathbb{R}^N. \tag{4.13}$$

convergen

Moreover, by using the Morrey's embedding $W^{1,\gamma'} \hookrightarrow C^{0,\theta}(\mathbb{R}^N)$ with $\theta\in(0,1-\frac{\gamma'}{N})$, one finds

$$m_{i,\varepsilon}\to m \text{ in } C^{0,\theta}_{\mathrm{loc}}(\mathbb{R}^N),\quad \sup_{\varepsilon\to 0^+}\|m_{i,\varepsilon}\|_{C^{0,\theta}(\mathbb{R}^N)} < +\infty,\ \ i=1,2. \tag{4.14}$$

convergen

Recall that $u_{1,\mathbf{a}}(x_{1,\varepsilon}) = \inf_{x\in\mathbb{R}^N} u_{1,\mathbf{a}} = 0$, then we have $u_{1,\varepsilon}(0) = \inf_{x\in\mathbb{R}^N} u_{1,\varepsilon}$. Moreover, by applying the maximum principle, one gets from the first equation of (4.9) that

$$\begin{aligned}\lambda_1\varepsilon^{\gamma'} \geq & -\alpha_1 m_{1,\varepsilon}^{\frac{\gamma'}{N}}(0) - \beta m_{2,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}}(0) m_{1,\varepsilon}^{\frac{\gamma'}{2N}-\frac{1}{2}}(0)\\ = & -(\alpha_1+\beta) m_{1,\varepsilon}^{\frac{\gamma'}{N}}(0) + o_\varepsilon(1),\end{aligned}$$

where we have used $m_{i,\varepsilon}\to m$ in $C_{\mathrm{loc}}(\mathbb{R}^N)$ shown in (4.14). In addition, noting (4.8), we have

$$\lim_{\varepsilon\to 0} m_{1,\varepsilon}^{\frac{\gamma'}{N}}(0) \geq \frac{\gamma'}{Na^*},$$

where we have used $\alpha_1+\beta \nearrow a^*$. Since $W^{1,\gamma'} \hookrightarrow C^{0,\theta}$, we have from (4.12) that for $i=1,2$, there exists $R_0>0$ and $C>0$ such that

$$m_{i,\varepsilon}(x) \geq C > 0,\ \ \forall |x|<R_0,\ \text{for } i=1,2. \tag{4.15}$$

mivarepsi

Moreover, we utilize (4.11) and (4.15) to get up to a subsequence,

$$\lim_{\varepsilon\to 0} x_{1,\varepsilon} = x_0,\ \text{s.t. } V_1(x_0) = 0 = V_2(x_0).$$

Combining (4.14) with (4.15), one also has

$$m(x) \geq C > 0, \ \forall |x| < R_0, \tag{4.16}$$ mlowerbou

where C and R_0 are positive constants.

Next, we study the regularity of the value function u. To this end, we rewrite the u_1-equation in (4.9) as

$$\begin{aligned} -\Delta u_{1,\varepsilon} + C_H|\nabla u_{1,\varepsilon}|^\gamma = & -\lambda_1 \varepsilon^{\gamma'} + \varepsilon^{\gamma'} V_1(\varepsilon x + x_{1,\varepsilon}) - \alpha_1 m_{1,\varepsilon}^{\frac{\gamma'}{N}} - \beta m_{1,\varepsilon}^{\frac{\gamma'}{2N}-\frac{1}{2}} m_{2,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}} \\ := & g_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap C^{0,\theta}_{\text{loc}}(\mathbb{R}^N). \end{aligned} \tag{4.17}$$ rewriteue

For $R > 0$ large enough, we have

$$\|g_\varepsilon(x)\|_{L^\infty(B_R(0))} < C_R < +\infty, \ \forall |x| < 2R,$$

where $C_R > 0$ is independent of ε. Then it follows from (4.17) and Sobolev embedding that

$$|\nabla u_{1,\varepsilon}(x)| \leq C_R, \ \forall |x| < 2R.$$

Since $u_{1,\varepsilon}(0) = 0$, we further have

$$|u_{1,\varepsilon}| \leq C_R, \ \forall |x| < 2R.$$

By using the $W^{2,p}$ estimate, one gets

$$\|u_{1,\varepsilon}\|_{W^{2,p}(B_{R+1}(0))} \leq C_{p,R}(\|u_{1,\varepsilon}\|_{L^p(B_{2R}(0))} + \|g_\varepsilon\|_{L^p(B_{2R}(0))} + \||\nabla u_{1,\varepsilon}|^\gamma\|_{L^p(B_{2R}(0))}), \ \forall p > 1,$$

where $C_{p,R} > 0$ is a constant depending on p and R. Let $p > N$, then we obtain

$$\|u_{1,\varepsilon}\|_{C^{1,\theta_1}(B_{R+1}(0))} \leq C_{\theta_1,R} < +\infty,$$

where some $\theta_1 \in (0,1)$. Moreover, we rewrite (4.17) as

$$-\Delta u_{1,\varepsilon} = -C_H|\nabla u_{1,\varepsilon}|^\gamma + g_\varepsilon \in C^{1,\theta_2}(B_{R+1}(0)).$$

One further deduces from the standard $W^{2,p}$ estimate that

$$\|u_{1,\varepsilon}\|_{C^{2,\theta_3}(B_R(0))} \leq C_{\theta_3,R} < +\infty,$$

where $C_{\theta_3,R} > 0$ is a constant. Then by the standard diagonal procedure and Arzelà-Ascoli theorem, we have from (4.9), (4.10), (4.13) and (4.14) that there exist $u_1 \in C^2(\mathbb{R}^N)$ and $w_1 \in L^{\gamma'}(\mathbb{R}^N)$ such that

$$u_{1,\varepsilon} \to u_1 \text{ in } C^2_{\text{loc}}(\mathbb{R}^N), \ \ w_{1,\varepsilon} \rightharpoonup w_1 \text{ in } L^{\gamma'}(\mathbb{R}^N), \tag{4.18}$$ eq4.180

and (u_1, m, w_1) satisfies

$$\begin{cases} -\Delta u_1 + C_H|\nabla u_1|^\gamma - \frac{\gamma'}{N} = -a^* m^{\frac{\gamma'}{N}}, & x \in \mathbb{R}^N, \\ -\Delta m = \gamma C_H \nabla \cdot (m|\nabla u_1|^{\gamma-2}\nabla u_1) = -\nabla \cdot w_1, & x \in \mathbb{R}^N, \\ 0 < \int_{\mathbb{R}^N} m\, dx \leq 1, \end{cases}$$

where we have used (4.8) and (4.16). In addition, by Lemma 2.6 and (1.12), one finds

$$\int_{\mathbb{R}^N} m\, dx = 1. \tag{4.19}$$ 418202410

Thus, with the aid of (4.13), we obtain for $i = 1, 2$, $m_{i,\varepsilon} \to m$ in $L^1(\mathbb{R}^N)$. Moreover, (4.14) indicates

$$m_{i,\varepsilon} \to m \text{ in } L^p(\mathbb{R}^N),\ \forall p \geq 1. \tag{4.20}$$ wefurther

This combine with (4.18) show that (1.26) holds for $i = 1$.

Next, we prove that (1.23). We first recall that $u_{2,\mathbf{a}}(x_{2,\varepsilon}) = 0 = \inf_{x\in\mathbb{R}^N} u_{2,\mathbf{a}}$. Then, we have from (4.9) and (1.25) that

$$\begin{aligned}\lambda_2 \geq & V_2(x_{2,\varepsilon}) - \alpha_2 m_{2,\mathbf{a}}^{\frac{\gamma'}{N}}(x_{2,\varepsilon}) - \beta m_{2,\mathbf{a}}^{\frac{\gamma'}{2N}-\frac{1}{2}}(x_{2,\varepsilon}) m_{1,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{N}}(x_{2,\varepsilon})\\ \geq & \varepsilon^{-\gamma'}\Big[- \alpha_2 m_{2,\varepsilon}^{\frac{\gamma'}{N}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) - \beta m_{2,\varepsilon}^{\frac{\gamma'}{2N}-\frac{1}{2}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) m_{1,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{N}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big)\Big],\end{aligned}$$

which implies

$$\alpha_2 m_{2,\varepsilon}^{\frac{\gamma'}{N}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) + \beta m_{2,\varepsilon}^{\frac{\gamma'}{2N}-\frac{1}{2}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) m_{1,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{N}}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) \geq \varepsilon^{\gamma'}\lambda_2 \geq \frac{\gamma'}{2N} \text{ as } \varepsilon \to 0. \tag{4.21}$$ lambda2th

Combining (4.14) with (4.20), one can easily check that for $i = 1, 2$

$$\lim_{|x|\to+\infty} m_{i,\varepsilon}(x) = 0 \text{ uniformly in } \varepsilon.$$

Combining this with (4.21), one has (1.23) holds.

We next similarly show that there exist $u_2 \in C^2(\mathbb{R}^N)$ and $w_2 \in L^{\gamma'}(\mathbb{R}^N)$ such that

$$u_{2,\varepsilon} \to u_2 \text{ in } C^2_{\text{loc}}(\mathbb{R}^N),\ \text{and } \mathbf{w}_{2,\varepsilon} \rightharpoonup \mathbf{w}_2 \text{ in } L^{\gamma'}(\mathbb{R}^N),$$

and (u_2, m, w_2) satisfies (1.27), in which (m, w_2) is a minimizer of (1.10). Indeed, we rewrite the u_2-equation in (4.9) as

$$\begin{aligned}-\Delta u_{2,\varepsilon} + C_H|\nabla u_{2,\varepsilon}|^{\gamma} &= -\lambda_2\varepsilon^{\gamma'} + \varepsilon^{\gamma'} V_2(\varepsilon x + x_{1,\varepsilon}) - \alpha_2 m_{2,\varepsilon}^{\frac{\gamma'}{N}} - \beta m_{2,\varepsilon}^{\frac{\gamma'}{2N}-\frac{1}{2}} m_{1,\varepsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}}\\ &:= h_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap C^{0,\theta}_{\text{loc}}(\mathbb{R}^N).\end{aligned} \tag{4.22}$$ argueforb

Moreover, by Lemma 2.1, one has for any $R > 0$ large enough,

$$|\nabla u_{2,\varepsilon}| \leq C_R < +\infty,\ \forall |x| < 2R. \tag{4.23}$$ gradiente

In light of

$$u_{2,\varepsilon}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big) = 0 = \inf_{x\in\mathbb{R}^N} u_{2,\varepsilon},$$

we use (1.23) and (4.23) to get

$$|u_{2,\varepsilon}(0)| \leq C_R\Big|\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big| + \Big|u_{2,\varepsilon}\Big(\frac{x_{2,\varepsilon}-x_{1,\varepsilon}}{\varepsilon}\Big)\Big| \leq \tilde{C}_R < +\infty,$$

where constant $\tilde{C}_R > 0$. Thus, thanks to (4.23), we find

$$|u_{2,\varepsilon}(x)| \leq C_R,\ \ \forall |x| < 2R. \tag{4.24}$$ 69innotes

Upon collecting (4.22), (4.23) and (4.24), one obtains

$$\|u_{2,\varepsilon}\|_{C^{2,\theta_5}(B_R(0))} \leq C_{\theta_5,R} < +\infty.$$

Moreover, we similarly get (u_2, m, w_2) satisfies (1.27), in which (m, w_2) is a minimizer of (1.10). To finish the proof of (1.26), it remains to show that $u_1 = u_2$ and $w_1 = w_2$, which can be obtained by following the argument shown in the proof of Theorem 2.4 in [9], Indeed, since (m, u_1, λ) and (m, u_2, λ) solve (1.27) with $w_i := \gamma m|\nabla u_i|^{\gamma-2}\nabla u_i$, we test the $u_1 - u_2$ equation and $m_1 - m_2$ equation against $m_1 - m_2$ and $u_1 - u_2$ and integrate them by parts, then subtract them to get a useful identity. With the aid of the strict convexity of $|p|^\gamma$, $\gamma > 1$, one has the conclusion $\nabla u_1 = \nabla u_2$ and then $w_1 = w_2$. By fixing the same minimum points of u_1 and u_2, we obtain $u_1 = u_2$.

Finally, proceeding the similar argument as shown in the proof of Case (ii), Theorem 1.4 in [5], we have (1.24) holds. □

Theorem 1.4 demonstrates that under mild assumptions (1.8) and (1.9), ground states $(m_{1,\mathbf{a}}, u_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{2,\mathbf{a}})$ are localized as $\mathbf{a} \nearrow \mathbf{a}^*_\beta$. We next discuss the proof of Theorem 1.5, which is for the refined asymptotic profiles of $(m_{1,\mathbf{a}}, u_{1,\mathbf{a}}, m_{2,\mathbf{a}}, u_{2,\mathbf{a}})$. First of all, we establish the following upper bound of $e_{\alpha_1,\alpha_2,\beta}$ given by (1.5):

emma4120240724⟩ **Lemma 4.1.** *Under the assumptions of Theorem 1.5, we have as $(\alpha_1, \alpha_2) \nearrow (a^* - \beta, a^* - \beta)$,*

$$0 \le e_{\alpha_1,\alpha_2,\beta} \le \Big(\frac{\gamma' + p_0}{p_0}\Big)\Big(\frac{\mu\bar\nu_{p_0} p_0}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+p_0}}\Big(\frac{2}{a^*}\Big)^{\frac{p_0}{\gamma'+p_0}}\Big(a^* - \frac{\alpha_1 + \alpha_2 + 2\beta}{2}\Big)^{\frac{p_0}{\gamma'+p_0}}(1 + o(1)). \tag{4.25}$$

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Proof. From the definition of $\bar\nu_{p_0}$ in (1.16), one can easily derive that, for any $\nu > \bar\nu_{p_0}$, there exist $(m_0, w_0) \in \mathcal{M}$ and $y \in \mathbb{R}^N$ such that

$$\bar\nu_{p_0} \le H_{m_0,p_0}(y) = \int_{\mathbb{R}^N} |x + y|^{p_0} m_0(x)dx \le \nu. \tag{4.26}$$

eq-nu

Since $(m_0, w_0) \in \mathcal{M}$ is a minimizer of (1.10), we have from (1.10) and Lemma 2.6 that

$$\int_{\mathbb{R}^N} m_0\, dx = 1,\ C_L \int_{\mathbb{R}^N} \Big|\frac{w_0}{m_0}\Big|^{\gamma'} m_0\, dx = 1,\ \text{and}\ \frac{N}{N+\gamma'}\int_{\mathbb{R}^N} m_0^{1+\frac{\gamma'}{N}}\, dx = \frac{1}{a^*}. \tag{4.27}$$

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Let $x_j \in Z_0$, and define

$$m_\tau(x) = \tau^N m_0(\tau(x - x_j) - y),\ \ w_\tau(x) = \tau^{N+1} w_0(\tau(x - x_j) - y), \tag{4.28}$$

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then one finds from (4.27) and (4.28) that

$$C_L \int_{\mathbb{R}^N} \Big|\frac{w_\tau}{m_\tau}\Big|^{\gamma'} m_\tau\, dx = \tau^{\gamma'} C_L \int_{\mathbb{R}^N} \Big|\frac{w_0}{m_0}\Big|^{\gamma'} m_0\, dx = \tau^{\gamma'},$$

$$\frac{N}{N+\gamma'}\int_{\mathbb{R}^N} m_\tau^{1+\frac{\gamma'}{N}}\, dx = \frac{N}{N+\gamma'}\tau^{\gamma'}\int_{\mathbb{R}^N} m_0^{1+\frac{\gamma'}{N}}\, dx = \frac{\tau^{\gamma'}}{a^*},$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}(V_1 + V_2)m_\tau\, dx &= \int_{\mathbb{R}^N}(V_1 + V_2)\Big(\frac{x+y}{\tau} + x_j\Big)m_0(x)\, dx\\ &= \frac{1}{\tau^{p_0}}\int_{\mathbb{R}^N}\frac{(V_1 + V_2)(\frac{x+y}{\tau} + x_j)}{|\frac{x+y}{\tau}|^{p_0}}|x + y|^p m_0\, dx.\end{aligned} \tag{4.29}$$

combining

Note that

$$\lim_{\tau\to+\infty}\frac{(V_1+V_2)(\frac{x+y}{\tau}+x_j)}{\left|\frac{x+y}{\tau}\right|^{p_0}}=\mu. \tag{4.30}$$

combining

Combining (4.26), (4.29) with (4.30), one can get

$$\int_{\mathbb{R}^N}(V_1+V_2)m_\tau(x)\,dx=\frac{\mu\nu}{\tau^{p_0}}+O\Big(\frac{1}{\tau^{p_0}}\Big).$$

Finally, by taking

$$\tau=\Bigg(\frac{\mu\nu p_0 a^*}{2\gamma'(a^*-\frac{\alpha_1+\alpha_2+2\beta}{2})}\Bigg)^{\frac{1}{\gamma'+p_0}} \tag{4.31}$$

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in (1.6), we obtain

$$\begin{aligned}0\le e_{\alpha_1,\alpha_2,\beta}\le\mathcal{E}(m_\tau,w_\tau,m_\tau,w_\tau)&=\tau^{\gamma'}\Big[2-\frac{\alpha_1+\alpha_2+2\beta}{a^*}\Big]+\frac{\mu\nu}{\tau^{p_0}}+O\Big(\frac{1}{\tau^{p_0}}\Big)\\&=\frac{\gamma'+p_0}{p_0}\Big(\frac{\mu\nu p_0}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+p_0}}\Big(\frac{2}{a^*}\Big)^{\frac{p_0}{\gamma'+p_0}}\Big(a^*-\frac{\alpha_1+\alpha_2+2\beta}{2}\Big)^{\frac{p_0}{\gamma'+p_0}}(1+o_\tau(1)),\end{aligned}$$

which indicates (4.25) since $\nu>\bar{\nu}_{p_0}$ is arbitrary. □

Now, we are ready to prove Theorem 1.5, which is

Proof of Theorem 1.5:

Proof. In light of (1.25), we compute

$$\begin{aligned}e_{\alpha_1,\alpha_2,\beta}=&\mathcal{E}(m_{1,\varepsilon},w_{1,\varepsilon},m_{2,\varepsilon},w_{2,\varepsilon})\\=&\sum_{i=1}^{2}\Big[\varepsilon^{-\gamma'}C_L\int_{\mathbb{R}^N}\Big|\frac{w_{i,\varepsilon}}{m_{i,\varepsilon}}\Big|^{\gamma'}m_{i,\varepsilon}\,dx-\frac{\alpha_i\varepsilon^{-\gamma'}}{1+\frac{\gamma'}{N}}\int_{\mathbb{R}^N}m_{i,\varepsilon}^{1+\frac{\gamma'}{N}}\,dx+\int_{\mathbb{R}^N}V(\varepsilon x+x_\varepsilon)m_{i,\varepsilon}\,dx\Big]\\&-\frac{2\beta\varepsilon^{-\gamma'}}{1+\frac{\gamma'}{N}}\int_{\mathbb{R}^N}m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}m_{2,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}\,dx,\end{aligned} \tag{4.32}$$

3point1no

where we redefine $x_{1,\varepsilon}$ as x_ε here and in the sequel for simplicity. Noting that $V_1(x_0)=V_2(x_0)$ shown in Theorem 1.4, we find there exists some j satisfying $1\le j\le l$ such that $x_0=x_j$. Then, we rewrite the potential energy as

$$\int_{\mathbb{R}^N}V_i(\varepsilon x+x_\varepsilon)m_{i,\varepsilon}(x)\,dx=\varepsilon^{p_j}\int_{\mathbb{R}^N}\frac{V_i(\varepsilon x+x_\varepsilon)}{|\varepsilon x+x_\varepsilon-x_j|^{p_j}}\Big|x+\frac{x_\varepsilon-x_j}{\varepsilon}\Big|^{p_j}m_{i,\varepsilon}\,dx, \tag{4.33}$$

3point2no

where $i=1,2$. In addition, since $x_\varepsilon\to x_j$, we obtain

$$\lim_{\varepsilon\to0^+}\sum_{i=1}^{2}\frac{V_i(\varepsilon x+x_\varepsilon)}{|\varepsilon x+x_\varepsilon-x_j|^{p_j}}=\mu_j\text{ a.e. in }\mathbb{R}^N, \tag{4.34}$$

{?}

where μ_j is defined in (1.29). Without loss of generality, we assume $p_{2j}\ge p_{1j}=p_j$ with p_{1j} and p_{2j} defined by (1.28).

Now, we claim that

$$p_j = p_0 = \max\{p_1, \cdots, p_l\}, \text{ and } \Big|\frac{x_\varepsilon - x_j}{\varepsilon}\Big| \text{ is uniformly bounded as } \varepsilon \to 0^+. \tag{4.35}$$
claimattra

To show (4.35), we argue by contradiction and obtain either $p_j < p_0$ or up to a subsequence,

$$\lim_{\varepsilon\to 0^+} \Big|\frac{x_\varepsilon - x_j}{\varepsilon}\Big| = +\infty.$$

By using (4.33) and $m_{i,\varepsilon} \to m_0$ in $L^1 \cap L^\infty$ shown in Theorem 1.4, one deduces that for any $\Gamma > 0$ large enough,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon^{-p_0} \int_{\mathbb{R}^N} V_1(\varepsilon x + x_\varepsilon) m_{1,\varepsilon}\, dx \\ =& \lim_{\varepsilon\to 0} \varepsilon^{p_j - p_0} \int_{\mathbb{R}^N} \frac{V_1(\varepsilon x + x_\varepsilon)}{|\varepsilon x + x_\varepsilon - x_j|^{p_j}} \Big|x + \frac{x_\varepsilon - x_j}{\varepsilon}\Big|^{p_j} m_{1,\varepsilon}\, dx \geq \Gamma. \end{aligned} \tag{4.36}$$
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Recall the definition of ε shown in (1.22) and the estimate of (1.19), then we find

$$\begin{aligned} \int_{\mathbb{R}^N} m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{2N}} m_{2,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}\, dx &= \int_{\mathbb{R}^N} m_{1,\varepsilon}^{1+\frac{\gamma'}{N}}\, dx + \int_{\mathbb{R}^N} \Big(m_{2,\varepsilon}^{\frac12+\frac{\gamma'}{2N}} - m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}\Big) m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}\, dx \\ &= \int_{\mathbb{R}^N} m_{1,\varepsilon}^{1+\frac{\gamma'}{N}}\, dx + o_\varepsilon(1). \end{aligned} \tag{4.37}$$
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Thus, one finds

$$\begin{aligned} &C_L \int_{\mathbb{R}^N} \Big|\frac{w_{1,\varepsilon}}{m_{1,\varepsilon}}\Big|^{\gamma'} m_{1,\varepsilon}\, dx - \frac{\alpha_1}{1+\frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_{1,\varepsilon}^{1+\frac{\gamma'}{N}}\, dx - \frac{2\beta}{1+\frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_{1,\varepsilon}^{\frac12+\frac{\gamma'}{2N}} m_{2,\varepsilon}^{\frac12+\frac{\gamma'}{2N}}\, dx \\ =& C_L \int_{\mathbb{R}^N} \Big|\frac{w_{1,\varepsilon}}{m_{1,\varepsilon}}\Big|^{\gamma'} m_{1,\varepsilon}\, dx - \frac{\alpha_1 + 2\beta}{1+\frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_{1,\varepsilon}^{1+\frac{\gamma'}{N}}\, dx + o_\varepsilon(1) \\ \geq& \Big(1 - \frac{\alpha_1 + 2\beta}{a^*}\Big) \int_{\mathbb{R}^N} \Big|\frac{w_{1,\varepsilon}}{m_{1,\varepsilon}}\Big|^{\gamma'} m_{1,\varepsilon}\, dx + o_\varepsilon(1). \end{aligned} \tag{4.38}$$
3p5notesb

In addition, in light of (1.21) and (1.22), one has

$$\int_{\mathbb{R}^N} \Big|\frac{w_{i,\varepsilon}}{m_{i,\varepsilon}}\Big|^{\gamma'} m_{i,\varepsilon}\, dx = 1 + o_\varepsilon(1),\ i = 1, 2, \tag{4.39}$$
alsohavef

and obtain from (4.38) that

$$\int_{\mathbb{R}^N} \Big|\frac{w_{2,\varepsilon}}{m_{2,\varepsilon}}\Big|^{\gamma'} m_{2,\varepsilon}\, dx - \frac{\alpha_2}{1+\frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_{2,\varepsilon}^{1+\frac{\gamma'}{N}}\, dx \geq \Big(1 - \frac{\alpha_2}{a^*}\Big) \int_{\mathbb{R}^N} \Big|\frac{w_{2,\varepsilon}}{m_{2,\varepsilon}}\Big|^{\gamma'} m_{2,\varepsilon}\, dx. \tag{4.40}$$
3p6notesb

Upon substituting (4.36), (4.37) and (4.40), (4.39), one finds from (4.32) that

$$\begin{aligned} \mathcal{E}(m_{1,\varepsilon}, w_{1,\varepsilon}, m_{2,\varepsilon}, w_{2,\varepsilon}) \geq& \varepsilon^{-\gamma'} \Big[1 - \frac{\alpha_1 + \alpha_2 + 2\beta}{a^*}\Big](1 + o(1)) + \Gamma \varepsilon^{-p_0} \\ \geq& (1 + o_\varepsilon(1)) \frac{p_0 + \gamma'}{p_0} \Big(\frac{p_0 \Gamma}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+p_0}} \Big(\frac{2}{a^*}\Big)^{\frac{\gamma'}{\gamma'+p_0}} \Big(a^* - \frac{\alpha_1 + \alpha_2 + 2\beta}{2}\Big)^{\frac{p_0}{\gamma'+p_0}}, \end{aligned}$$

which is contradicted to Lemma 4.1. This completes the proof of claim (4.35). Hence, we obtain $\exists y_0 \in \mathbb{R}^N$ such that

$$\lim_{\varepsilon\to 0}\frac{x_\varepsilon - x_j}{\varepsilon} = y_0.$$

We next show that y_0 satisfies (1.31). Since $p_i = p_0$, it follows from Theorem 1.4 that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\varepsilon^{-p_0}\int_{\mathbb{R}^N}\sum_{i=1}^{2}V_i(\varepsilon x + x_\varepsilon)m_{i,\varepsilon}(x)\,dx\\
=&\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\frac{\sum_{i=1}^{2}V_i\Big(\varepsilon(x+\frac{x-x_j}{\varepsilon})+x_j\Big)}{\big|\varepsilon(x+\frac{x-x_j}{\varepsilon})\big|^{p_0}}\Big|x+\frac{x-x_\varepsilon}{\varepsilon}\Big|^{p_0}m_{i,\varepsilon}\,dx\\
\geq&\mu_i\int_{\mathbb{R}^N}|x+y_0|^{p_0}m_0\,dx \geq \mu\bar{\nu}_{p_0},
\end{aligned} \tag{4.41}$$ 4100negat

where the last two inequalities hold if and only if one has (1.31). As a consequence, we deduce from (4.38) and (4.40) that

$$\begin{aligned}
e_{\alpha_1,\alpha_2,\beta} \geq&\varepsilon^{-\gamma'}\Big[2-\frac{\alpha_1+\alpha_2+2\beta}{a^*}\Big][1+o(1)]+\varepsilon^{p_0}\mu\nu[1+o(1)]\\
\geq&[1+o(1)]\Big[\frac{\gamma'+p_0}{p_0}\Big(\frac{\mu\bar{\nu}_{p_0}p_0}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+p_0}}\Big(\frac{2}{a^*}\Big)^{\frac{p_0}{\gamma'+p_0}}\Big(a^*-\frac{\alpha_1+\alpha_2+2\beta}{2}\Big)^{\frac{p_0}{\gamma'+p_0}}\Big],
\end{aligned} \tag{4.42}$$ 4101negat

where the equality in the second inequality holds if and only if

$$\varepsilon = \Big(\frac{\mu\bar{\nu}_{p_0}p_0a^*}{2\gamma'(a^*-\frac{\alpha_1+\alpha_2+2\beta}{2})}\Big)^{-\frac{1}{\gamma'+p_0}} = \Big(\frac{2\gamma'}{p_0\mu\bar{\nu}_{p_0}a^*}\Big)^{\frac{1}{\gamma'+p_0}}\Big(a^*-\frac{\alpha_1+\alpha_2+2\beta}{2}\Big)^{\frac{1}{\gamma'+p_0}}.$$

Combining the lower bound (4.42) with the upper bound (4.25), we find the equalities in (4.41) and (4.42) hold. As a consequence, we obtain (1.30) and (1.31) and finish the proof of this theorem. □

5 Asymptotic Profiles of Ground States with $\beta < 0$

⟨sect520240929⟩ In this section, we shall discuss the concentration phenomena within (1.2) under the repulsive case with $\beta < 0$. Similarly as shown in Section 4, we first investigate the basic blow-up profiles of ground states with some assumptions imposed on the potentials, which is summarized as Theorem 1.6. Then, we investigate the refined blow-up profiles shown in Theorem 1.7 when potentials satisfy local polynomial expansions.

Proof of Theorem 1.6:

Proof. As shown in the proof of Theorem 1.1, we have proved that when $\beta < 0$,

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*} e_{\alpha_1,\alpha_2,\beta} = 0. \tag{5.1}$$ C4negativ

In addition, one obtains from (1.12) that

$$\mathcal{E}^i_{\alpha_i}(m_i,w_i)\geq 0 \text{ if } \alpha_i < a^*,$$

where $\mathcal{E}^i_{\alpha_i}(m_i, w_i)$ are given by (3.2). Moreover, noting that $\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1, w_1, m_2, w_2)$ defined by (1.6) can be written as

$$\mathcal{E}_{\alpha_1,\alpha_2,\beta}(m_1, w_1, m_2, w_2) = \sum_{i=1}^{2} \mathcal{E}^i_{\alpha_i}(m_i, w_i) - \frac{2\beta N}{N+\gamma'} \int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx,$$

we find from (5.1) that (1.33), (1.34) and (1.35) hold.

Next, we shall prove (1.36) and argue by contradiction. Assume

$$\limsup_{\mathbf{a}\nearrow\mathbf{a}^*} \int_{\mathbb{R}^N} C_L \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx < +\infty,$$

then it follows from (1.12) that

$$\limsup_{\mathbf{a}\nearrow\mathbf{a}^*} \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx < +\infty.$$

Therefore, we deduce from (1.33) that

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*} \mathcal{E}^i_{\mathbf{a}^*}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) = \lim_{\mathbf{a}\nearrow\mathbf{a}^*} \mathcal{E}^i_{\alpha_i}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) = 0 = e^i_{a^*},\ i = 1, 2,$$

where $e^i_{\alpha_i}$ is defined by $e^i_{\alpha_i} = \inf_{(m,w)\in\mathcal{K}_i} \mathcal{E}^i_{\alpha_i}(m, w)$. Recall that $\{(m_{1,\mathbf{a}}, w_{1,\mathbf{a}})\}$ is a bounded minimizing sequence of $e^i_{a^*}$ given by (3.1) and its limit is a minimizer of $e^i_{a^*}$, i.e.

$$\lim_{\alpha_i\nearrow a^*} e^i_{\alpha_i} = e^i_{a^*}.$$

This is a contradiction to the fact that $e^i_{a^*}$ does not admit any minimizer as shown in [5]. Hence, one finds (1.36) holds.

Let

$$\hat{\varepsilon}_i := \left(C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx\right)^{-\frac{1}{\gamma'}} \to 0 \text{ as } \mathbf{a} \nearrow \mathbf{a}^*.$$

Recall that $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) \in \mathcal{K}$ is a minimizer and by using Lemma 3.2, one has for $i = 1, 2$,

$$\begin{aligned}
\lambda_{i,\mathbf{a}} =& C_L \int_{\mathbb{R}^N} \left|\frac{w_{i,\mathbf{a}}}{m_{i,\mathbf{a}}}\right|^{\gamma'} m_{i,\mathbf{a}}\, dx + \int_{\mathbb{R}^N} V_i m_{i,\mathbf{a}}\, dx - \alpha_i \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx - \beta \int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx\\
=& \mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}}, w_{i,\mathbf{a}}) - \frac{N\alpha_i}{N+\gamma'} \int_{\mathbb{R}^N} m_{i,\mathbf{a}}^{1+\frac{\gamma'}{N}}\, dx - \beta \int_{\mathbb{R}^N} m_{1,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\mathbf{a}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\, dx\\
=& -\frac{\gamma'}{N}\hat{\varepsilon}_i^{-\gamma'} + o_{\varepsilon_i}(1),
\end{aligned}$$

which implies

$$\lambda_{i,\mathbf{a}}\hat{\varepsilon}_i^{\gamma'} \to -\frac{\gamma'}{N} \text{ as } \hat{\varepsilon}_i \to 0^+,\ i = 1, 2. \tag{5.2}$$

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Since $(u_{1,\mathbf{a}}, u_{2,\mathbf{a}})$ is bounded from below, we have $u_{i,\mathbf{a}} \to +\infty$ as $|x| \to +\infty$. Thus, there exist $x_{i,\hat{\varepsilon}}$, $i = 1, 2$ such that

$$u_{i,\hat{\varepsilon}}(0) = u_{i,\mathbf{a}}(x_{i,\hat{\varepsilon}}) = \inf_{x\in\mathbb{R}^N} u_{i,\mathbf{a}}.$$

By using (1.37) and (3.12), we find $(m_{1,\hat\varepsilon}, u_{1,\hat\varepsilon}, m_{2,\hat\varepsilon}, u_{2,\hat\varepsilon})$ satisfies

$$\begin{cases} -\Delta u_{1,\hat\varepsilon} + C_H|\nabla u_{1,\hat\varepsilon}|^\gamma + \lambda_{1,\mathbf{a}}\hat\varepsilon_1^{\gamma'} = \hat\varepsilon_1^{\gamma'} V_1(\hat\varepsilon_1 x + x_{1,\hat\varepsilon}) - \alpha_1 m_{1,\hat\varepsilon}^{\frac{\gamma'}{N}} - \beta(\frac{\hat\varepsilon_1}{\hat\varepsilon_2})^{\frac{\gamma'}{2}+\frac{N}{2}} m_{1,\hat\varepsilon}^{\frac{\gamma'}{2N}-\frac12} m_{2,\hat\varepsilon}^{\frac{\gamma'}{2N}+\frac12}(\frac{\hat\varepsilon_1 x + x_{1,\hat\varepsilon} - x_{2,\hat\varepsilon}}{\hat\varepsilon_2}), \\ -\Delta m_{1,\hat\varepsilon} = C_H\gamma\nabla\cdot(m_{1,\hat\varepsilon}|\nabla u_{1,\hat\varepsilon}|^{\gamma-2}\nabla u_{1,\hat\varepsilon}) = -\nabla\cdot w_{1,\hat\varepsilon}, \\ -\Delta u_{2,\hat\varepsilon} + C_H|\nabla u_{2,\hat\varepsilon}|^\gamma + \lambda_{2,\mathbf{a}}\hat\varepsilon_2^{\gamma'} = \hat\varepsilon_2^{\gamma'} V_2(\hat\varepsilon_2 x + x_{2,\hat\varepsilon}) - \alpha_2 m_{2,\hat\varepsilon}^{\frac{\gamma'}{N}} - \beta(\frac{\hat\varepsilon_2}{\hat\varepsilon_1})^{\frac{\gamma'}{2}+\frac{N}{2}} m_{2,\hat\varepsilon}^{\frac{\gamma'}{2N}-\frac12} m_{1,\hat\varepsilon}^{\frac{\gamma'}{2N}+\frac12}(\frac{\hat\varepsilon_2 x + x_{2,\hat\varepsilon} - x_{1,\hat\varepsilon}}{\hat\varepsilon_1}), \\ -\Delta m_{2,\hat\varepsilon} = C_H\gamma\nabla\cdot(m_{2,\hat\varepsilon}|\nabla u_{2,\hat\varepsilon}|^{\gamma-2}\nabla u_{2,\hat\varepsilon}) = -\nabla\cdot w_{2,\hat\varepsilon}. \end{cases} \tag{5.3}$$

Then by applying the maximum principle on (5.3), one finds for $i, j = 1, 2$ and $i \neq j$ that

$$\lambda_{i,\mathbf{a}}\hat\varepsilon_i^{\gamma'} \geq -\alpha_i m_{i,\hat\varepsilon}^{\frac{\gamma'}{N}}(0) + \hat\varepsilon_i^{\gamma'} V_i(\hat\varepsilon_i x + x_{i,\hat\varepsilon}) - \beta\Big(\frac{\hat\varepsilon_i}{\hat\varepsilon_j}\Big)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{i,\hat\varepsilon}^{\frac{\gamma'}{2N}-\frac12}(0) m_{j,\hat\varepsilon}^{\frac{\gamma'}{2N}+\frac12}\Big(\frac{x_{i,\hat\varepsilon} - x_{j,\hat\varepsilon}}{\hat\varepsilon_j}\Big),$$

Noting that $\alpha_i > 0, \beta < 0$ and $V_i \geq 0$ with $i = 1, 2$, we further have when $\alpha_i \nearrow a^*$,

$$C \geq m_{i,\varepsilon}^{\frac{\gamma'}{N}}(0) > \frac{\gamma'}{2a^* N} > 0, \tag{5.4}$$

where $C > 0$ is a constant. Invoking (1.35) and (1.36), we obtain

$$\int_{\mathbb{R}^N} V_i(\hat\varepsilon_i x + x_{i,\hat\varepsilon}) m_{i,\hat\varepsilon}(x)\, dx \to 0 \text{ as } \mathbf{a} \nearrow \mathbf{a}^*, \tag{5.5}$$

and

$$\int_{\mathbb{R}^N} C_L\Big|\frac{w_{i,\hat\varepsilon}}{m_{i,\hat\varepsilon}}\Big|^{\gamma'} m_{i,\hat\varepsilon}\, dx = 1, \quad \int_{\mathbb{R}^N} m_{i,\hat\varepsilon}^{1+\frac{\gamma'}{N}}\, dx \to \frac{N+\gamma'}{Na^*}. \tag{5.6}$$

Now, we claim up to a subsequence,

$$x_{i,\hat\varepsilon} \to x_i \text{ with } V_i(x_i) = 0, \ i = 1, 2. \tag{5.7}$$

Indeed, we have from (5.6) and Lemma 2.4 that

$$\limsup_{\hat\varepsilon_1,\hat\varepsilon_2\to 0^+} \|m_{i,\hat\varepsilon}\|_{W^{1,\gamma'}(\mathbb{R}^N)} < +\infty. \tag{5.8}$$

Moreover, since $\gamma' > N$, one gets from Morrey's estimate that

$$\limsup_{\hat\varepsilon_1,\hat\varepsilon_2\to 0^+} \|m_{i,\hat\varepsilon}\|_{C^{0,\frac{N}{\gamma'}}(\mathbb{R}^N)} < +\infty. \tag{5.9}$$

(5.9) together with (5.4) gives us that there exists $R > 0$ such that

$$m_{i,\hat\varepsilon}(x) \geq \frac{C}{2} > 0, \ \forall |x| < R, \ i = 1, 2, \tag{5.10}$$

where $C > 0$ is a constant independent of $\hat\varepsilon_i$. As a consequence, we obtain claim (5.7) thanks to (5.5) and (5.10). In light of (1.32) and (5.7), one finds

$$\lim_{\hat\varepsilon_1,\hat\varepsilon_2\to 0^+} \frac{|x_{1,\hat\varepsilon} - x_{2,\hat\varepsilon}|}{\hat\varepsilon_i} = +\infty, \ i = 1, 2.$$

Next, we study the convergence of $(m_{1,\hat\varepsilon}, u_{1,\hat\varepsilon}, m_{2,\hat\varepsilon}, u_{2,\hat\varepsilon})$ as $\hat\varepsilon_i \to 0$ with $i = 1, 2$. First of all, we have from (5.8) and (5.10) that there exist $0 \not\equiv, \leq m_i \in W^{1,\gamma'}(\mathbb{R}^N)$ with $i = 1, 2$ such that

$$m_{i,\hat\varepsilon} \rightharpoonup m_i \text{ in } W^{1,\gamma'}(\mathbb{R}^N).$$

Without loss of the generality, we assume

$$\hat{\varepsilon}_1 \geq \hat{\varepsilon}_2. \tag{5.11}$$ assumenot

Since (5.9) and (5.11), one has

$$\beta\Big(\frac{\hat{\varepsilon}_2}{\hat{\varepsilon}_1}\Big)^{\frac{N}{2}+\frac{\gamma'}{2}} m_{1,\hat{\varepsilon}}^{\frac{1}{2}+\frac{\gamma'}{2N}}\Big(\frac{\hat{\varepsilon}_2 x + x_{2,\hat{\varepsilon}} - x_{1,\hat{\varepsilon}}}{\hat{\varepsilon}_1}\Big) m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{2N}-\frac{1}{2}}(x) \leq C,$$

where constant $C > 0$ is independent of $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$. In addition, by using Lemma 2.1, one obtains for any $x \in B_R(0)$,

$$|\nabla u_{2,\hat{\varepsilon}}(x)| \leq C_R, \tag{5.12}$$ 446notene

where $C_R > 0$ is a constant. Moreover, the $u_{2,\hat{\varepsilon}}$-equation in (5.3) becomes

$$-\Delta u_{2,\hat{\varepsilon}} = -C_H|\nabla u_{2,\hat{\varepsilon}}|^\gamma + g_{\hat{\varepsilon}}(x),$$

where $g_{\hat{\varepsilon}}(x)$ is given by (4.17) with ε replaced by $\hat{\varepsilon}$. We further find from (5.12) that $|-C_H|\nabla u_{2,\hat{\varepsilon}}|^\gamma + g_{\hat{\varepsilon}}| \leq \tilde{C}_R$ with $\tilde{C}_R > 0$. Then we apply the standard elliptic regularity to get $\|u_{2,\hat{\varepsilon}}\|_{C^{2,\theta}(B_R)} \leq C_R$, where $C_R > 0$ is a constant and $\theta \in (0,1)$. Thus, we take the limit in the $u_{2,\hat{\varepsilon}}$-equation and $m_{2,\hat{\varepsilon}}$-equation of (5.3), use the diagonalization procedure and Arzelà-Ascoli theorem to deduce that as $\hat{\varepsilon}_1, \hat{\varepsilon}_2 \to 0^+$,

$$u_{2,\hat{\varepsilon}} \to u_2 \text{ in } C^{2,\hat{\theta}}_{\rm loc}(\mathbb{R}^N)$$

with $\hat{\theta} \in (0,1)$, and (m_2, u_2) satisfies

$$\begin{cases} -\Delta u_2 + C_H|\nabla u_2|^\gamma - \frac{\gamma'}{N} = a^* m_2, \\ -\Delta m_2 = C_H\gamma\nabla\cdot(m_2|\nabla u_2|^{\gamma-2}\nabla u_2) = -\nabla\cdot w_2, \\ 0 < \int_{\mathbb{R}^N} m_2\,dx \leq 1. \end{cases}$$

Similar as the derivation of (4.19), one uses Lemma 2.6 to get $\int_{\mathbb{R}^N} m_2\,dx = 1$. It follows that $m_{2,\hat{\varepsilon}} \to m_2$ in $L^1(\mathbb{R}^N)$. Combining this with (5.9) , we deduce

$$m_{2,\hat{\varepsilon}} \to m_2 \text{ in } L^q(\mathbb{R}^N), \forall q \geq 1. \tag{5.13}$$ 4551qbeta

Invoking Lemma 2.2, (5.2) and (5.9), one has

$$u_{2,\hat{\varepsilon}}(x) \geq C\max\{|x|, (\varepsilon_2^{\gamma'} V_2(\hat{\varepsilon}_2 x + x_{2,\hat{\varepsilon}}))^{\frac{1}{\gamma}}\}, \text{ if } |x| > R, \tag{5.14}$$ 478202408

where $C > 0$ and $R > 0$ are constants independent of $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$. Indeed, it suffices to prove $u_{2,\hat{\varepsilon}}(x) \geq C|x|$ for some constant $C > 0$ when $|x| > R$. To this end, we find from (5.3) that when $\hat{\varepsilon}_i$, $i = 1, 2$ are small,

$$-\Delta u_{2,\hat{\varepsilon}} + C_H|\nabla u_{2,\hat{\varepsilon}}|^\gamma + \lambda_0 \geq \frac{\gamma'}{3N} - \alpha_2 m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{N}} - \beta\Big(\frac{\hat{\varepsilon}_2}{\hat{\varepsilon}_1}\Big)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{2N}-\frac{1}{2}} m_{1,\hat{\varepsilon}}^{\frac{\gamma'}{2N}+\frac{1}{2}}\Big(\frac{\hat{\varepsilon}_2 x + x_{2,\hat{\varepsilon}} - x_{1,\hat{\varepsilon}}}{\hat{\varepsilon}_1}\Big), \tag{5.15}$$ 20240811n

where $\lambda_0 := -\frac{\gamma'}{2N}$ and we have used (5.2) and the positivity of V_2. In addition, (5.11) and (5.13) indicate that as $|x| \to +\infty$,

$$-\alpha_2 m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{N}} - \beta\Big(\frac{\hat{\varepsilon}_2}{\hat{\varepsilon}_1}\Big)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{2N}-\frac{1}{2}} m_{1,\hat{\varepsilon}}^{\frac{\gamma'}{2N}+\frac{1}{2}}\Big(\frac{\hat{\varepsilon}_2 x + x_{2,\hat{\varepsilon}} - x_{1,\hat{\varepsilon}}}{\hat{\varepsilon}_1}\Big) \to 0 \text{ uniformly in } \hat{\varepsilon}_1 \text{ and } \hat{\varepsilon}_2. \tag{5.16}$$ 20240811n

Thus, one further obtains from (5.15) and (5.16) that

$$-\Delta u_{2,\hat{\varepsilon}} + C_H|\nabla u_{2,\hat{\varepsilon}}|^{\gamma} + \lambda_0 > 0 \text{ when } |x| \gg 1. \tag{5.17}$$ 481202408

Now, we fix any $|\tilde{x}|$ large enough and define

$$h(x) := K_1|\tilde{x}|\chi\Big(\frac{x}{|\tilde{x}|}\Big),$$

where constant $K_1 > 0$ will be chosen later and $\chi \geq 9$ denotes the smooth cut-off function satisfying $\chi \equiv 0$ when $x \in (0, \frac{1}{2}) \cup (\frac{3}{2}, +\infty)$. We compute to get

$$-\Delta h + C_H|\nabla h|^{\gamma} + \lambda_0 \leq \frac{K_1}{|\tilde{x}|} + C_H K_1^{\gamma} + \lambda_0 < 0, \tag{5.18}$$ 482202408

if we choose K_1 small enough. Applying the comparison principle into (5.17) and (5.18), one has

$$u_{2,\hat{\varepsilon}}(x) \geq h(x) \text{ for } \frac{1}{2}|\tilde{x}| < |x| < \frac{3}{2}|\tilde{x}|,$$

which finishes the proof of (5.14).

Next, we claim that for any $p > 1$, there exist $R > 0$ and $C > 0$ such that

$$m_{2,\hat{\varepsilon}}(x) \leq C|x|^{-p}, \ \forall |x| > R.$$

Indeed, let $\phi = u_{2,\hat{\varepsilon}}^{p}$, then we have

$$\begin{aligned}
&-\Delta\phi + C_H\gamma|\nabla u_{2,\hat{\varepsilon}}|^{\gamma-2}\nabla u_{2,\hat{\varepsilon}}\cdot\nabla\phi\\
=&pu_{2,\hat{\varepsilon}}^{p-1}[-\Delta u_{2,\hat{\varepsilon}} - (p-1)\frac{|\nabla u_{2,\hat{\varepsilon}}|^2}{u_{2,\hat{\varepsilon}}} + C_H\gamma|\nabla u_{2,\hat{\varepsilon}}|^{\gamma}]\\
=&pu_{2,\hat{\varepsilon}}^{p-1}\Big[C_H(\gamma-1)|\nabla u_{2,\hat{\varepsilon}}|^{\gamma} - \lambda_2\hat{\varepsilon}_2^{\gamma'} - (p-1)\frac{|\nabla u_{2,\hat{\varepsilon}}|^2}{u_{2,\hat{\varepsilon}}}\\
&+\hat{\varepsilon}_2^{\gamma'}V_2(\hat{\varepsilon}_2x + x_{2,\hat{\varepsilon}}) - \alpha_2 m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{N}} - \beta\Big(\frac{\hat{\varepsilon}_2}{\hat{\varepsilon}_1}\Big)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{2N}-\frac{1}{2}} m_{1,\hat{\varepsilon}}^{\frac{\gamma'}{2N}+\frac{1}{2}}\Big(\frac{\hat{\varepsilon}_2x + x_{2,\hat{\varepsilon}} - x_{1,\hat{\varepsilon}}}{\hat{\varepsilon}_1}\Big)\Big]\\
:=&pu_{2,\hat{\varepsilon}}^{p-1}G_{\hat{\varepsilon}}(x).
\end{aligned} \tag{5.19}$$ C34notesb

Lemma 2.1 implies

$$|\nabla u_{2,\hat{\varepsilon}}| \leq C[1 + \hat{\varepsilon}_2^{\gamma'}V_2(\hat{\varepsilon}_2x + x_{2,\hat{\varepsilon}})]^{\frac{1}{\gamma}}. \tag{5.20}$$ 456betane

Hence, we deduce from (5.14) that

$$\frac{|\nabla u_{2,\hat{\varepsilon}}|^{2-\gamma}}{u_{2,\hat{\varepsilon}}} \leq C\frac{\Big[1 + \hat{\varepsilon}_2^{\gamma'}V_2(\hat{\varepsilon}_2x + x_{2,\hat{\varepsilon}})^{\frac{2-\gamma}{\gamma}}\Big]}{\max\{|x|, [\hat{\varepsilon}_2^{\gamma'}V_2(\hat{\varepsilon}_2x + x_{2,\hat{\varepsilon}})]^{\frac{1}{\gamma}}\}} \leq \frac{C_H(\gamma-1)}{2(p-1)}, \text{ for } |x| > R.$$

Thus,

$$\begin{aligned}
&C_H(\gamma-1)|\nabla u_{2,\hat{\varepsilon}}|^{\gamma} - (p-1)\frac{|\nabla u_{2,\hat{\varepsilon}}|^2}{u_{2,\hat{\varepsilon}}}\\
=&|\nabla u_{2,\hat{\varepsilon}}|^{\gamma}\Big[C_H(\gamma-1) - (p-1)\frac{|\nabla u_{2,\hat{\varepsilon}}|^{2-\gamma}}{u_{2,\hat{\varepsilon}}}\Big] > 0 \text{ for } |x| > R.
\end{aligned}$$

In light of (5.19), we further find

$$-\Delta\phi + C_H\gamma|\nabla u_{2,\hat{\varepsilon}}|^{\gamma-2}\nabla u_{2,\hat{\varepsilon}}\cdot\nabla\phi \geq Cpu_{2,\hat{\varepsilon}}^{p-1}, \text{ for } |x|>R. \tag{5.21}$$ 459notesb

By using Theorem 3.1 in [11], one gets

$$\int_{\mathbb{R}^N} m_{2,\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx < +\infty.$$

Noting that $x_{2,\hat{\varepsilon}}$ is uniformly bounded, we have R is independent of $\hat{\varepsilon}_1$ in (5.21). Thus,

$$\limsup_{\hat{\varepsilon}_1,\hat{\varepsilon}_2\to 0^+}\int_{\mathbb{R}^N} m_{2,\varepsilon}u_{2,\hat{\varepsilon}}^{p-1}\,dx < +\infty. \tag{5.22}$$ 461betane

Indeed, we test the $m_{2,\hat{\varepsilon}}$-equation in (5.3) against ϕ and integrate it by parts to obtain

$$0=\int_{\mathbb{R}^N} m_{2,\hat{\varepsilon}}[-\Delta\phi + C_H\gamma|\nabla u_{2,\hat{\varepsilon}}|^{\gamma-2}\nabla u_{2,\hat{\varepsilon}}\cdot\nabla\phi]\,dx = p\int_{\mathbb{R}^N} m_{2,\hat{\varepsilon}}G_{\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx.$$

It follows that for some large $R_1>0$ independent of $\hat{\varepsilon}_i$, $i=1,2$,

$$\int_{\{x||x|>R_1\}} m_{2,\hat{\varepsilon}}G_{\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx = -\int_{\{x||x|\leq R_1\}} m_{2,\hat{\varepsilon}}G_{\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx. \tag{5.23}$$ collectin

On one hand, in light of (5.21), one has

$$\int_{\{x||x|>R_1\}} m_{2,\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx \leq C\int_{\{x||x|>R_1\}} m_{2,\hat{\varepsilon}}G_{\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx, \tag{5.24}$$ collectin

where $C>0$ is some constant independent of $\hat{\varepsilon}_1$. On the other hand, by fixing $\inf\limits_{x\in\mathbb{R}^N} u_{2,\hat{\varepsilon}}=1$ in (5.19), we get $G_{\hat{\varepsilon}}\geq -C$ for some constant $C>0$ independent of $\hat{\varepsilon}$. Combining this with (5.20), one has from the boundedness of $|x_{2,\hat{\varepsilon}}|$ that

$$\left|\int_{\{x||x|\leq R_1\}} m_{2,\hat{\varepsilon}}G_{\hat{\varepsilon}}u_{2,\hat{\varepsilon}}^{p-1}\,dx\right| \leq C\int_{\mathbb{R}^N} m_{2,\hat{\varepsilon}}\,dx \leq \tilde{C}, \tag{5.25}$$ collectin

where C and $\tilde{C}$ are positive constants independent of $\hat{\varepsilon}$. Collecting (5.23), (5.24) and (5.25), one finds (5.22) holds. Moreover, (5.22) indicates

$$m_{2,\hat{\varepsilon}} \leq C|x|^{1-p},\ \forall p>1,$$

where $C>0$ is a constant independent of $\hat{\varepsilon}_1$. As a consequence, for any fixed $x\in\mathbb{R}^N$, we have

$$\left|\frac{\hat{\varepsilon}_1 x + x_{1,\hat{\varepsilon}} - x_{2,\hat{\varepsilon}}}{\hat{\varepsilon}_2}\right| \geq \frac{\hat{\varepsilon}_1|x|}{\hat{\varepsilon}_2} + \frac{1}{2}\frac{|x_{1,\hat{\varepsilon}}-x_{2,\hat{\varepsilon}}|}{\hat{\varepsilon}_2} \geq \frac{C}{\hat{\varepsilon}_2},$$

where $C>0$ is a constant. It follows that

$$\left(\frac{\hat{\varepsilon}_1}{\hat{\varepsilon}_2}\right)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{2,\hat{\varepsilon}}^{\frac{\gamma'}{2N}+\frac{1}{2}}\left(\frac{\hat{\varepsilon}_1 x + x_{1,\hat{\varepsilon}} - x_{2,\hat{\varepsilon}}}{\hat{\varepsilon}_2}\right) \leq \left(\frac{\hat{\varepsilon}_1}{\hat{\varepsilon}_2}\right)^{\frac{\gamma'}{2}+\frac{N}{2}}\hat{\varepsilon}_2^p \leq \hat{\varepsilon}_1^{\frac{\gamma'}{2}-\frac{N}{2}} \text{ by choosing } p>\frac{\gamma'}{2}+\frac{N}{2}. \tag{5.26}$$ 464implie

We rewrite the $u_{1,\hat{\varepsilon}}$-equation in (5.3) as

$$\begin{aligned}&-\Delta u_{1,\hat{\varepsilon}} + C_H|\nabla u_{1,\hat{\varepsilon}}|^{\gamma} + \lambda_{1,\mathbf{a}}\varepsilon_1^{\gamma'}\\ =&\hat{\varepsilon}_1 V_1(\hat{\varepsilon}_1 x + x_{1,\hat{\varepsilon}}) - \alpha_1 m_1^{\frac{\gamma'}{N}} - \beta\left(\frac{\hat{\varepsilon}_1}{\hat{\varepsilon}_2}\right)^{\frac{\gamma'}{2}+\frac{N}{2}} m_{1,\hat{\varepsilon}}m_{2,\hat{\varepsilon}}\left(\frac{\hat{\varepsilon}_1 x + x_{1,\hat{\varepsilon}} - x_{2,\hat{\varepsilon}}}{\hat{\varepsilon}_2}\right) := IV_{\hat{\varepsilon}}.\end{aligned} \tag{5.27}$$ takelimit

Since (5.26) indicates for any $\hat{R} > 0$,

$$|IV_{\hat{\varepsilon}}| \leq C_{\hat{R}}, \text{ for } |x| < \hat{R},$$

we have from Lemma 2.1 that

$$|\nabla u_{1,\hat{\varepsilon}}| \leq C_{\hat{R}}, \text{ for } |x| < \hat{R}.$$

Thus, we find by the standard diagonal procedure that

$$u_{1,\varepsilon} \to u_1 \text{ in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \text{ with } \alpha \in (0,1),$$

then take the limit in (5.27) to obtain u_1 satisfies

$$\begin{cases} -\Delta u_1 + C_H |\nabla u_1|^\gamma - \frac{\gamma'}{N} = a^* m_1^{\frac{\gamma'}{N}}, \\ -\Delta m_1 = C_H \gamma \nabla \cdot (m_1 |\nabla u_1|^{\gamma-2} \nabla u_1), \\ 0 < \int_{\mathbb{R}^N} m_1 \, dx \leq 1. \end{cases}$$

Similarly, Lemma 2.6 implies

$$\int_{\mathbb{R}^N} m_1 \, dx = 1.$$

We further deduce from (5.8) that

$$m_{1,\hat{\varepsilon}} \to m_1 \text{ in } L^p(\mathbb{R}^N), \ \forall p \geq 1,$$

which finishes the proof of this theorem. □

Next, we focus on the refined blow-up rate of minimizers under the case $\beta \leq 0$ and proceed to complete the proof of Theorem 1.7. Before proving Theorem 1.7, we collect the results of the existence of minimizers to (3.1) and the corresponding asymptotic profiles as follows

2notesbetacopy⟩ **Lemma 5.1.** *Define K_i and $\mathcal{E}^i_{\alpha_i}(m,w)$, $i = 1,2$ as (1.7). Then we have problem (3.1) admit minimizers $(m_i, w_i, u_i) \in W^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$, $i = 1,2$ with $p > 1$. Moreover, $w_i = -C_H \gamma m_i |\nabla u_i|^{\gamma-2} \nabla u_i$ and the following conclusions hold for $i = 1,2$:*

(i). $\epsilon_i := \left(C_L \int_{\mathbb{R}^N} \left|\frac{w_i}{m_i}\right|^{\gamma'} m_i \, dx\right)^{-\frac{1}{\gamma'}} \to 0$ as $\alpha_i \nearrow a^*$;

(ii). Let x_{i,ϵ_i} be a global minimum point of u_i, then

$$u_{i,\epsilon} := \epsilon_i^{\frac{2-\gamma}{\gamma-1}} u_i(\epsilon_i x + x_{i,\epsilon_i}), \ m_{i,\epsilon} := \epsilon_i^N m_i(\epsilon_i x + x_{i,\epsilon_i}), \ w_{i,\epsilon} := \epsilon_i^{N+1} w(\epsilon_i x + x_{i,\epsilon_i}) \tag{5.28}$$ 5p6notebe

satisfies up to a subsequence,

$$u_{i,\epsilon_i} \to \bar{u}_i \text{ in } C^2_{loc}(\mathbb{R}^N), \ m_{i,\epsilon_i} \to \bar{m}_i \text{ in } L^p(\mathbb{R}^N), \forall p \in [1,+\infty], \ w_{i,\epsilon_i} \to \bar{w}_i \text{ in } L^{\gamma'}(\mathbb{R}^N), \tag{5.29}$$ 5p7betano

where $(\bar{m}_i, \bar{w}_i)$ is a minimizer of (1.10) and $(\bar{m}_i, \bar{u}_i)$ satisfies (1.11);

(iii). if V_i satisfies (1.39) and set

$$\nu_{p_i} := \inf_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} |x+y|^{p_i} \bar{m}_i \, dx, \tag{5.30}$$
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then $\nu_{p_i} = \bar{\nu}_{p_i}$ with $\bar{\nu}_{p_i}$ given in (1.16) and

$$e^i_{\alpha_i} := (1+o(1))\frac{p_i+\gamma'}{p_i}\Big(\frac{p_i\bar{\nu}_{p_i}b_i}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+1}}\Big(\frac{a^*-\alpha_i}{a^*}\Big)^{\frac{p_i}{\gamma'+p_i}}, \ \epsilon_i = (1+o(1))\Big(\frac{\gamma'(a^*-\alpha_i)}{a^* b_i \bar{\nu}_{p_i} p_i}\Big)^{\frac{1}{\gamma'+p_i}}. \tag{5.31}$$
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Moreover, we have

$$\frac{x_{i,\epsilon_i}-x_i}{\epsilon_i} \to y_i,$$

where $y_i \in \mathbb{R}^N$ satisfies

$$H_{\bar{m}_i,p_i}(y_i) = \inf_{y\in\mathbb{R}^N} H_{\bar{m}_i,p_i}(y) = \bar{\nu}_{p_i}.$$

In particular, there exist $R>0$, $C>0$ and $\kappa_1, \delta_0 > 0$ small such that

$$0 < m_{i,\epsilon_i} \le Ce^{-\frac{\kappa_1}{2}|x|^{\delta_0}} \text{ when } |x|>R, \tag{5.32}$$
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Proof. Proceeding the similar arguments shown in [5], we are able to show Conclusion (i), (ii) and (iii) with $\bar{\nu}_{p_i}$ replaced by ν_{p_i}. (5.32) follows directly from Proposition A.1 shown in Appendix A. It is left to show $\bar{\nu}_{p_i} = \nu_{p_i}$ with ν_{p_i} defined by (5.30). First of all, it is straightforward to see that $\bar{\nu}_{p_i} \le \nu_{p_i}$. Then, we argue by contradiction and assume

$$\bar{\nu}_{p_i} < \nu_{p_i}, \ i = 1 \text{ or } 2.$$

In light of the definition of $\bar{\nu}_{p_i}$ given in (1.16), we find that there exists $(m,w,u)\in\mathcal{M}$ with $\mathcal{M}$ defined by (1.17) and $y_i\in\mathbb{R}^N$ such that

$$\nu_{i0} := \inf_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N}|x+y|^{p_i}m(x)\,dx = \int_{\mathbb{R}^N}|x+y_i|^{p_i}m(x)\,dx < \nu_{p_i}. \tag{5.33}$$
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Let

$$m_\tau := \tau^N m(\tau(x-x_i)-y_i), \ w_\tau := \tau^{N+1}w(\tau(x-x_i)-y_i),$$

where τ is defined by (4.31) and y_i is the minimum point of ν_{i0} given in (5.33). Then one can obtain

$$e^i_{\alpha_i} \le (1+o(1))\frac{p_i+\gamma'}{p_i}\Big(\frac{p_i\nu_{i0}b_i}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+1}}\Big(\frac{a^*-\alpha_i}{a^*}\Big)^{\frac{p_i}{\gamma'+p_i}}. \tag{5.34}$$
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Whereas, (5.31) gives that

$$e^i_{\alpha_i} = (1+o(1))\frac{p_i+\gamma'}{p_i}\Big(\frac{p_i\nu_{p_i}b_i}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+1}}\Big(\frac{a^*-\alpha_i}{a^*}\Big)^{\frac{p_i}{\gamma'+p_i}},$$

which reaches a contradiction to (5.34). □

Now, we establish the lower and upper bounds of $\mathcal{E}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ in the following lemma, where $(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}})$ denotes the minimizer of (1.5).

⟨emma4320240809⟩ **Lemma 5.2.** *Assume that each V_i satisfies (1.39) with $x_1 \neq x_2$ and (1.40) holds. Let $(m_{1,\boldsymbol{a}}, w_{1,\boldsymbol{a}}, m_{2,\boldsymbol{a}}, w_{2,\boldsymbol{a}})$ be a minimizer of $e_{\alpha_1,\alpha_2,\beta}$ defined by (1.5) with $\beta < 0$. Then for any $q > \max\{p_1, p_2\}$, we have there exists $C_q > 0$ such that*

$$\begin{aligned} e^1_{\alpha_1} + e^2_{\alpha_2} \leq & e_{\alpha_1,\alpha_2,\beta} = \mathcal{E}(m_{1,\boldsymbol{a}}, w_{1,\boldsymbol{a}}, m_{2,\boldsymbol{a}}, w_{2,\boldsymbol{a}}) \\ \leq & e^1_{\alpha_1} + e^2_{\alpha_2} + C_q \tilde{\epsilon}_2^q, \ \text{as } (\alpha_1, \alpha_2) \nearrow (a^*, a^*), \end{aligned} \tag{5.35}$$ 513notebe

where $e^i_{\alpha_i}$, $i = 1, 2$ are defined by (3.1). In particular, the following estimates hold:

$$e^i_{\alpha_i} \leq \mathcal{E}^i_{\alpha_i}(m_{i,\boldsymbol{a}}.w_{i,\boldsymbol{a}}) \leq e^i_{\alpha_i} + C_q \tilde{\epsilon}_2^q, \ i = 1, 2. \tag{5.36}$$ 514notebe

Proof. Noting that $\beta < 0$, we deduce from (3.1) and (3.2) that

$$\mathcal{E}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}, m_{2,\mathbf{a}}, w_{2,\mathbf{a}}) \geq \sum_{i=1}^{2} \mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}}, w_{i,\mathbf{a}}) \geq e^1_{\alpha_1} + e^2_{\alpha_2}. \tag{5.37}$$ 515notebe

Moreover, let (m_i, w_i) be the minimizers of $e^i_{\alpha_i}$, $i = 1, 2$ obtained in Lemma 5.1, then one has

$$\begin{aligned} e_{\alpha_1,\alpha_2,\beta} \leq \mathcal{E}(m_1, w_1, m_2, w_2) = & \sum_{i=1}^{2} \mathcal{E}^i_{\alpha_i}(m_i, w_i) - \frac{2\beta}{1 + \frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_1^{\frac12 + \frac{\gamma'}{2N}} m_2^{\frac12 + \frac{\gamma'}{2N}} \, dx \\ = & e^1_{\alpha_1} + e^2_{\alpha_2} - \frac{2\beta}{1 + \frac{\gamma'}{N}} \int_{\mathbb{R}^N} m_1^{\frac12 + \frac{\gamma'}{2N}} m_2^{\frac12 + \frac{\gamma'}{2N}} \, dx. \end{aligned} \tag{5.38}$$ 516noteco

By using (5.28), one finds

$$\int_{\mathbb{R}^N} m_1^{\frac12 + \frac{\gamma'}{2N}} m_2^{\frac12 + \frac{\gamma'}{2N}} \, dx = (\epsilon_1 \epsilon_2)^{-N(\frac12 + \frac{\gamma'}{2N})} \epsilon_1^N \int_{\mathbb{R}^N} m_{1,\epsilon}^{\frac12 + \frac{\gamma'}{2N}}(x) m_{2,\epsilon}^{\frac12 + \frac{\gamma'}{2N}} \Big(\frac{\epsilon_1}{\epsilon_2} x + \frac{x_{1,\epsilon_1} - x_{2,\epsilon_2}}{\epsilon_2} \Big) dx. \tag{5.39}$$ 517notebe

Since $x_{i,\epsilon_i} \to x_i$, $i = 1, 2$ and $x_1 \neq x_2$, we take $R := \frac14 |x_1 - x_2|$ and obtain for any $x \in B_{R/\epsilon_1}(0)$,

$$\begin{aligned} \Big| \frac{x_{1,\epsilon_1} - x_{2,\epsilon_2}}{\epsilon_2} + \frac{\epsilon_1}{\epsilon_2} x \Big| \geq \frac{|x_{1,\epsilon_1} - x_{2,\epsilon_2}|}{\epsilon_2} - \frac{\epsilon_1}{\epsilon_2}|x| \geq & \frac34 \frac{|x_1 - x_2|}{\epsilon_2} - \frac{R}{\epsilon_2} \\ = & \frac12 \frac{|x_1 - x_2|}{\epsilon_2} = O\Big(\frac{1}{\epsilon_2}\Big) \to +\infty. \end{aligned}$$

It then follows from (5.32) that there exists constant $C > 0$ such that

$$m_{2,\epsilon}\Big(\frac{\epsilon_1}{\epsilon_2} x + \frac{x_{1,\epsilon_1} - x_{2,\epsilon_2}}{\epsilon_2} \Big) \leq C \Big| \frac{x_1 - x_2}{\epsilon_2} \Big|^{-\hat{q}}, \ \forall x \in B_{R/\epsilon_1}(0),$$

which implies

$$\int_{|x| < \frac{R}{\epsilon_1}} m_{1,\epsilon}^{\frac12 + \frac{\gamma'}{2N}}(x) m_{2,\epsilon}^{\frac12 + \frac{\gamma'}{2N}} \Big(\frac{\epsilon_1}{\epsilon_2} x + \frac{x_{1,\epsilon_1} - x_{2,\epsilon_2}}{\epsilon_2} \Big) dx \leq C \epsilon_2^{\hat{q}(\frac12 + \frac{\gamma'}{2N})} \int_{\mathbb{R}^N} m_{1,\epsilon}^{\frac12 + \frac{\gamma'}{2N}} \, dx \leq C \epsilon_2^{\hat{q}(\frac12 + \frac{\gamma'}{2N})}, \tag{5.40}$$ 518betano

where we have used (5.29). In addition, invoking (5.32), we obtain $m_{1,\epsilon}$ satisfies for any $\hat{q} > 0$,

$$m_{1,\epsilon}(x) \leq C_\epsilon |x|^{-\hat{q}} \text{ for } |x| > R, \tag{5.41}$$ by519notel

where $C_\epsilon > 0$ is some constant. On the other hand, (5.29) indicates that

$$\limsup_{\epsilon_2 \to 0^+} \|m_{2,\epsilon}\|_{L^\infty} < +\infty. \tag{5.42}$$
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Combining (5.41) with (5.42), we deduce that

$$\begin{aligned}\int_{|x|>\frac{R}{\epsilon_1}} m_{1,\epsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}} m_{2,\epsilon}^{\frac{1}{2}+\frac{\gamma'}{2N}}\Big(\frac{\epsilon_1}{\epsilon_2}x + \frac{x_{1,\epsilon_1}-x_{2,\epsilon_2}}{\epsilon_2}\Big)dx &\le C\int_{|x|>\frac{R}{\epsilon_1}} m_{1,\epsilon_1}^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx \\ &\le C_{\hat q}\int_{\frac{R}{\epsilon_1}}^{+\infty} r^{-\hat q\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)} r^{N-1}\,dr \le C_{\hat q}\epsilon_1^{\hat q\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)-N},\end{aligned} \tag{5.43}$$
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where we have used (5.41). Upon collecting (5.39), (5.40) and (5.43), we deduce that

$$\int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx \le C_{\hat q}(\epsilon_1\epsilon_2)^{-N(\frac{1}{2}+\frac{\gamma'}{2N})}\epsilon_1^N\Big(\epsilon_2^{\hat q(\frac{1}{2}+\frac{\gamma'}{2N})} + \epsilon_1^{\hat q(\frac{1}{2}+\frac{\gamma'}{2N})-N}\Big). \tag{5.44}$$
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Since (1.40) and (5.31) imply up to a subsequence,

$$\lim_{\alpha_i \nearrow a^*} \frac{\epsilon_i}{\tilde\epsilon_i} = C_i,\ \ i = 1,2, C_i > 0 \text{ are constants},$$

we have from (5.44) that

$$\begin{aligned}\int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx \le& C_{\hat q}\tilde\epsilon_1^{-N(\frac{\gamma'}{2N}-\frac{1}{2})}\tilde\epsilon_2^{(\hat q-N)\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)} + C_{\hat q}\tilde\epsilon_1^{(\hat q-N)\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)}\tilde\epsilon_2^{-N\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)} \\ =& C_{\hat q}\tilde\epsilon_2^{(\hat q-N)\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)-sN\left(\frac{\gamma'}{2N}-\frac{1}{2}\right)} + \tilde\epsilon_2^{s(\hat q-N)\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)-N\left(\frac{1}{2}+\frac{\gamma'}{2N}\right)}.\end{aligned} \tag{5.45}$$
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By choosing $\hat q > 0$ large enough, one finds for any $q > \max\{p_1, p_2\}$,

$$(\hat q - N)\Big(\frac{1}{2}+\frac{\gamma'}{2N}\Big) - sN\Big(\frac{\gamma'}{2N}-\frac{1}{2}\Big) > q,\ \ s(\hat q - N)\Big(\frac{1}{2}+\frac{\gamma'}{2N}\Big) - N\Big(\frac{1}{2}+\frac{\gamma'}{2N}\Big) > q.$$

Thus, we obtain from (5.45) that

$$\int_{\mathbb{R}^N} m_1^{\frac{1}{2}+\frac{\gamma'}{2N}} m_2^{\frac{1}{2}+\frac{\gamma'}{2N}}\,dx \le C_q\tilde\epsilon_2^q,\ \ \forall q > \max\{p_1, p_2\}, \tag{5.46}$$
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where $C_q > 0$ is a constant. Finally, (5.46) together with (5.37) and (5.38) implies (5.35).

We next show estimate (5.36). First of all, it is straightforward to obtain from the definitions of $e^i_{\alpha_i}$ that

$$e^i_{\alpha_i} \le \mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}}, w_{i,\mathbf{a}}),\ \ i = 1,2. \tag{5.47}$$
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Then, we argue by contradiction to establish the estimate shown in the right hand side of (5.36). Without loss of generality, we assume for $i = 1$,

$$\mathcal{E}^1_{\alpha_1}(m_{1,\mathbf{a}}, w_{1,\mathbf{a}}) \ge e^1_{\alpha_i} + \Gamma\tilde\epsilon_2^q,$$

where $\Gamma > 0$ is large enough and $\tilde\epsilon_2^q \ll \min\{e^1_{\alpha_1}, e^2_{\alpha_2}\}$ thanks to (5.31). Whereas, by using (5.47), one has

$$e^1_{\alpha_1} + e^2_{\alpha_2} + \Gamma\tilde\epsilon_2^q \le \sum_{i=1}^{2}\mathcal{E}_{\alpha_i}(m_{i,\mathbf{a}}, w_{i,\mathbf{a}}) \le e_{\alpha_1,\alpha_2,\beta},$$

which is contradicted to (5.35). Therefore, we find (5.36) holds for $i = 1$. Proceeding the similar argument, we can show (5.36) holds for $i = 2$.

□

Remark 5.1. *We remark that by using the exponential decay properties of m shown in Proposition A.1, the conclusion in Lemma 5.2 holds when (1.40) is replaced by the following condition*

$$\lim_{\mathbf{a}\nearrow\mathbf{a}^*}\frac{e^{-\tilde{\epsilon}_1^{\hat{\delta}}}}{\tilde{\epsilon}_2^{p_2}}=0, \tag{5.48}$$

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where p_2 *is given in* (1.39) *and constant* $\hat{\delta}>0$ *depends on* δ_0 *and* κ_1*, which are defined in Proposition A.1. Moreover, with the aid of* (5.48)*, one can show all conclusions of Theorem 1.7. In other words, assumption* (1.40) *can be relaxed as* (5.48) *if the exponential decay properties of* m_1 *and* m_2 *are established.*

Now, we are ready to prove Theorem 1.7, which is
Proof of Theorem 1.7:

Proof. First of all, we have the fact that

$$\mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}},w_{i,\mathbf{a}})\geq\hat{\varepsilon}_i^{\gamma'}\Big(1-\frac{\alpha_i}{a^*}\Big)+\int_{\mathbb{R}^N}V_i(\hat{\varepsilon}_ix+x_{i,\hat{\varepsilon}})m_{i,\hat{\varepsilon}}\,dx.$$

We compute to get

$$\begin{aligned}\int_{\mathbb{R}^N}V_i(\hat{\varepsilon}_ix+x_{i,\hat{\varepsilon}})m_{i,\hat{\varepsilon}}\,dx&=\hat{\varepsilon}_i^{p_i}\int_{\mathbb{R}^N}\frac{V_i(\hat{\varepsilon}_ix+x_{i,\hat{\varepsilon}})}{|\hat{\varepsilon}_ix+x_{i,\hat{\varepsilon}}-x_i|^{p_i}}\Big|x+\frac{x_{i,\hat{\varepsilon}}-x_i}{\hat{\varepsilon}_i}\Big|m_{i,\hat{\varepsilon}}\,dx\\&:=\hat{\varepsilon}_i^{p_i}I_{\hat{\varepsilon}}.\end{aligned} \tag{5.49}$$

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By using (3.1), (5.36) and (5.31), we proceed the similar argument shown in Theorem 1.5, then obtain up to a subsequence,

$$\frac{x_{i,\hat{\varepsilon}}-x_i}{\hat{\varepsilon}_i}\to y_{i0}\text{ for some }y_{i0}\in\mathbb{R}^N.$$

Hence, one has $I_{\hat{\varepsilon}}$ defined in (5.49) satisfies

$$\lim_{\hat{\varepsilon}\to0}I_{\hat{\varepsilon}}\geq b_i\int_{\mathbb{R}^N}|x+y_{i0}|^{p_i}m_i(x)\,dx\geq\bar{\nu}_{p_i}b_i.$$

It then follows that

$$\begin{aligned}\mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}},w_{i,\mathbf{a}})\geq&\hat{\varepsilon}_i^{\gamma'}\frac{a^*-\alpha_i}{a^*}+\hat{\varepsilon}_i^{p_i}b_i\bar{\nu}_{p_i}(1+o(1))\\\geq&(1+o(1))\frac{p_i+\gamma'}{p_i}\Big(\frac{p_i\bar{\nu}_{p_i}b_i}{\gamma'}\Big)^{\frac{\gamma'}{\gamma'+1}}\Big(\frac{a^*-\alpha_i}{a^*}\Big)^{\frac{p_i}{\gamma'+p_i}},\end{aligned}$$

and the equality holds if and only if

$$\hat{\varepsilon}_i^{\gamma'}=(1+o(1))\Big(\frac{\gamma'(a^*-\alpha_i)}{a^*b_i\bar{\nu}_{p_i}p_i}\Big)^{\frac{1}{\gamma'+p_i}}.$$

Comparing the lower bound and the upper bound of $\mathcal{E}^i_{\alpha_i}(m_{i,\mathbf{a}},w_{i,\mathbf{a}})$ with $i=1,2$ shown in Lemma 5.2, we finish the proof of this theorem. □

Theorem 1.7 exhibits the refined blow-up profiles of ground states when interaction coefficient $\beta<0$ under some technical assumptions (1.39) and (1.40). It is worthy mentioning that with the aid of Proposition A.1, we are able to improve the condition (1.40) such that the conclusion shown in Theorem 1.7 still holds.

6 Conclusions

In this paper, we have studied the stationary multi-population Mean-field Games system (1.2) with decreasing cost self-couplings and interactive couplings under critical mass exponents via variational methods. Concerning the existence of ground states, we classified the existence of minimizers to constraint minimization problem (1.5) in terms of self-focusing coefficients and interaction coefficients, in which the attractive and repulsive interactions were discussed, respectively. In particular, when all coefficients are subcritical, we showed the existence of ground states to (1.2) by the duality argument. Then, the basic and refined blow-up profiles of ground states were studied under some mild assumptions of potential functions V_i, $i = 1, 2$.

We would like to mention that there are also some open problems deserve explorations in the future. In this paper, we focus on the existence and asymptotic profiles of ground states to (1.2) with mass critical local couplings under the case of $\gamma < N'$ with γ given in (1.3) since population density m can be shown in some Hölder space by using Morrey's estimate and system (1.2) enjoys the better regularity. Whereas, if $\gamma \geq N'$, nonlinear terms (1.4) in (1.2) become singular and one can only show $m \in L^p(\mathbb{R}^N)$ for some $p > 1$ by standard Sobolev embedding. Correspondingly, the positivities of m_1 and m_2 given in (1.2) can not be shown due to the worse regularities. Hence, when $\gamma \geq N'$, it seems a challenge but interesting to prove the existence of ground states even under the mass subcritical local couplings. On the other hand, while discussing the concentration phenomena in (1.2), we impose some assumptions on potential functions V_i, $i = 1, 2$. In detail, when the interaction coefficient β satisfies $\beta > 0$, (1.14) is assumed for the convenience of analysis. However, when V_1 and V_2 satisfy $\inf_{x\in\mathbb{R}^N}(V_1(x) + V_2(x)) > 0$, the classification of the existence of minimizers is more intriguing and the corresponding blow-up profiles analysis might be more complicated. Similarly, if the interaction is repulsive, the investigation of the concentration property of global minimizers is also challenging when V_1 and V_2 have common global minima.

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Appendix A Exponential Decay Estimates of Population Densities

⟨appendixA⟩ In this appendix, we investigate the exponential decay property of population density m. More precisely, we consider the following system:

$$\begin{cases} -\Delta u_\varepsilon + C_H|\nabla u_\varepsilon|^\gamma + \lambda_\varepsilon = \varepsilon^\gamma V(\varepsilon x + x_\varepsilon) + g_\varepsilon(x), & x \in \mathbb{R}^N, \\ -\Delta m_\varepsilon + C_H\gamma\nabla\cdot(m_\varepsilon|\nabla u_\varepsilon|^{\gamma-2}\nabla u_\varepsilon) = 0, & x \in \mathbb{R}^N, \end{cases}$$

where $\gamma > 1$, V and g_ε are given. Under some assumptions of g_ε and λ_ε, one can show m_ε satisfies the exponential decay property, which is

⟨appenexp⟩ **Proposition A.1.** *Denote $(m_\varepsilon, u_\varepsilon, \lambda_\varepsilon) \in W^{1,p}(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \times \mathbb{R}$ as the solution to*

$$\begin{cases} -\Delta u_\varepsilon + C_H|\nabla u_\varepsilon|^\gamma + \lambda_\varepsilon = \varepsilon^\gamma V(\varepsilon x + x_\varepsilon) + g_\varepsilon(x), & x \in \mathbb{R}^N, \\ -\Delta m_\varepsilon + C_H\gamma\nabla\cdot(m_\varepsilon|\nabla u_\varepsilon|^{\gamma-2}\nabla u_\varepsilon) = 0, & x \in \mathbb{R}^N, \end{cases}$$

where $m_\varepsilon > 0$ in $\mathbb{R}^N$, u_ε is uniformly bounded from below and Hölder continuous function V satisfies (1.8) and (1.9); moreover, g_ε is assumed to satisfy $g_\varepsilon \in C^{0,\theta}(\mathbb{R}^N)$ with $\theta \in (0,1)$ independent of ε. Suppose that

(i). $\lambda_\varepsilon \to \lambda_0$ up to a subsequence with $\lambda_0 < 0$;

(ii). $g_\varepsilon(x) \to 0$ uniformly as $|x| \to +\infty$,

then we have there exist constants $C > 0$ and $R > 0$ independent of ε such that

$$0 < m_\varepsilon \leq Ce^{-\frac{\kappa_1}{2}|x|^{\delta_0}} \text{ when } |x| > R, \tag{A.1}$$
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where constant $\delta_0 \in (0, \min\{\gamma - 1, 1\})$, constant $\kappa_1 > 0$ and they are independent of ε.

Proof. By following the same argument shown in the proof of Theorem 1.6, one has

$$u_\varepsilon(x) \geq C|x| \text{ for } |x| > \hat{R}, \tag{A.2}$$
appendixA

where $\hat{R} > 0$ is some constant. Then we define the Lyapunov function $\Phi = e^{\kappa u_\varepsilon^{\delta_0}}$ with $0 < \kappa < 1$ and $0 < \delta_0 < 1$ will be determined later. We compute to get

$$\begin{aligned}
&-\Delta\Phi + C_H\gamma|\nabla u_\varepsilon|^{\gamma-2}\nabla u_\varepsilon \cdot \nabla\Phi \\
=&\kappa\delta_0\Phi u_\varepsilon^{\delta_0-1}[-\Delta u_\varepsilon - (\kappa\delta_0 u_\varepsilon^{\delta_0-1} + (\delta_0 - 1)u_\varepsilon^{-1})|\nabla u_\varepsilon|^2 + C_H\gamma|\nabla u_\varepsilon|^\gamma] \\
=&\kappa\delta_0\Phi u_\varepsilon^{\delta_0-1}[C_H(\gamma-1)|\nabla u_\varepsilon|^\gamma - \lambda_\varepsilon + \varepsilon^\gamma V(\varepsilon x + x_\varepsilon) + g_\varepsilon(x) - (\kappa\delta_0 u_\varepsilon^{\delta_0-1} + (\delta_0 - 1)u_\varepsilon^{-1})|\nabla u_\varepsilon|^2].
\end{aligned}$$

Without loss of generality, we assume $u_\varepsilon \geq 1$ by fixing $u_\varepsilon(0)$. Then it is straightforward to show that

$$(\kappa\delta_0 u_\varepsilon^{\delta_0-1} + (\delta_0 - 1)u_\varepsilon^{-1})|\nabla u_\varepsilon|^2 \leq 2\kappa\delta_0 u_\varepsilon^{\delta_0-1}|\nabla u_\varepsilon|^2, \ |x| > R,$$

where $R > 0$ is a large constant and we have used $u^{\alpha-1} \geq u^{-1}$. In addition, by using Lemma 2.1 and Lemma 2.2, we have facts that

$$|\nabla u|^{2-\gamma} \leq C(1 + \varepsilon^\gamma V)^{\frac{2-\gamma}{\gamma}}, \text{ and } u^{1-\delta_0} \geq C(1 + \varepsilon^\gamma V)^{\frac{1-\delta_0}{\gamma}} \text{ for } |x| > R,$$

where $C > 0$ is a constant and $0 < \delta_0 < 1$.

Next, we would like to prove there exists $R > 0$ independent of ε such that

$$\frac{C_H(\gamma-1)}{2}|\nabla u_\varepsilon|^\gamma \geq 2\kappa\delta_1 u_\varepsilon^{\delta_0-1}|\nabla u_\varepsilon|^2, \ \forall |x| > R. \tag{A.3}$$
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When $\gamma \geq 2$, it is easy to show (A.3) holds by choosing κ small enough. When $1 < \gamma < 2$, by taking δ_0 and κ such that $2 - \gamma \leq 1 - \delta_0$ and κ small, one finds (A.3) holds. In summary, upon choosing $\delta_0 \in (0, \gamma - 1)$ and κ small enough, we apply Condition (i) and (ii) to get

$$-\Delta\Phi + C_H\gamma|\nabla u_\varepsilon|^{\gamma-2}\nabla u_\varepsilon \cdot \nabla\Phi \geq C\kappa\delta_1 u_\varepsilon^{\delta_0-1}\Phi, \text{ if } |x| > R,$$

where $\delta_1 > 0$ is some constant. Proceeding the similar argument shown in the proof of (5.22), one finds

$$\sup_\varepsilon \int_{\mathbb{R}^N} e^{\kappa u_\varepsilon^{\delta_0}} u_\varepsilon^{\delta_0-1} m_\varepsilon \, dx < +\infty.$$

Therefore, by using the uniformly Hölder continuity of m_ε and the fact that $u_\varepsilon \geq 1$, we obtain for $|x| > R$ with constant $R > 0$ independent of ε,

$$0 < m_\varepsilon \leq Ce^{-\frac{\kappa}{2}u_\varepsilon^{\delta_0}}, \ \delta_0 \in (0, \gamma - 1), \tag{A.4}$$
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where $C > 0$, $\kappa > 0$ is small and $\delta_0 \in (0, \min\{\gamma - 1, 1\})$, which are all independent of ε. Moreover, in light of (A.2), one has from (A.4) that (A.1) holds. □

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