

Entire solutions of the magnetic Ginzburg-Landau equation in \mathbb{R}^4

Yong Liu

Department of Mathematics,
University of Science and Technology of China, Hefei, China,
Email: yliumath@ustc.edu.cn

Xinan Ma

Department of Mathematics,
University of Science and Technology of China, Hefei, China,
Email: xinan@ustc.edu.cn

Juncheng Wei

Department of Mathematics,
University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2
Email: jcwei@math.ubc.ca

Wangze Wu

Department of Mathematics,
University of Science and Technology of China, Hefei, China,
Email: wuwz18@mail.ustc.edu.cn

August 5, 2021

Abstract

We construct entire solutions of the magnetic Ginzburg-Landau equations in dimension 4 using Lyapunov-Schmidt reduction. The zero set of these solutions are close to the minimal submanifolds studied by Arezzo-Pacard[1]. We also show the existence of a saddle type solution to the equations, whose zero set consists of two vertical planes in \mathbb{R}^4 . These two types of solutions are believed to be energy minimizers of the corresponding energy functional and lie in the same connect component of the moduli space of entire solutions.

1 Introduction

In this paper, we consider the following system of magnetic Ginzburg-Landau equations:

$$\begin{cases} -\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0 \text{ in } \mathbb{R}^n, \\ d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}) = 0 \text{ in } \mathbb{R}^n. \end{cases} \quad (1)$$

Here $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is a complex valued function, $A \in \Omega^1(\mathbb{R}^n, \mathbb{R})$ is a one form, and $\lambda > 0$ is a parameter. We use $\nabla_A = \nabla - iA$ to denote the covariant gradient and $\Delta_A = \nabla_A \cdot \nabla_A$, $d^* = (-1)^{n(k+1)+1} * d * : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k-1}(\mathbb{R}^n)$, where the operator $* : \Omega^k \rightarrow \Omega^{n-k}$ is the usual Hodge star operator on \mathbb{R}^n with the standard metric.

The standard macroscopic theory of superconductivity is due to Ginzburg and Landau in [10]. The equations can be derived from the microscopic theory by the works of Bardeen, Cooper and Schrieffer [9], [11]. The constant $\lambda > 0$ is depending on the material in question: when $\lambda < 1$, the material is of type I; when $\lambda > 1$, the material is of type II.

In the case of dimension $n = 2$, the system (1) has a family of “radial” j -vortex solutions of the form $u_j = (\psi^{(j)}, A^{(j)})$, $j = 1, \dots$, with

$$\psi^{(j)}(x) = U_j(r)e^{ij\phi}, \text{ and } A^{(j)} = V_j(r)\nabla(j\phi). \quad (2)$$

Here (r, ϕ) is the polar coordinate and $j = \text{deg } \psi^{(j)}$ is an integer, and U_j, V_j are real valued functions determined by a system of ODE. Moreover, both U_j and V_j tend to 1 as $r \rightarrow \infty$ with exponential rate. Existence of this kind of solutions was proved in [2] using variational methods. The stability properties of these solutions were established in [12]. More precisely, we now know that for $\lambda \leq 1$, u_j is stable; for $\lambda > 1$, u_j is stable provided that $j = \pm 1$. Using these j -vortex radial solutions as basic blocks, nonradial solutions in \mathbb{R}^2 of equation (1) has been constructed in [24] using a finite dimensional Lyapunov-Schmidt reduction procedure. Using similar methods, solutions to the magnetic Ginzburg-Landau equation with external potential have been constructed in [20].

The magnetic Ginzburg-Landau equations (1) is the Euler-Lagrange equations for the following energy functional:

$$E(A, \psi) = \int_{\mathbb{R}^n} |\nabla_A \psi|^2 + |dA|^2 + \frac{\lambda}{4}(1 - |\psi|^2)^2. \quad (3)$$

Note that after a rescaling by a parameter ϵ , the magnetic Ginzburg-Landau

equations become

$$\begin{cases} -\epsilon^2 \Delta_A \psi + \frac{\lambda}{2} (|\psi|^2 - 1) \psi = 0, \\ \epsilon^2 d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}) = 0. \end{cases} \quad (4)$$

The corresponding energy functional is

$$E_\epsilon(A, \psi) = \int_{\mathbb{R}^n} |\nabla_A \psi|^2 + \epsilon^2 |dA|^2 + \frac{\lambda}{4\epsilon^2} (1 - |\psi|^2)^2.$$

For $\lambda = 1$, it is called the self-dual case. In this case, the system can be reduced to first order equations. In this special case, it is proved in [21] that if $(A_\epsilon, \psi_\epsilon)$ is a family of critical points for E_ϵ with a uniform finite energy bound, then as $\epsilon \rightarrow 0$, the energy measures

$$\mu_\epsilon := \frac{1}{2\pi} [|\nabla_{A_\epsilon} \psi_\epsilon|^2 + \epsilon^2 |dA_\epsilon|^2 + \frac{\lambda}{4\epsilon^2} (1 - |\psi_\epsilon|^2)^2] \mathcal{H}^n,$$

(where \mathcal{H}^n means n -dimensional Hausdorff measures) converge subsequentially to the weight measure μ of a stationary, integral $(n - 2)$ -varifold. In other words, when ϵ tends to 0, the zero sets of ψ_ϵ are close to a codimension two minimal submanifold. They also proved existence of varifold type solutions using variational methods.

In this paper, we shall investigate the general case of $\lambda > 0$. We would like to construct some special smooth solutions from certain minimal submanifold with nice geometrical structure. We will focus on the case of $n = 4$. Let us consider the following two planes:

$$\begin{aligned} \Pi_1 &:= \{(x_1, y_1, 0, 0) \in \mathbb{R}^4 : x_1, y_1 \in \mathbb{R}\}, \\ \Pi_2 &:= \{(0, 0, x_2, y_2) \in \mathbb{R}^4 : x_2, y_2 \in \mathbb{R}\}. \end{aligned}$$

They intersect at the origin. Let (r_1, θ_1) and (r_2, θ_2) be the polar coordinates in Π_1 and Π_2 , respectively. Our first result states that there exists a saddle type solution. This is the content of the following

Theorem 1.1. *For $n = 4$, there exists a solution (A, ψ) to (1), satisfying*

$$\begin{aligned} \psi &= f(r_1, r_2) e^{i(\theta_1 + \theta_2)}, \\ A &= g(r_1, r_2) d\theta_1 + h(r_1, r_2) d\theta_2, \end{aligned}$$

where f, g, h are three real valued functions satisfying a system of second order PDE in the (r_1, r_2) variables. The function f vanish at the axes of the r_1 - r_2 plane.

We shall call the solution stated in Theorem 1 as a saddle type solution. This is motivated by the saddle solution of the Allen-Cahn equation

$$-\Delta u + u^3 - u = 0, \text{ in } \mathbb{R}^n. \quad (5)$$

Indeed, for $n = 2$, the Allen-Cahn equation has a solution u which vanishes precisely at the two axes of the plane.

To proceed, let us recall the following minimal submanifold studied in Arezzo-Pacard[1]. Let

$$\Gamma = \{(\rho e^{i\theta}, \rho^{-1} e^{i\theta}) : \rho, \theta \in \mathbb{R}\}.$$

Consider the scaled surface $\Gamma_\epsilon := \frac{1}{\epsilon} \Gamma$. As ϵ tends to ∞ , formally Γ_ϵ tends to $\Pi_1 \cup \Pi_2$. Our second result is the following theorem, where we use ± 1 vortex solutions to construct solutions in \mathbb{R}^4 .

Theorem 1.2. *For $\epsilon > 0$ small enough, there exists a sequence of solutions $(A_\epsilon, \psi_\epsilon)$ to (4). As $\epsilon \rightarrow 0$, whose zero sets of ψ_ϵ and A_ϵ converge to Γ uniformly in the entire space.*

We use infinite dimensional Lyapunov-Schmidt method to construct these solutions. More precise asymptotic behavior of these solutions will be provided in the subsequent sections. We point out Brendle has done some related works for the self-dual magnetic Ginzburg-Landau equation in the unpublished paper[5]. Part of our arguments in this paper are inspired by his work.

It is expected that the saddle type solution described in Theorem 1.1 and the solutions in Theorem 1.2 lie on the same connected component of the moduli space of entire solutions of the magnetic Ginzburg-Landau equation. This is also analogous to the case of Allen-Cahn equation. Indeed, for any even dimension not less than 8, the Allen-Cahn equation has a saddle type equation which vanishes on the Simons' cone. Moreover, there is family of solutions whose zero sets are asymptotic to the Simons' cone, See [6, 19, 18, 23] for related results. These solutions are conjectured to be global minimizers of the corresponding Allen-Cahn energy functional, since the Simons' cone is area minimizing when the dimension $n \geq 8$. One also know that dimension 8 is the critical dimension. Since the union of the two planes Π_1 and Π_2 are also area minimizing(See [15]) in \mathbb{R}^4 , we formulate the following

Conjecture 1.1. *The solutions of the magnetic Ginzburg-Landau equations provided in Theorem 1.1 and Theorem 1.2 are energy minimizers of the energy functional.*

This paper is organized as follows. In Section 2, we show the existence of saddle type solution and proved Theorem 1.1. Sections 3-8 will be devoted to the proof of Theorem 1.2. Section 3 studies the Fermi coordinate with respect to the minimal submanifold. Section 4 deals with the Jacobi operator. In Section 5 we define the approximate solution and estimate its error. Section 6 is devoted to the linear theory of the linearized operator around the approximate solution. In Section 7, we study a projected nonlinear problem. Then in Section 8 we show the existence of true solutions and finish the proof of Theorem 1.2. We put some of the tedious computations in Appendix.

Acknowledgement Y. Liu is partially supported by “The Fundamental Research Funds for the Central Universities WK3470000014,” and NSFC no. 11971026. J. Wei is partially supported by NSERC of Canada.

2 A saddle type solution whose zero set are two planes

In this section, we show the existence of saddle type solution and prove Theorem 1.1. We seek for a solution to the magnetic Ginzburg Landau equations (1) with the form:

$$\begin{aligned}\psi &= f(r_1, r_2)e^{i(\theta_1+\theta_2)}, \\ A &= g(r_1, r_2)d\theta_1 + h(r_1, r_2)d\theta_2 \\ &= -\frac{g \sin \theta_1}{r_1}dx_1 + \frac{g \cos \theta_1}{r_1}dy_1 - \frac{h \sin \theta_2}{r_2}dx_2 + \frac{h \cos \theta_2}{r_2}dy_2,\end{aligned}$$

and $\{\psi = 0\} = \Pi_1 \cup \Pi_2$.

Let us denote $g_j := \partial_{r_j}g$, $h_j := \partial_{r_j}h$ and $f_j := \partial_{r_j}f$.

Lemma 2.1. *If (A, ψ) is the solution to (1), then (f, g, h) satisfies*

$$\begin{cases} f_{11} + f_{22} + \frac{f_1}{r_1} + \frac{f_2}{r_2} - \frac{f(1-g)^2}{r_1^2} - \frac{f(1-h)^2}{r_2^2} - \frac{\lambda}{2}(f^2 - 1)f = 0, & (6) \end{cases}$$

$$\begin{cases} g_{11} + g_{22} - \frac{g_1}{r_1} + \frac{g_2}{r_2} + f^2 - f^2g = 0, & (7) \end{cases}$$

$$\begin{cases} h_{11} + h_{22} - \frac{h_2}{r_2} + \frac{h_1}{r_1} + f^2 - f^2h = 0, & (8) \end{cases}$$

with the energy functional

$$\begin{aligned}
E(f, g, h) &= \frac{1}{2} \int_{\mathbb{R}^4} f_1^2 + f_2^2 + [(f - fg)^2 + g_1^2 + g_2^2] \frac{1}{r_1^2} + [(f - fh)^2 + h_1^2 + h_2^2] \frac{1}{r_2^2} + \frac{\lambda}{4} (1 - f^2)^2 \\
&= 2\pi^2 \int_{r_1, r_2 > 0} \{ |\nabla f|^2 + [(f - fg)^2 + |\nabla g|^2] \frac{1}{r_1^2} + [(f - fh)^2 + |\nabla h|^2] \frac{1}{r_2^2} \\
&\quad + \frac{\lambda}{4} (1 - f^2)^2 \} r_1 r_2 dr_1 dr_2.
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
\nabla_A \psi &= \nabla \psi - iA\psi \\
&= \nabla f \cdot e^{i(\theta_1 + \theta_2)} + ie^{i(\theta_1 + \theta_2)} \nabla(\theta_1 + \theta_2) \cdot f - ie^{i(\theta_1 + \theta_2)} f \cdot (g\nabla\theta_1 + h\nabla\theta_2).
\end{aligned}$$

We also have

$$\begin{aligned}
dA &= dg \wedge d\theta_1 + dh \wedge d\theta_2 \\
&= (\cos \theta_1 \cdot g_1 dx_1 + \sin \theta_1 \cdot g_1 dy_1 + \cos \theta_2 \cdot g_2 dx_2 + \sin \theta_2 \cdot g_2 dy_2) \wedge d\theta_1 \\
&\quad + (\cos \theta_1 \cdot h_1 dx_1 + \sin \theta_1 \cdot h_1 dy_1 + \cos \theta_2 \cdot h_2 dx_2 + \sin \theta_2 \cdot h_2 dy_2) \wedge d\theta_2 \\
&= \frac{g_1}{r_1} dx_1 \wedge dy_1 + \left(\frac{g_2 \sin \theta_1 \cos \theta_2}{r_1} - \frac{h_1 \cos \theta_1 \sin \theta_2}{r_2} \right) dx_1 \wedge dx_2 \\
&\quad + \left(\frac{g_2 \sin \theta_1 \sin \theta_2}{r_1} + \frac{h_1 \cos \theta_1 \cos \theta_2}{r_2} \right) dx_1 \wedge dy_2 + \left(-\frac{g_2 \cos \theta_1 \cos \theta_2}{r_1} - \frac{h_1 \sin \theta_1 \sin \theta_2}{r_2} \right) dy_1 \wedge dx_2 \\
&\quad + \left(-\frac{g_2 \cos \theta_1 \sin \theta_2}{r_1} + \frac{h_1 \sin \theta_1 \cos \theta_2}{r_2} \right) dy_1 \wedge dy_2 + \frac{h_2}{r_2} dx_2 \wedge dy_2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Delta_A \psi &= \Delta \psi + id^* A \cdot \psi - 2i \langle A, d\psi \rangle - |A|^2 \psi, \\
e^{-i(\theta_1 + \theta_2)} \Delta \psi &= f_{11} + \frac{f_1}{r_1} - \frac{f}{r_1^2} + f_{22} + \frac{f_2}{r_2} - \frac{f}{r_2^2}, \\
d^* A &= \partial_{x_1} \left(\frac{g \sin \theta_1}{r_1} \right) - \partial_{y_1} \left(\frac{g \cos \theta_1}{r_1} \right) + \partial_{x_2} \left(\frac{h \sin \theta_2}{r_2} \right) - \partial_{y_2} \left(\frac{h \cos \theta_2}{r_2} \right) = 0, \\
e^{-i(\theta_1 + \theta_2)} \langle A, d\psi \rangle &= if(gr_1^{-2} + hr_2^{-2}).
\end{aligned}$$

It follows that

$$\begin{aligned}
e^{-i(\theta_1 + \theta_2)} \Delta_A \psi &= e^{-i(\theta_1 + \theta_2)} (\Delta \psi + id^* A \cdot \psi - 2i \langle A, d\psi \rangle - |A|^2 \psi) \\
&= f_{11} + \frac{f_1}{r_1} - \frac{f}{r_1^2} + f_{22} + \frac{f_2}{r_2} - \frac{f}{r_2^2} + 2f(gr_1^{-2} + hr_2^{-2}) - f(g^2 r_1^{-2} + h^2 r_2^{-2}).
\end{aligned}$$

Let $d^*dA = (d^*dA)_1dx_1 + (d^*dA)_2dy_1 + (d^*dA)_3dx_2 + (d^*dA)_4dy_2$. Then there holds

$$\begin{aligned}(d^*dA)_1 &= \partial_{y_1}\left(\frac{g_1}{r_1}\right) + \partial_{x_2}\left(-\frac{h_1 \cos \theta_1 \sin \theta_2}{r_2} + \frac{g_2 \sin \theta_1 \cos \theta_2}{r_1}\right) \\ &\quad + \partial_{y_2}\left(\frac{h_1 \cos \theta_1 \cos \theta_2}{r_2} + \frac{g_2 \sin \theta_1 \sin \theta_2}{r_1}\right) \\ &= \left(\frac{g_{11}}{r_1} - \frac{g_1}{r_1^2} + \frac{g_2}{r_1 r_2} + \frac{g_{22}}{r_1}\right) \sin \theta_1,\end{aligned}$$

$$\begin{aligned}(d^*dA)_2 &= -\partial_{x_1}\left(\frac{g_1}{r_1}\right) + \partial_{x_2}\left(-\frac{h_1 \sin \theta_1 \sin \theta_2}{r_2} - \frac{g_2 \cos \theta_1 \cos \theta_2}{r_1}\right) \\ &\quad + \partial_{y_2}\left(\frac{h_1 \sin \theta_1 \cos \theta_2}{r_2} - \frac{g_2 \cos \theta_1 \sin \theta_2}{r_1}\right) \\ &= -\left(\frac{g_{11}}{r_1} - \frac{g_1}{r_1^2} + \frac{g_2}{r_1 r_2} + \frac{g_{22}}{r_1}\right) \cos \theta_1,\end{aligned}$$

$$\begin{aligned}(d^*dA)_3 &= -\partial_{x_1}\left(-\frac{h_1 \cos \theta_1 \sin \theta_2}{r_2} + \frac{g_2 \sin \theta_1 \cos \theta_2}{r_1}\right) \\ &\quad - \partial_{y_1}\left(-\frac{h_1 \sin \theta_1 \sin \theta_2}{r_2} - \frac{g_2 \cos \theta_1 \cos \theta_2}{r_1}\right) + \partial_{y_2}\left(\frac{h_2}{r_2}\right) \\ &= \left(\frac{h_{22}}{r_2} - \frac{h_2}{r_2^2} + \frac{h_1}{r_1 r_2} + \frac{h_{11}}{r_2}\right) \sin \theta_2,\end{aligned}$$

$$\begin{aligned}(d^*dA)_4 &= -\partial_{x_1}\left(\frac{h_1 \cos \theta_1 \cos \theta_2}{r_2} + \frac{g_2 \sin \theta_1 \sin \theta_2}{r_1}\right) \\ &\quad - \partial_{y_1}\left(\frac{h_1 \sin \theta_1 \cos \theta_2}{r_2} - \frac{g_2 \cos \theta_1 \sin \theta_2}{r_1}\right) - \partial_{x_2}\left(\frac{h_2}{r_2}\right) \\ &= -\left(\frac{h_{22}}{r_2} - \frac{h_2}{r_2^2} + \frac{h_1}{r_1 r_2} + \frac{h_{11}}{r_2}\right) \cos \theta_2.\end{aligned}$$

Besides, we have:

$$\begin{aligned}Im(\nabla_A \psi \cdot \bar{\psi}) &= (f^2 - f^2 g)d\theta_1 + (f^2 - f^2 h)d\theta_2 \\ &= (f^2 - f^2 g)\left(-\frac{\sin \theta_1}{r_1}dx_1 + \frac{\cos \theta_1}{r_1}dy_1\right) + (f^2 - f^2 h)\left(-\frac{\sin \theta_2}{r_2}dx_2 + \frac{\cos \theta_2}{r_2}dy_2\right).\end{aligned}$$

Therefore, the energy functional for f and g, h has the form

$$\begin{aligned} E(f, g, h) &= \frac{1}{2} \int_{\mathbb{R}^4} f_1^2 + f_2^2 + [(f - fg)^2 + g_1^2 + g_2^2] \frac{1}{r_1^2} + [(f - fh)^2 + h_1^2 + h_2^2] \frac{1}{r_2^2} + \frac{\lambda}{4} (1 - f^2)^2 \\ &= 2\pi^2 \int_{r_1, r_2 > 0} \{ |\nabla f|^2 + [(f - fg)^2 + |\nabla g|^2] \frac{1}{r_1^2} + [(f - fh)^2 + |\nabla h|^2] \frac{1}{r_2^2} \\ &\quad + \frac{\lambda}{4} (1 - f^2)^2 \} r_1 r_2 dr_1 dr_2. \end{aligned}$$

Then the equations (1) with respect to f and g, h in the form

$$\begin{cases} f_{11} + f_{22} + \frac{f_1}{r_1} + \frac{f_2}{r_2} - \frac{f(1-g)^2}{r_1^2} - \frac{f(1-h)^2}{r_2^2} - \frac{\lambda}{2} (f^2 - 1)f = 0, & (9) \\ g_{11} - \frac{g_1}{r_1} + \frac{g_2}{r_2} + g_{22} + f^2 - f^2 g = 0, & (10) \\ h_{22} - \frac{h_2}{r_2} + \frac{h_1}{r_1} + h_{11} + f^2 - f^2 h = 0. & (11) \end{cases}$$

□

Theorem 2.1. *There exists a triple (f, g, h) , such that the equations in Lemma 2.1 hold, and satisfies the boundary condition:*

$$f = 0, \text{ on } \Pi_1 \cup \Pi_2.$$

Moreover, $g(r_1, r_2) = h(r_2, r_1)$ and g vanishes at the r_2 axis.

Proof. We minimize the energy functional.

Although r_1, r_2 are polar coordinates and they are nonnegative, for convenience, during the proof we shall also consider the negative r_1 region by reflection. Let us consider the bounded region: $\Omega_k := \{(r_1, r_2) : r_1 \in \mathbb{R}, r_2 > 0\} \cap B_k$, where B_k is the ball of radius k in the r_1 - r_2 plane, centered at the origin.

Consider the corresponding energy functional

$$\begin{aligned} E_k(f, g, h) &:= \int_{\Omega_k} \{ |\nabla f|^2 + [(f - fg)^2 + |\nabla g|^2] \frac{1}{r_1^2} + [(f - fh)^2 + |\nabla h|^2] \frac{1}{r_2^2} \\ &\quad + \frac{\lambda}{4} (1 - f^2)^2 \} r_1 r_2 dr_1 dr_2. \end{aligned}$$

Define the space

$$\begin{aligned} H_k &:= \{ (f, g, h) : f, g, h \in \tilde{H}_0^1(\Omega_k); g(r_1, r_2) = h(r_2, r_1), \\ &\quad f(r_1, r_2) = -f(-r_1, r_2), g(r_1, r_2) = g(-r_1, r_2), a.e. \}. \end{aligned}$$

Here $\tilde{H}_0^1(\Omega_k)$ is obtained by replacing the weight by $r_1 r_2$ in the usual $H_0^1(\Omega_k)$ space. Let us consider the variational problem:

$$\inf_{(f,g,h) \in H_k} E_k(f, g, h).$$

The existence of a minimizer of the action functional E_k in this space follows from a straight forward modification of standard argument, by using the fact that the functional E_k is coercive and satisfies the P.S. condition.

We can assume that the minimizer satisfies $0 \leq f_k, g_k, h_k \leq 1$. So using the interior estimates, we know (f_k, g_k, h_k) converges to some (f, g, h) in $H_{loc}^1(\{r_1, r_2 > 0\})$. Hence (f, g, h) is smooth in $\mathbb{R}^4 \setminus (\Pi_1 \cup \Pi_2)$.

Next, we need to show the (ψ, A) corresponding to (f, g, h) is smooth in \mathbb{R}^4 . In fact, let us first consider (ψ_k, A_k) corresponding to (f_k, g_k, h_k) and rewrite (1) as

$$\begin{cases} -\Delta \psi = \frac{\lambda}{2}(1 - |\psi|^2)\psi + 2\psi(gr_1^{-2} + hr_2^{-2}) - \psi(g^2r_1^{-2} + h^2r_2^{-2}), \\ -\Delta A = (f^2 - f^2g)d\theta_1 + (f^2 - f^2h)d\theta_2. \end{cases}$$

Note that $(f_k, g_k, h_k) \in H_{k+1}$ and $E_{k+1}(f_k, g_k, h_k) \geq E_{k+1}(f_{k+1}, g_{k+1}, h_{k+1})$, so we always have:

$$E_k(f, g, h) \leq C_1,$$

where C_1 is a constant.

Thus we know

$$\|(f_k^2 - f_k^2g_k)d\theta_1\|_{L^2(\mathbb{R}^4)} \leq \|(f_k - f_kg_k)r_1^{-1}\|_{L^2(\mathbb{R}^4)} \leq \sqrt{C_1}.$$

Then $\|A_k\|_{W^{2,2}(\mathbb{R}^4)} \leq C_2$.

Similarly, we can also get that

$$\left\| \frac{\lambda}{2}(1 - |\psi_k|^2)\psi_k + 2\psi_k(gr_1^{-2} + hr_2^{-2}) - \psi_k(g^2r_1^{-2} + h^2r_2^{-2}) \right\|_{L^2(\mathbb{R}^4)} \leq C_3.$$

Then $\|\psi_k\|_{W^{2,2}(\mathbb{R}^4)} \leq C_4$.

Therefore, (A_k, ψ_k) converges to some (A, ψ) in $W^{1,2}(\mathbb{R}^4)$ norm. Hence (A, ψ) is one solution to (1) and it is smooth. By minimum principle, we know in $\mathbb{R}^4 \setminus (\Pi_1 \cup \Pi_2)$, $f > 0$. \square

3 Fermi coordinates

Instead of working in the usual Euclidean coordinate, we shall work in the coordinate adapted to the minimal submanifold, which we call Fermi

coordinate here. Let us recall the minimal submanifold Γ studied in [1]:

$$\Gamma = \left(\frac{\sqrt{2}}{2}\rho e^{i\theta}, \frac{\sqrt{2}}{2}\rho^{-1}e^{i\theta} \right).$$

We rewrite the minimal submanifold Γ as:

$$\Gamma = \frac{e^{is}}{\sqrt{\sin 2s}}\Theta,$$

where $s \in (0, \frac{\pi}{2})$, $\Theta = (\cos \theta, \sin \theta) \in S^1$. The rescaled manifold Γ_ϵ is given by

$$\left(\frac{\sqrt{2}}{2\epsilon}\rho e^{i\theta}, \frac{\sqrt{2}}{2\epsilon}\rho^{-1}e^{i\theta} \right),$$

The tangent vector of Γ can be written in the form

$$\partial_s \Gamma = \frac{-e^{-is}}{(\sin 2s)^{\frac{3}{2}}}\Theta, \quad \partial_\theta \Gamma = \frac{e^{is}}{\sqrt{\sin 2s}}\Theta^\perp,$$

where $\Theta^\perp = (-\sin \theta, \cos \theta)$. We also have the unit tangent vector $e_1 = e^{-is}\Theta$, $e_2 = e^{is}\Theta^\perp$, and unit normal vector $m = ie^{-is}\Theta$, $n = ie^{is}\Theta^\perp$.

Define a map $T : (s, \theta, a, b) \longrightarrow (z_1, z_2) \in \mathbb{C}^2$:

$$\begin{aligned} Y &= \Gamma_\epsilon + am + bn \\ &= \left(\frac{\cos s}{\epsilon\sqrt{\sin 2s}}\Theta + a \sin s \cdot \Theta - b \sin s \cdot \Theta^\perp, \frac{\sin s}{\epsilon\sqrt{\sin 2s}}\Theta + a \cos s \cdot \Theta + b \cos s \cdot \Theta^\perp \right). \end{aligned}$$

Denote $\Sigma_\epsilon := \{(s, \theta, a, b) : a^2 + b^2 < \frac{1}{\epsilon^2 \sin 2s}\}$. We would like to show that if $(s, \theta, a, b) \in \Sigma_\epsilon$, then $Y = \Gamma_\epsilon + am + bn$ defined a smooth coordinate system around the minimal submanifold. To see this, we need several lemmas.

Lemma 3.1. *Let $P = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})$, $\rho_1 \neq \rho_2$. If there are two points $P_1, P_2 \in \Gamma$ satisfying $\text{dist}(P_j, P) = \text{dist}(\Gamma_\epsilon, P)$, then $P_1 = P_2$.*

Proof. The distance between P and P_1 is

$$\begin{aligned} d^2 &= \inf_{\rho, \theta} \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + \rho_1^2 + \rho_2^2 - \sqrt{2}\rho_1\rho \cos(\theta - \theta_1) - \sqrt{2}\rho_2\rho^{-1} \cos(\theta - \theta_2) \\ &= \inf_{\rho} \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + \rho_1^2 + \rho_2^2 - \sqrt{2}\sqrt{\rho_1^2\rho^2 + \rho_2^2\rho^{-2} + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}. \end{aligned}$$

Set $f(\rho) = \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} - \sqrt{2}\sqrt{\rho_1^2\rho^2 + \rho_2^2\rho^{-2} + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}$.

Suppose $\rho_1 > \rho_2$. We observe that if $\rho \geq 1$, then

$$f(\rho) \leq f(\rho^{-1}),$$

Moreover, “=” holds iff $\rho = 1$. So $\inf_{\rho>0} f(\rho) = \inf_{\rho\geq 1} f(\rho)$.

Suppose there exist $r_1, r_2 \geq 1$, satisfying

$$f(r_1) = f(r_2) = \inf_{\rho>0} f(\rho),$$

then $f'(r_1) = f'(r_2) = 0$. Let us define $\alpha = \theta_1 - \theta_2$, we have

$$\begin{aligned} f'(\rho) &= \rho - \rho^{-3} - \sqrt{2} \frac{\rho_1^2 \rho - \rho_2^2 \rho^{-3}}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\alpha)}}, \\ f'(1) &= \frac{-\sqrt{2}(\rho_1^2 - \rho_2^2)}{\sqrt{(\rho_1 - \rho_2)^2 + 2\rho_1 \rho_2(1 + \cos \alpha)}} \\ &> \frac{-\sqrt{2}(\rho_1^2 - \rho_2^2)}{\rho_1 - \rho_2} \\ &= -\sqrt{2}(\rho_1 + \rho_2) < 0. \end{aligned}$$

So $\rho = 1$ is not an extreme point. As a result, we obtain $r_1 > r_2 > 1$. If $f'(\rho) = 0$, then

$$\begin{aligned} f(\rho) &= \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} - 2\frac{\rho_1^2 \rho - \rho_2^2 \rho^{-3}}{\rho - \rho^{-3}} \\ &= \frac{\rho^6 - \rho^{-2} - 4\rho_1^2 \rho^4 + 4\rho_2^2}{2(\rho^4 - 1)} =: g(\rho). \end{aligned}$$

Since

$$g'(\rho) = \frac{2(\rho^3 - \rho^{-1})^3 + 16\rho^3(\rho_1^2 - \rho_2^2)}{2(\rho^4 - 1)^2} > 0,$$

g is strictly increasing. If $g(\rho_1) = g(\rho_2)$, we have $r_1 = r_2$, which is a contradiction. So when $\rho_1 \neq \rho_2$, the closest point is unique, which completes the proof. \square

The proof of the next three lemmas relies on careful analysis of the geometric structure around the minimal submanifold. The computation is straight forward but tedious, we put the proof in the Appendix.

Lemma 3.2. *Suppose $\rho_1 > 0$, then $(\rho_1 e^{i\theta_1}, \rho_1 e^{i\theta_2})$ has two different closest points to Γ if and only if $\rho_1 > \sqrt{2 + 2\cos \alpha}$.*

Next, we want to show what the conditions on a, b are needed, if $y = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) \in Y$ has a unique closest point to Γ_ϵ .

Lemma 3.3. *If $y = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) = T(s, \theta, a, b)$, then y has a unique closest point to Γ_ϵ if and only if $a^2 + b^2 < \frac{1}{\epsilon^2 \sin 2s}$.*

Lemma 3.4. *If $y = T(s, \theta, a, b)$, and $a^2 + b^2 < \frac{1}{\epsilon^2 \sin 2s}$, then $\text{dist}(y, \Gamma_\epsilon) = \sqrt{a^2 + b^2}$.*

From previous arguments, we know that if

$$\begin{aligned} y &= T(s, \theta, a, b) \\ &= \left(\frac{\cos s}{\epsilon \sqrt{\sin 2s}} \Theta + a \sin s \cdot \Theta - b \sin s \cdot \Theta^\perp, \frac{\sin s}{\epsilon \sqrt{\sin 2s}} \Theta + a \cos s \cdot \Theta + b \cos s \cdot \Theta^\perp \right) \\ &\in T(\Sigma_\epsilon). \end{aligned}$$

We can define a map $S: T(\Sigma_\epsilon) \rightarrow \Sigma_\epsilon$ as follows: if $P \in T(\Sigma_\epsilon)$, then we suppose $P = T(s_0, \theta_0, a_0, b_0)$ and $Q_0 = \frac{e^{is_0}}{\epsilon \sqrt{\sin 2s_0}} \Theta_0 \in \Gamma_\epsilon$, satisfying $a_0^2 + b_0^2 < \frac{1}{\epsilon^2 \sin 2s_0}$. By Lemma 3.4, Q_0 is the closest point. But we know $P \in T(\Sigma_\epsilon)$, so the closest point is unique. As a result, let $S(P) = (s_0, \theta_0, a_0, b_0)$,

$$\begin{aligned} a &= (P - Q_0) \cdot m_0, \\ b &= (P - Q_0) \cdot n_0, \end{aligned}$$

where m_0, n_0 are the normal vector at Q_0 . Then T is well-defined and $T \circ S = Id$, $S \circ T = Id$. So $Y = \Gamma_\epsilon + am + bn$ is a coordinate in Σ_ϵ .

Denote $a = r \cos \phi$, $b = r \sin \phi$. By the definition, we know r and ϕ are smooth in Σ_ϵ . Finally in this section, we show the regularity of r and ϕ in \mathbb{R}^4 .

Lemma 3.5. *ϕ is Lipschitz continuous near $\mathbb{C}^2 \setminus \Sigma_\epsilon$ and r is Lipschitz continuous in \mathbb{C}^2 . Especially ϕ is $C^{1,1}$ near 0.*

Proof. As $|\nabla r| \leq 1$, r is Lipschitz continuous, but not C^1 . Because when $x \in \mathbb{C}^2 \setminus \Sigma_\epsilon$, the nearest point is not unique. So r is continuous but not smooth. Next we will show that ϕ is continuous.

Consider $\Gamma = \frac{e^{is}}{\sqrt{\sin 2s}} \Theta$. Denote $P = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) \neq 0$. Suppose Q_1 is a closest point to P :

$$Q_1 = \left(\frac{\sqrt{2}}{2} \rho e^{i\theta}, \frac{\sqrt{2}}{2} \rho^{-1} e^{i\theta} \right).$$

Then the normal vectors at P are:

$$\begin{aligned} m_1 &= (\sin s \cdot e^{i\theta}, \cos s \cdot e^{i\theta}), \\ n_1 &= (-\sin s \cdot i e^{i\theta}, \cos s \cdot i e^{i\theta}), \\ \frac{\sqrt{2}}{2} \rho &= \frac{\cos s}{\sqrt{\sin 2s}} \Rightarrow \rho^2 = \frac{\cos s}{\sin s}. \end{aligned}$$

From previous arguments, we get

$$\begin{aligned}
\cos \theta &= \frac{\rho_1 \rho \cos \theta_1 + \rho_2 \rho^{-1} \cos \theta_2}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\sin \theta &= \frac{\rho_1 \rho \sin \theta_1 + \rho_2 \rho^{-1} \sin \theta_2}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\cos(\theta - \theta_1) &= \frac{\rho_1 \rho + \rho_2 \rho^{-1} \cos(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\sin(\theta - \theta_1) &= \frac{\rho_2 \rho^{-1} \sin(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\cos(\theta - \theta_2) &= \frac{\rho_2 \rho^{-1} + \rho_1 \rho \cos(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\sin(\theta - \theta_2) &= \frac{-\rho_1 \rho \sin(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}, \\
\rho \sin s &= \sqrt{\frac{\cos s}{\sin s}} \cdot \sin s = \sqrt{\frac{\sin 2s}{2}}, \\
\rho^{-1} \cos s &= \sqrt{\frac{\sin 2s}{2}},
\end{aligned}$$

$$\begin{aligned}
a_1 &= (P - Q_1) \cdot m \\
&= \rho_1 \sin s \cdot \frac{\rho_1 \rho + \rho_2 \rho^{-1} \cos(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}} - \frac{\sqrt{2}}{2} \rho \sin s + \\
&\quad \rho_2 \cos s \cdot \frac{\rho_2 \rho^{-1} + \rho_1 \rho \cos(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}} - \frac{\sqrt{2}}{2} \rho^{-1} \cos s, \\
b_1 &= \rho_1 \sin s \cdot \frac{\rho_2 \rho^{-1} \sin(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}} + \\
&\quad \rho_2 \cos s \cdot \frac{\rho_1 \rho \sin(\theta_2 - \theta_1)}{\sqrt{\rho_1^2 \rho^2 + \rho_2^2 \rho^{-2} + 2\rho_1 \rho_2 \cos(\theta_1 - \theta_2)}}.
\end{aligned}$$

If $\rho_1 = \rho_2 > 0$, then there are two closest points to P , one of which is Q_1 . Let the other be Q_2 . By symmetry, we know:

$$Q_2 = \left(\frac{\sqrt{2}}{2} \rho^{-1} e^{i\theta}, \frac{\sqrt{2}}{2} \rho e^{i\theta} \right).$$

We observe that, when $Q_1 \rightarrow Q_2$, we have $\rho \rightarrow \rho^{-1}$, $s \rightarrow (\frac{\pi}{2} - s)$. So $a_1 = a_2$, $b_1 = b_2$.

If $\rho_1 \neq \rho_2$, let ρ_1, ρ_2 tend to 0. As a result, ρ will tend to 1, and

$$\begin{aligned} a_1 &= -1, \\ b_1 &= 0. \end{aligned}$$

So $\phi(0) = \pi$. Therefore ϕ is continuous around $\mathbb{C}^2 \setminus \Sigma_\epsilon$. So it is Lipschitz continuous around $\mathbb{C}^2 \setminus \Sigma_\epsilon$. At the origin, we consider two directions: $(\rho_1, \theta_1, \rho_2, \theta_2)$, $(\rho_1, \theta_1 + \pi, \rho_2, \theta_2 + \pi)$. As ρ is Lipschitz continuous near 0, we deduce $\nabla\phi(0) = 0$. So ϕ is $C^{1,1}$ near the origin. \square

We remark that from the expression of a_1, b_1 , we deduce that

$$\begin{aligned} \phi(\rho_1, \theta_1, \rho_2, \theta_2) &= -\phi(\rho_1, \theta_2, \rho_2, \theta_1) = \phi(\rho_2, \theta_1, \rho_1, \theta_2) = -\phi(\rho_1, -\theta_1, \rho_2, -\theta_2); \\ r(\rho_1, \theta_1, \rho_2, \theta_2) &= r(\rho_2, \theta_1, \rho_1, \theta_2) = r(\rho_1, \theta_2, \rho_2, \theta_1) = r(\rho_1, -\theta_1, \rho_2, -\theta_2). \end{aligned}$$

4 The Jacobi operator

In this section, we show that the Jacobi operator naturally arise from the projection of the error of a naturally defined approximate solution around the manifold. We recall that the energy functional in \mathbb{R}^4 is:

$$E(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^4} \{|\nabla_A \psi|^2 + |dA|^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2\},$$

for the fields

$$\begin{aligned} A &= (A_1, A_2, A_3, A_4) : \mathbb{R}^4 \rightarrow \mathbb{R}^4, \\ \psi &= \psi_1 + i\psi_2 : \mathbb{R}^4 \rightarrow \mathbb{C}. \end{aligned}$$

And $\nabla_A = \nabla - iA$ is the covariant gradient. As we can consider A as a 1-form on \mathbb{R}^4 , then dA is well-defined.

$$\begin{aligned} |\nabla_A \psi|^2 &= \sum_{j=1}^4 |\partial_j \psi - iA_j \psi|^2, \\ |dA|^2 &= |\partial_j A_k - \partial_k A_j|^2. \end{aligned}$$

Critical point $u = (\psi, A)$ of $E(\psi, A)$ satisfies the Ginzburg-Landau (GL) equations

$$-\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0, \quad (12)$$

$$\sum_{k \neq j} \partial_k (\partial_k A_j - \partial_j A_k) + \text{Im}((\nabla_A)_{e_j} \psi \cdot \bar{\psi}) = 0, j = 1, 2, 3, 4. \quad (13)$$

Equations (13) has the form

$$d^* dA = \text{Im}(\nabla_A \psi \cdot \bar{\psi}). \quad (14)$$

We also have

$$\begin{aligned}
\Delta_A \psi &= \nabla_A \nabla_A \psi \\
&= \Delta \psi - i \partial_j A_j \psi - 2i A_j \partial_j \psi - |A|^2 \psi \\
&= \Delta \psi + i d^* A \psi - 2i \langle A, d\psi \rangle - |A|^2 \psi.
\end{aligned}$$

Let us define

$$F(u) = \left(-\Delta_A \psi + \frac{\lambda}{2} (|\psi|^2 - 1) \psi, d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}) \right).$$

Let $W := (A^{(1)}, \psi^{(1)}) = (V(r)\nabla\phi, U(r)e^{i\phi})$ be the +1 vortex solution. Then we have:

$$\begin{aligned}
A_3 &= -\frac{V}{r} \sin \phi, \quad A_4 = \frac{V}{r} \cos \phi, \\
\partial_3 A_3 &= \frac{V}{r^2} \sin 2\phi - \frac{V'}{2r} \sin 2\phi, \\
\partial_4 A_3 &= -\frac{V'}{r} \sin^2 \phi - \frac{V}{r^2} \cos 2\phi, \\
\partial_3 A_4 &= \frac{V'}{r} \cos^2 \phi - \frac{V}{r^2} \cos 2\phi, \\
\partial_4 A_4 &= \frac{V'}{2r} \sin 2\phi - \frac{V}{r^2} \sin 2\phi.
\end{aligned}$$

In the following context, we use the following properties (see [2], [22]):

1. $-V'' + \frac{V'}{r} - U^2 + VU^2 = 0$,
2. $U = cr, V = dr^2$, as $r \rightarrow 0$,
3. $U = 1 + O(e^{-m_\lambda r}), V = 1 + O(e^{-r})$ as $r \rightarrow \infty$, with $m_\lambda := \min(\sqrt{\lambda}, 2)$.

Let ϵ be a small parameter and denote $t_1 = a - \epsilon f$, $t_2 = b - \epsilon g$, $t_1 + it_2 = \tilde{r}e^{i\tilde{\phi}}$. When it is close to Γ_ϵ , we define the following (ψ, A) as the

approximation solution:

$$\begin{aligned}
\psi(a, b) &:= \psi^{(1)}(a - \epsilon f, b - \epsilon g), \\
A &= A_j dy^j \\
&:= V(\tilde{r}) \nabla(\phi(a - \epsilon f, b - \epsilon g)) \\
&= (-\epsilon f_s A_3 - \epsilon g_s A_4) dy^1 + (-\epsilon f_\theta A_3 - \epsilon g_\theta A_4) dy^2 \\
&\quad + A_3 dy^3 + A_4 dy^4 \\
&= (\epsilon f_s \cdot \frac{V}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_s \cdot \frac{V}{\tilde{r}} \cos \tilde{\phi}) dy^1 \\
&\quad + (\epsilon f_\theta \cdot \frac{V}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_\theta \cdot \frac{V}{\tilde{r}} \cos \tilde{\phi}) dy^2 \\
&\quad - \frac{V}{\tilde{r}} \sin \tilde{\phi} dy^3 + \frac{V}{\tilde{r}} \cos \tilde{\phi} dy^4.
\end{aligned}$$

Denote $y = (\frac{\cos s}{\sqrt{\sin 2s}} \Theta + a \sin s \cdot \Theta - b \sin s \cdot \Theta^\perp, \frac{\sin s}{\sqrt{\sin 2s}} \Theta + a \cos s \cdot \Theta + b \cos s \cdot \Theta^\perp)$, and $\Gamma_\epsilon = \frac{1}{\epsilon} \Gamma = (\frac{\cos s}{\epsilon \sqrt{\sin 2s}} \Theta, \frac{\sin 2}{\epsilon \sqrt{\sin 2s}} \Theta)$. Then we can get a coordinate with respect to (s, θ, a, b) near Γ_ϵ . Denote $\rho = (\sin 2s)^{-\frac{1}{2}}$ for simplicity.

Recall the Jacobian operator of Γ (see [1]): For normal vector field $N = ie^{-is} f \cdot \Theta + ie^{is} g \Theta^\perp = fm + gn$, we have

$$\begin{aligned}
L_H N &= ie^{-is} \Theta [(\sin 2s)^3 \partial_s^2 f + 2(\sin 2s)^2 \cos 2s \cdot \partial_s f + \sin 2s \cdot \partial_\theta^2 f - \\
&\quad 2 \sin 2s \cos 2s \cdot \partial_\theta g + 2(\sin 2s)^3 \cdot f - \sin 2s (\cos 2s)^2 \cdot f] + ie^{is} \Theta^\perp \cdot \\
&\quad [(\sin 2s)^3 \partial_s^2 g + 2(\sin 2s)^2 \cos 2s \cdot \partial_s g + \sin 2s \cdot \partial_\theta^2 g + 2 \sin 2s \cos 2s \cdot \partial_\theta f + \\
&\quad 2(\sin 2s)^3 \cdot g - \sin 2s (\cos 2s)^2 \cdot g].
\end{aligned} \tag{15}$$

In this section, we show

Theorem 4.1. *When ϵ is small enough, the equations*

$$\begin{cases} \langle F(u), (\partial_3 \psi - i A_3 \psi, -\partial_3 A + \nabla A_3) \rangle = 0, \\ \langle F(u), (\partial_4 \psi - i A_4 \psi, -\partial_4 A + \nabla A_4) \rangle = 0, \end{cases}$$

can be written as

$$L_H(f, g) = (G_1, G_2)$$

where the inner product is the integral over the whole space $(a, b) \in \mathbb{R}^2$. $G_1, G_2 = O(\epsilon)$ is of higher order in f, g

Before we prove Theorem 4.1, we first give several lemmas, whose proof can be found in the Appendix, since again they follows from tedious but routine computations.

Denote $dA = \Sigma_{j < k} A_{jk} dy^j \wedge dy^k$, then we have:

$$\begin{aligned}
A_{jk} &= \partial_j A_k - \partial_k A_j, \\
A_{12} &= O(\epsilon^2), \\
A_{13} &= \epsilon g_s (\partial_3 A_4 - \partial_4 A_3), \\
A_{14} &= -\epsilon f_s (\partial_3 A_4 - \partial_4 A_3), \\
A_{23} &= \epsilon g_\theta (\partial_3 A_4 - \partial_4 A_3), \\
A_{24} &= -\epsilon f_\theta (\partial_3 A_4 - \partial_4 A_3), \\
A_{34} &= \partial_3 A_4 - \partial_4 A_3 = \frac{V'}{r}.
\end{aligned}$$

Lemma 4.1. *There holds*

$$\begin{aligned}
-\Delta\psi &= 2\epsilon^2 (\sin 2s)^2 \cos 2s (\epsilon f_s \partial_3 \psi + \epsilon g_s \partial_4 \psi) + \epsilon^2 (\sin 2s)^3 (\epsilon f_{ss} \partial_3 \psi + \epsilon g_{ss} \partial_4 \psi) + \\
&\epsilon^2 \sin 2s (\epsilon f_{\theta\theta} \partial_3 \psi + \epsilon g_{\theta\theta} \partial_4 \psi) + 2b\epsilon^2 \sin 2s \cos 2s (\epsilon f_\theta \partial_3 \partial_3 \psi + \epsilon g_\theta \partial_3 \partial_4 \psi) + \\
&2a\epsilon^2 \sin 2s \cos 2s (-\epsilon f_\theta \partial_3 \partial_4 \psi - \epsilon g_\theta \partial_4 \partial_4 \psi) + 2a\epsilon^2 (\sin 2s)^3 \partial_3 \psi - \partial_3 \partial_3 \psi - \\
&b^2 \epsilon^2 \sin 2s (\cos 2s)^2 \partial_3 \partial_3 \psi + a\epsilon^2 \sin 2s (\cos 2s)^2 \partial_3 \psi + 2ab\epsilon^2 \sin 2s (\cos 2s)^2 \partial_3 \partial_4 \psi + \\
&b\epsilon^2 \sin 2s (\cos 2s)^2 \partial_4 \psi + 2b\epsilon^2 (\sin 2s)^3 \partial_4 \psi - \partial_4 \partial_4 \psi - a^2 \epsilon^2 \sin 2s (\cos 2s)^2 \partial_4 \partial_4 \psi + \\
&O(\epsilon^4).
\end{aligned}$$

Lemma 4.2. *We have:*

$$\begin{aligned}
d^* A &= \epsilon^2 \rho^{-6} \cdot (\epsilon f_{ss} \cdot A_3 + \epsilon g_{ss} \cdot A_4) + \epsilon^2 \rho^{-2} \cdot (\epsilon f_{\theta\theta} A_3 + \epsilon g_{\theta\theta} A_4) \\
&+ 2b \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta \partial_3 A_3 + b \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta (\partial_4 A_3 + \partial_3 A_4) \\
&- a \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta (\partial_3 A_4 + \partial_4 A_3) - 2a \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta \partial_4 A_4 \\
&- (-ab (\cos 2s)^2 \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \epsilon^3 \rho^{-5}) (\partial_4 A_3 + \partial_3 A_4) \\
&- (1 + b^2 (\cos 2s)^2 \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_3 A_3 \\
&- (1 + a^2 (\cos 2s)^2 \epsilon^2 \rho^{-2} - 2a^3 (\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_4 A_4 \\
&+ 2\epsilon^2 \rho^{-4} \cos 2s \cdot (\epsilon f_s A_3 + \epsilon g_s A_4) \\
&+ (2a\epsilon^2 \rho^{-6} + a (\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2) (\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_3 \\
&+ (2b\epsilon^2 \rho^{-6} + b (\cos 2s)^2 \epsilon^2 \rho^{-2} + 2ab (\cos 2s)^2 \epsilon^3 \rho^{-5} - 4ab\epsilon^3 \rho^{-9}) A_4 + O(\epsilon^4).
\end{aligned}$$

Lemma 4.3. *We have:*

$$\begin{aligned}
\langle A, d\psi \rangle &= -2b \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta A_3 \partial_3 \psi - b \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta (A_3 \partial_4 \psi + A_4 \partial_3 \psi) \\
&\quad + a \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta (A_4 \partial_3 \psi + A_3 \partial_4 \psi) + 2a \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta A_4 \partial_4 \psi \\
&\quad + (1 + b^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) A_3 \cdot \partial_3 \psi \\
&\quad + (-ab (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) (A_3 \partial_4 \psi + A_4 \partial_3 \psi) + \\
&\quad + (1 + a^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2a^3 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) A_4 \partial_4 \psi \\
&\quad + O(\epsilon^4).
\end{aligned}$$

Lemma 4.4. *If we denote $d^*dA = (d^*dA)_\tau dy^\tau$, then we have:*

$$\begin{aligned}
(d^*dA)_1 &= -\partial_3 A_{31} - \partial_4 A_{41} + O(\epsilon^2) \\
&= \cos \tilde{\phi} \cdot \epsilon g_s \left(\frac{V''}{\tilde{r}} - \frac{V'}{\tilde{r}^2} \right) - \sin \tilde{\phi} \cdot \epsilon f_s \left(\frac{V''}{\tilde{r}} - \frac{V'}{\tilde{r}^2} \right) \\
&\quad + O(\epsilon^2), \\
(d^*dA)_2 &= -\partial_3 A_{32} - \partial_4 A_{42} + O(\epsilon^2) \\
&= \cos \tilde{\phi} \cdot \epsilon g_\theta \left(\frac{V''}{r} - \frac{V'}{r^2} \right) - \sin \tilde{\phi} \cdot \epsilon f_\theta \left(\frac{V''}{r} - \frac{V'}{r^2} \right) \\
&\quad + O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
(d^*dA)_3 &= -\epsilon^2 \rho^{-6} \partial_1 A_{13} - \epsilon^2 \rho^{-2} \partial_2 A_{23} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_3 A_{23} \\
&\quad + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_4 A_{23} + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_2 A_{43} \\
&\quad - (-ab (\cos 2s)^2 \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_3 A_{43} \\
&\quad - (1 + a^2 (\cos 2s)^2 \epsilon^2 \rho^{-2} - 2a^3 (\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_4 A_{43} \\
&\quad - 2\epsilon^2 \rho^{-4} \cos 2s \cdot A_{13} + (\epsilon^4) \\
&\quad + (2b\epsilon^2 \rho^{-6} + b(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2ab(\cos 2s)^2 \epsilon^3 \rho^{-5} - 4ab\epsilon^3 \rho^{-9}) A_{43} + O(\epsilon^4),
\end{aligned}$$

$$\begin{aligned}
(d^*dA)_4 &= -\epsilon^2 \rho^{-6} \cdot \partial_1 A_{14} - \epsilon^2 \rho^{-2} \partial_2 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_3 A_{24} \\
&\quad + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_4 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_2 A_{34} \\
&\quad - (1 + b^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_3 A_{34} \\
&\quad - (-ab (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_4 A_{34} \\
&\quad + (-2\epsilon^2 \rho^{-4} \cos 2s) A_{14} + O(\epsilon^4) \\
&\quad + (2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2) (\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_{34} + O(\epsilon^4).
\end{aligned}$$

Lemma 4.5. *After calculation, we have:*

$$\begin{aligned} \text{Im}(\nabla_A \psi \cdot \bar{\psi}) &= (\epsilon f_s \cdot \frac{U^2 - VU^2}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_s \cdot \frac{U^2 - VU^2}{\tilde{r}} \cos \tilde{\phi}) dy^1 \\ &\quad + (\epsilon f_\theta \cdot \frac{U^2 - VU^2}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_\theta \cdot \frac{U^2 - VU^2}{\tilde{r}} \cos \tilde{\phi}) dy^2 \\ &\quad + (-\frac{U^2}{\tilde{r}} \sin \tilde{\phi} - A_3 |\psi|^2) dy^3 + (\frac{U^2}{\tilde{r}} \cos \tilde{\phi} - A_4 |\psi|^2) dy^4. \end{aligned}$$

We use the kernels $(\partial_k \psi - iA_k \psi, \partial_k A - \nabla A_k), k = 3, 4$, rather than $(\partial_k \psi, \partial_k A)$ because the inner products of them with the equations are invariant after gauge transformation.

Now we can prove Theorem 4.1:

Proof. By Lemma 4.1, we get:

$$\begin{aligned} &\langle -\Delta \psi, \partial_3 \psi - iA_3 \psi \rangle \\ &= [\epsilon^3 \rho^{-6} f_{ss} + \epsilon^3 \rho^{-2} f_{\theta\theta} + 2\epsilon^3 \rho^{-4} \cos 2s \cdot f_s - 2\epsilon^3 \rho^{-2} \cos 2s \cdot g_\theta \\ &\quad + 2\epsilon^3 \rho^{-6} f - \epsilon^3 \rho^{-2} (\cos 2s)^2 f] \cdot [\int_{\tilde{r}} \pi \tilde{r} (U')^2 d\tilde{r} + \pi \frac{V'}{\tilde{r}} |_{\tilde{r}=0}] \\ &\quad - (2\epsilon^3 \rho^{-2} \cos 2s \cdot g_\theta + 2\epsilon^3 \rho^{-2} (\cos 2s)^2 \cdot f) \cdot \int_{\tilde{r}} (-2\pi U U' + \pi V U U') d\tilde{r} + O(\epsilon^4). \end{aligned}$$

On the other hand, from Lemma 4.2, we deduce

$$\begin{aligned} &\langle -i d^* A \psi, \partial_3 \psi - iA_3 \psi \rangle = \int_{r, \phi} d^* A \left(\frac{U^2}{\tilde{r}} - \frac{VU^2}{\tilde{r}} \right) \sin \tilde{\phi} \\ &= \int_{r, \phi} d^* A \cdot \frac{\sin \tilde{\phi}}{\tilde{r}} \left(-V'' + \frac{V'}{\tilde{r}} \right) \\ &= \int_{\tilde{r}} \frac{\pi (V')^2}{\tilde{r}} \cdot (-\epsilon^3 \rho^{-6} f_{ss} - \epsilon^3 \rho^{-2} f_{\theta\theta} - 2\epsilon^3 \rho^{-4} \cos 2s \cdot f_s - 2\epsilon^3 \rho^{-6} f + \epsilon^3 \rho^{-2} \cos 2s \cdot g_\theta) \\ &\quad + \int_{\tilde{r}} 4\pi \tilde{r} V \left(-V'' + \frac{V'}{\tilde{r}} \right) (\cos 2s)^2 \epsilon^3 \rho^{-5} - \int_{\tilde{r}} 4\pi \tilde{r} V \epsilon^3 \rho^{-9} \left(-V'' + \frac{V'}{\tilde{r}} \right) + O(\epsilon^4). \end{aligned}$$

By lemma 4.3, we have:

$$\begin{aligned} &\langle 2i(A, d\psi), \partial_3 \psi - iA_3 \psi \rangle \\ &= \int_{\tilde{r}} \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta \cdot (-6\pi V U U' + 2\pi V^2 U U') \\ &\quad + \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-2} f \cdot (-6\pi V U U' + 2\pi V^2 U U') \\ &\quad + \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-5} \cdot 4\pi \tilde{r}^2 V U U' + O(\epsilon^4). \end{aligned}$$

By direct calculation and using the metric matrix of Fermi coordinate, we get:

$$\begin{aligned}
& \langle |A|^2 \psi, \partial_3 \psi - i A_3 \psi \rangle = \int_{\tilde{r}, \tilde{\phi}} |A|^2 U U' \cos \tilde{\phi} \\
& = \int_{\tilde{r}, \tilde{\phi}} [2A_2 A_3 \cdot b \cos 2s \cdot \epsilon^2 \rho^{-2} - 2A_2 A_4 \cdot a \cos 2s \cdot \epsilon^2 \rho^{-2} \\
& + A_3^2 (1 + b^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \\
& + 2A_3 A_4 (-ab (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \\
& + A_4^2 (1 + a^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2a^3 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5})] \cdot U U' \cos \tilde{\phi} \\
& = \int_{\tilde{r}} 2\pi V^2 U U' \cdot (\cos 2s)^2 \cdot \epsilon^3 \rho^{-2} f - \int_{\tilde{r}} 2\pi \tilde{r}^2 V^2 U U' (\cos 2s)^2 \epsilon^3 \rho^{-5} \\
& + \int_{\tilde{r}} 2\pi V^2 U U' \cdot \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta + O(\epsilon^4).
\end{aligned}$$

From Lemma 4.4 and Lemma 4.5, we can get that:

$$\begin{aligned}
& \langle d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}), \partial_3 A - \nabla A_3 \rangle \\
& = \int (\partial_3 A_4 - \partial_4 A_3) \cdot [-\epsilon^2 \rho^{-6} \cdot \partial_1 A_{14} - \epsilon^2 \rho^{-2} \partial_2 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_3 A_{24} \\
& + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_4 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_2 A_{34} \\
& - (1 + b^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_3 A_{34} \\
& - (-ab (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_4 A_{34} \\
& + (-2\epsilon^2 \rho^{-4} \cos 2s) A_{14} \\
& + (2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2)(\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_{34}] \\
& = \int (\frac{V'}{\tilde{r}}) \cdot [\epsilon^3 \rho^{-6} f_{ss} \cdot \frac{V'}{\tilde{r}} + \epsilon^3 \rho^{-2} f_{\theta\theta} \cdot \frac{V'}{\tilde{r}} + b \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta \sin \phi \cdot (\frac{V'}{\tilde{r}})' \\
& - (1 + b^2 (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2 (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_3 A_{34} \\
& - (-ab (\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b (\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_4 A_{34} \\
& + (2\epsilon^3 \rho^{-4} \cos 2s) f_s \cdot \frac{V'}{\tilde{r}} \\
& + (2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2)(\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_{34}] \\
& = \int_{\tilde{r}} \pi \epsilon^3 \frac{(V')^2}{\tilde{r}} \cdot (2\rho^{-6} f_{ss} + 2\rho^{-2} f_{\theta\theta} + 4\rho^{-4} \cos 2s \cdot f_s \\
& - \rho^{-2} \cos 2s \cdot g_\theta + 4\rho^{-6} f + \rho^{-2} (\cos 2s)^2 f - 6(\cos 2s)^2 \rho^{-5} \tilde{r}^2 + 4\rho^{-9} \tilde{r}^2) + O(\epsilon^4).
\end{aligned}$$

Finally, from the above results, we deduce:

$$\begin{aligned}
& \langle -\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi, \partial_3 \psi - iA_3 \psi \rangle - \langle d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}), \partial_3 A - \nabla A_3 \rangle \\
&= [\epsilon^3 \rho^{-6} f_{ss} + \epsilon^3 \rho^{-2} f_{\theta\theta} + 2\epsilon^3 \rho^{-4} \cos 2s \cdot f_s - 2\epsilon^3 \rho^{-2} \cos 2s \cdot g_\theta \\
&+ 2\epsilon^3 \rho^{-6} f - \epsilon^3 \rho^{-2} (\cos 2s)^2 f] \cdot \left[\int \pi \tilde{r} (U')^2 d\tilde{r} + \pi \frac{V'}{\tilde{r}} \Big|_{\tilde{r}=0} - 3\pi \frac{(V')^2}{\tilde{r}} \right] \\
&+ \int_{\tilde{r}} \pi \epsilon^3 \frac{(V')^2}{\tilde{r}} \cdot (6(\cos 2s)^2 \rho^{-5} \tilde{r}^2 - 4\rho^{-9} \tilde{r}^2) \\
&+ \int_{\tilde{r}} 4\pi \tilde{r} V (-V'' + \frac{V'}{\tilde{r}}) (\cos 2s)^2 \epsilon^3 \rho^{-5} - \int_{\tilde{r}} 4\pi \tilde{r} V \epsilon^3 \rho^{-9} (-V'' + \frac{V'}{\tilde{r}}) \\
&+ \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-5} \cdot 4\pi \tilde{r}^2 V U U' - \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-5} \cdot 2\pi \tilde{r}^2 V^2 U U' + O(\epsilon^4).
\end{aligned}$$

□

Noting the exponential decay of U and V , we can define the cut-off function χ :

$$\begin{cases} \chi = 1, & \text{if } a^2 + b^2 \leq \frac{1}{4\epsilon^2 \sin 2s}, \\ \chi = 0, & \text{if } a^2 + b^2 \geq \frac{1}{2\epsilon^2 \sin 2s}, \end{cases}$$

and get the conclusion:

Corollary 4.1. *When ϵ is small enough, the equations*

$$\begin{cases} \langle F(u), \chi(\partial_3 \psi - iA_3 \psi, -\partial_3 A + \nabla A_3) \rangle = 0, \\ \langle F(u), \chi(\partial_4 \psi - iA_4 \psi, -\partial_4 A + \nabla A_4) \rangle = 0, \end{cases}$$

has the form

$$L_H(f, g) = (M_1, M_2).$$

$M_1, M_2 = O(\epsilon)$ is of higher order in f, g .

We remark that the principal curvatures of Γ_ϵ in $\pm\epsilon(\sin 2s)^{\frac{3}{2}}$. Indeed,

$$\begin{aligned}
\Gamma_\epsilon &= \left(\frac{\cos s}{\epsilon\sqrt{\sin 2s}} \Theta, \frac{\sin s}{\epsilon\sqrt{\sin 2s}} \Theta \right), \\
e_1 &= e^{-is} \Theta = -\epsilon(\sin 2s)^{\frac{3}{2}} \partial_s, \quad e_2 = e^{is} \Theta^\perp = \epsilon\sqrt{\sin 2s} \cdot \partial_\theta, \\
m &= ie^{-is}, \quad n = ie^{is} \Theta^\perp, \quad \nabla_{e_1} m = -\epsilon(\sin 2s)^{\frac{3}{2}} e_1, \\
\nabla_{e_2} m &= \epsilon\sqrt{\sin 2s} (\sin 2s \cdot e_2 + \cos 2s \cdot n), \\
\nabla_{e_1} n &= \epsilon(\sin 2s)^{\frac{3}{2}} e_2, \\
\nabla_{e_2} n &= -\epsilon\sqrt{\sin 2s} (-\sin 2s \cdot e_1 + \cos 2s \cdot m).
\end{aligned}$$

So we observe that

$$\begin{aligned}\nabla_{e_1+e_2}^\tau n &= \epsilon(\sin 2s)^{\frac{3}{2}}(e_1 + e_2), \\ \nabla_{e_1-e_2}^\tau n &= -\epsilon(\sin 2s)^{\frac{3}{2}}(e_1 - e_2).\end{aligned}$$

For any normal vector $\nu = am + bn$, $a^2 + b^2 = 1$, we have

$$\begin{aligned}\nabla_{e_1}^\tau \nu &= \epsilon(\sin 2s)^{\frac{3}{2}}(ae_1 + be_2), \\ \nabla_{e_2}^\tau \nu &= \epsilon(\sin 2s)^{\frac{3}{2}}(be_1 - ae_2).\end{aligned}$$

Because

$$\det \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -a^2 - b^2 = -1.$$

and mean curvature is 0, we find that for any normal direction ν , the principal curvature is always $\pm\epsilon(\sin 2s)^{\frac{3}{2}}$.

5 The approximation solution and errors

In this section, we will give the approximation solution and calculate the errors of it. Recall in section 1, we have already defined the solution when it is near Γ_ϵ : $\psi = \psi^{(1)}(a-\epsilon f, b-\epsilon g)$, $A = A^{(1)}(a-\epsilon f, b-\epsilon g)$. We hope in the whole space, the approximation solution has the form $u = (\psi, A) = (We^{i\varphi}, Z\nabla\varphi)$.

Firstly, we will define W and Z :

$$\begin{aligned}W &= \begin{cases} U(a - \epsilon f, b - \epsilon g), & r \leq \frac{1}{4\epsilon\sqrt{\sin 2s}}, \\ 1, & r \geq \frac{1}{2\epsilon\sqrt{\sin 2s}}, \end{cases} \\ Z &= \begin{cases} V(a - \epsilon f, b - \epsilon g), & r \leq \frac{1}{4\epsilon\sqrt{\sin 2s}}, \\ 1, & r \geq \frac{1}{2\epsilon\sqrt{\sin 2s}}. \end{cases}\end{aligned}$$

In fact, let ζ_1, ζ_2 be a smooth partition of unity on \mathbb{R}^4 , subordinate to the open sets $\{r < \frac{1}{2\epsilon\sqrt{\sin 2s}}\}, \{r > \frac{1}{4\epsilon\sqrt{\sin 2s}}\}$. Let $W = \zeta_1 U + \zeta_2$, $Z = \zeta_1 V + \zeta_2$ then we have

$$W' = O(e^{-m\lambda r}), \quad Z' = O(e^{-r}), \quad \text{when } r > 2.$$

Following [17], we will "extend" the angle function ϕ to \mathbb{R}^4 in the following way. We first consider the following equation:

$$\Delta\omega = 0, \quad \text{in } \mathbb{R}^4 \setminus \Gamma. \tag{16}$$

ω is a 1-form and satisfies for any oriented closed curve C , $\int_C \omega = 2\pi$ (the winding number of C around Γ). Then we follow the thoughts of Lin F.H. and Rivière. T in [17] . Denote:

$G(x) = \frac{1}{8\omega_4|x|^2}$, the fundamental solution of the laplacian operator in \mathbb{R}^4 .

$\omega_4 =$ the volume of $B_1 \subset \mathbb{R}^4$.

$\Omega = \mathbb{R}^4 \setminus \Gamma$.

For a locally integrable k -form, we have a $(4 - k)$ -current T_ω given by $T_\omega(\tau) = \int_\Omega \omega \wedge \tau$, for any $\tau \in D^{4-k}(\Omega)$. For any k -current T on Ω , σT is a $(k + 1)$ -current defined by $(\sigma T)(\tau) = T(\delta\tau)$, for any $\tau \in D^{k+1}(\Omega)$. And for any k -current T on Ω , ∂T is a $(k - 1)$ -current defined by $(\partial T)(\tau) = T(d\tau)$, for any $\tau \in D^{k-1}(\Omega)$.

We can get such a 1-form ω :

$$\begin{cases} \sigma(T_\omega) = 0, \\ \partial(T_\omega) = 2\pi\Gamma, \end{cases} \quad (17)$$

and for any oriented closed curve C , $\int_C \omega = 2\pi$ (the winding number of C around Γ).

In fact, if we suppose for $x \in \Gamma$, $e_1(x), \dots, e_4(x)$ is a positive orthonormal base of \mathbb{R}^4 , s.t. $e_1(x), e_2(x)$ is a positive base for $T_x\Gamma$. Let $e^1(x), \dots, e^4(x)$ be the dual base, then we call $\chi(x) = e^1(x) \wedge e^2(x)$ the orientation form of Γ .

Let

$$\omega(x) = -2\pi \int_\Gamma * [d_x(G(x-y)) \wedge \chi(y)] dH^2(y),$$

for $x \in \mathbb{R}^4$.

From (5), we know

$$\begin{cases} \delta\omega = 0, \\ \delta d\omega = 0. \end{cases} \quad (18)$$

So $\Delta\omega = 0$.

Then, we have $\int_C \omega - d\phi = 0$, for any oriented closed curve $C \subset \{\rho_1 = \rho_2\}$, so it is equal to dh for some smooth function h . It follows from the growth of ω and $d\phi$ near Γ that dh is $C^{0,1}$ on $\{\rho_1 = \rho_2\}$. Observe that $\omega - d\phi$ satisfies:

$$\begin{cases} \Delta(\omega - d\phi) = -d(\Delta\phi), \text{ in } \{\rho_1 \leq \rho_2\}, \\ (\omega - d\phi)_\top = dh, \text{ on } \{\rho_1 = \rho_2\}, \\ (\delta(\omega - d\phi))_\top = \Delta\phi, \text{ on } \{\rho_1 = \rho_2\}. \end{cases} \quad (19)$$

Let us solve the Dirichlet problem

$$\begin{cases} \Delta\tilde{\varphi}_1 = -\Delta\phi, \text{ in } \{\rho_1 < \rho_2\}, \\ \tilde{\varphi}_1|_{\{\rho_1=\rho_2\}} = h, \text{ in } \{\rho_1 \leq \rho_2\}. \end{cases}$$

Then $d\tilde{\varphi}_1$ satisfies (9), so $\omega = d\phi + d\tilde{\varphi}_1$. Similarly, we can get $\omega = d\phi + d\tilde{\varphi}_2$ in $\{\rho_1 \geq \rho_2\}$, where $\tilde{\varphi}_2$ satisfies:

$$\begin{cases} \Delta\tilde{\varphi}_2 = -\Delta\phi, & \text{in } \{\rho_1 > \rho_2\}, \\ \tilde{\varphi}_2|_{\{\rho_1=\rho_2\}} = h, & \text{in } \{\rho_1 \geq \rho_2\}. \end{cases}$$

Denote $\tilde{\varphi} =$

$$\begin{cases} \phi + \tilde{\varphi}_1, & \text{in } \{\rho_1 \leq \rho_2\}, \\ \phi + \tilde{\varphi}_2, & \text{in } \{\rho_1 \geq \rho_2\}. \end{cases}$$

then $\tilde{\varphi}$ is a solution of (5), and $d\tilde{\varphi} = \omega \in C^\infty(\mathbb{R}^4 \setminus \Gamma)$, $\tilde{\varphi} \in C_{loc}^\infty(\mathbb{R}^4 \setminus \Gamma)$.

We have the following results:

Lemma 5.1. *When it is on $\{\rho_1 = \rho_2\}$, we have*

$$\begin{aligned} \omega_\top(x) &= \frac{\pi}{8\omega_4} \int_{\rho \geq 1, 0 \leq \theta < 2\pi} |x - y|^{-4} \cdot \{[-2\rho^{-3}\rho_1^2 + 2\sqrt{2}\rho^{-2}\rho_1 \cos(\theta - \theta_1)] \\ &+ 2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2)]d\theta_1(x) \\ &+ [2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2) - 2\rho_1^2\rho^{-3} + 2\sqrt{2}\rho_1\rho^{-2} \cos(\theta - \theta_1)]d\theta_2(x)\}d\rho d\theta. \end{aligned}$$

The proof will be given in the Appendix. Next we estimate $\tilde{\varphi}_1$.

Theorem 5.1. $|\tilde{\varphi}_1| < \infty$ in \mathbb{R}^4 .

Proof. Firstly, we will estimate $\Delta\phi$. By previous arguments, we know that when ρ is large enough, $\Delta\phi = O(\rho^{-4})$. Besides, ϕ is smooth away from Γ . So we only need to estimate $\Delta\phi$ near Γ in B_R .

$$\begin{aligned} \Delta\phi &= (w_{3,3} + w_{3,4} + w_{4,3} + w_{4,4})\phi \\ &= 2b(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(r^{-2}). \end{aligned}$$

Therefore, $|\Delta\phi|_{C^{0,\alpha}} < \infty$.

Next we will show that $|h| < \infty$.

By Lemma 5.1, we can get that in $\{\rho_1 = \rho_2\}$, $\omega(x) \in \text{span}\{d\rho_1 - d\rho_2, d\theta_1 + d\theta_2\}$. Because of $(\omega - d\phi)_\top = dh$ on $\{\rho_1 = \rho_2\}$, we know:

$$|h_{\theta_1}| \leq C, |h_{\theta_2}| \leq C, |h_{\rho_1}| = 0.$$

If we fix $h = 0$ at $(1, 0, 1, 0)$, then $h < \infty$ on $\{\rho_1 = \rho_2\}$. So by standard elliptic estimates, we have $|\tilde{\varphi}_1| < \infty$. □

Therefore, we know that the same results hold for $\tilde{\varphi}_2$ and $\tilde{\varphi}$.

After rescaling, we can get similar results for Γ_ϵ . For simplicity, we still denote it by $\tilde{\varphi}$.

$$\varphi = \zeta_1 \phi(a - \epsilon f, b - \epsilon g) + \zeta_2 \tilde{\varphi}.$$

In order to estimate $F(u)$, we define the following weighted norm:

$$\|h\|_{**} = \sup_P \rho(P)^2 e^{\delta r} \|h\|_{L^q(B_1(P))},$$

where $1 < q < 2$.

For 1-form B , we define:

$$\|(B, h)\|_{**} = \|B\|_{**} + \|h\|_{**}.$$

In order to control f, g , we define the following norm $W_k^{j,q}$:

$$\begin{aligned} \|f\|_{0,k,q} &:= \sup_{P \in \Gamma_\epsilon} \rho(P)^k \|f\|_{L^q(\rho(P) \leq \rho \leq \rho(P)+1)}, \\ \|f\|_{2,k,q} &:= \sup_{P \in \Gamma_\epsilon} \rho(P)^k \|f\|_{L^q(\rho(P) \leq \rho \leq \rho(P)+1)} + \sup_{P \in \Gamma_\epsilon} \rho(P)^{k+1} \|\nabla f\|_{L^q(\rho(P) \leq \rho \leq \rho(P)+1)} \\ &\quad + \sup_{P \in \Gamma_\epsilon} \rho(P)^{k+2} \|\nabla^2 f\|_{L^q(\rho(P) \leq \rho \leq \rho(P)+1)}, \\ \|(f, g)\|_{2,k,q} &:= \|f\|_{2,k,q} + \|g\|_{2,k,q}. \end{aligned}$$

And we always assume that

$$\|(f, g)\|_{2,1,q} \leq C. \tag{20}$$

Now, we have the following results for the error:

Theorem 5.2.

$$\|F(u)\|_{**} < C\epsilon^2. \tag{21}$$

Proof. • 1. When $r > \frac{1}{2\epsilon\sqrt{\sin 2s}}$, we can get

$$F(u) = 0.$$

• 2. When $\frac{1}{4\epsilon\sqrt{\sin 2s}} < r < \frac{1}{2\epsilon\sqrt{\sin 2s}}$,

$$1 - Z, 1 - W = O(e^{-\delta r}), \text{ where } 0 < \delta < \min\{m_\lambda, 1\}.$$

Thus we can get that

$$F(u) = O(e^{-\delta r}).$$

- 3. When $2 < r < \frac{1}{4\epsilon\sqrt{\sin 2s}}$, then

$$W = U, Z = V, \varphi = \phi.$$

Similarly, we know at least $F(u) = O(e^{-\delta r})$. So if we want to get more accurate estimates, we only need to consider the exponential decay items in $F(u)$, which we have already calculated in section 1. Then we get the errors:

$$F(u) = O(\epsilon^2 \rho^{-2} e^{-\delta r}).$$

- 4. When $r < 2$, observe that the approximate solution may not be C^∞ smooth near Γ_ϵ . For example, when it is close to Γ_ϵ , we have:

$$\Delta\psi = \Delta(We^{i\varphi}) = \Delta W \cdot e^{i\varphi} - W e^{i\varphi} \cdot |\nabla\varphi|^2 + 2ie^{i\varphi} \nabla W \cdot \nabla\varphi + ie^{i\varphi} W \Delta\varphi.$$

Then the item $W e^{i\varphi} \cdot |\nabla\varphi|^2 = O(\tilde{r}^{-1})$. This is different from the case in Allen-Cahn equation. As a result, we use the L^p norm rather than Hölder norm.

We only need to care about the $O(\tilde{r}^{-1})$ items, and get:

$$\begin{aligned} -\Delta_A \psi + \frac{\lambda}{2} (|\psi|^2 - 1)\psi &= O(\epsilon^2 \rho^{-2} \tilde{r}^{-1}), \\ d^* dA - \text{Im}(\nabla_A \psi \cdot \bar{\psi}) &= O(\epsilon^2 \rho^{-2}). \end{aligned}$$

Therefore, we get that:

$$\|F(u)\|_{**} < C\epsilon^2.$$

□

Remark: We distinguish r and \tilde{r} only when it is close to Γ_ϵ . Otherwise, we see them as the same.

6 The linearized operator

From now on, we will apply standard infinite dimensional Lyapunov-Schmidt reduction. See [19], [7].

Denote L as the linearization of the Ginzburg-Landau equations at an approximate solution $u = (A, \psi)$. Then

$$L(B, \eta) = \left(\begin{array}{c} d^* dB - \text{Im}(\nabla\psi \cdot \bar{\eta} + \nabla\eta \cdot \bar{\psi}) + A(\bar{\psi}\eta + \psi\bar{\eta}) + B|\psi|^2 \\ -\Delta_A \eta - id^* B \cdot \psi + 2i \langle B, d\psi \rangle + 2 \langle A, B \rangle \psi + \frac{\lambda}{2} (\psi^2 \bar{\eta} + 2|\psi|^2 \eta - \eta) \end{array} \right).$$

Define $v = (\tilde{A}, \tilde{\psi}) := u + w = (A, \psi) + (B, \eta)$. Note that the first equation is not elliptic, so we always need to add some items so that the new operator is uniformly elliptic.

Denote the new operator $T : \Omega^0(\mathbb{R}^4, \mathbb{R}) \rightarrow \Omega^1(\mathbb{R}^4, \mathbb{R}) \oplus \Omega^0(\mathbb{R}^4, \mathbb{C})$:

$$T(v) = (dv, iv\tilde{\psi}).$$

We remark that in fact that the operator is the kernel of L induced by the gauge transformation:

$$\begin{aligned}\psi &\rightarrow e^{iv}\psi, \\ A &\rightarrow A + \nabla v.\end{aligned}$$

The adjoint of T is given by:

$$T^*(B, \eta) = d^*B - \text{Im}(\bar{\eta}\tilde{\psi}).$$

And

$$TT^*(B, \eta) = \begin{pmatrix} dd^*B - d[\text{Im}(\bar{\eta}\psi)] \\ i\tilde{\psi} \cdot (d^*B - \text{Im}(\bar{\eta}\psi)) \end{pmatrix}.$$

Finally we define $\mathbb{L}(B, \eta) := L(B, \eta) + TT^*(B, \eta)$:

$$\mathbb{L}(B, \eta) = \begin{pmatrix} -\Delta B + 2\text{Im}(\overline{\nabla_A \psi} \cdot \eta) + B|\psi|^2 \\ -\Delta_A \eta + 2i \langle B, \nabla_A \psi \rangle + \frac{1}{2}(\lambda - 1)\psi^2 \bar{\eta} + (\lambda + \frac{1}{2})|\psi|^2 \eta - \frac{\lambda}{2}\eta \end{pmatrix}.$$

Here we remark that, we only use the linear part of $TT^*(B, \eta)$ and the nonlinear part will be added to the following $N(w)$ part.

So by (21), we can define the following weighted norm:

$$\|\eta\|_* = \sup_P \rho(P)^2 e^{\delta r} \|\eta\|_{W^{2,q}(B_1(P))},$$

where $1 < q < 2$.

Similarly, we define:

$$\|(B, \eta)\|_* = \|B\|_* + \|\eta\|_*.$$

We hope the "true" solution of the Ginzburg-Landau equation is $u + w := (A, \psi) + (B, \eta)$, then we get the new equations for $w = (B, \eta)$:

$$F(u + w) + TT^*(B, \eta) = F(u) + \mathbb{L}w + N(w) = 0,$$

where

$$N(w) = \begin{pmatrix} -\text{Im}(\bar{\eta}\nabla_A \eta) + B(2\text{Re}(\bar{\psi}\eta) + |\eta|^2) \\ \lambda(2\psi\bar{\eta} + \bar{\psi}\eta + |\eta|^2)\eta + |B|^2(\psi + \eta) + i[-\eta\text{Im}(\bar{\eta}\psi) + 2B \cdot \nabla_A \eta] \end{pmatrix}. \quad (22)$$

We now recall the nondegeneracy property of the linearized operator.

Theorem 6.1 (See [12]). *For all $\lambda > 0$, the ± 1 -vortex is stable.*

Naturally, we then consider the linearized operator in the $\mathbb{R}^2 \times \mathbb{R}^2$ case. By Fourier transform, we can get:

Corollary 6.1. *Let $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ and denote $(A^{(1)}, \psi^{(1)})$ the $+1$ vortex solution to the Ginzburg-Landau equations. Define the linear operator*

$$L^{(1)}(B, \eta) := \begin{pmatrix} -\Delta B + 2\text{Im}(\overline{\nabla_A \psi} \cdot \eta) + B|\psi|^2 \\ -\Delta_A \eta + 2i \langle B, \nabla_A \psi \rangle + \frac{1}{2}(\lambda - 1)\psi^2 \bar{\eta} + (\lambda + \frac{1}{2})|\psi|^2 \eta - \frac{\lambda}{2}\eta \end{pmatrix}.$$

where $(A, \psi) = (A^{(1)}(y), \psi^{(1)}(y))$. Then if $\|(B, \eta)\|'_* < \infty$, we have $(B, \eta) = 0$.

Here $\|\eta\|'_* := \sup_P e^{\delta|y|} \|\eta\|_{W^{2,q}(B_1(P))}$. With this, we can give the most important result in this section.

Theorem 6.2. *Let $\epsilon > 0$ be small. Suppose $\|(B, \eta)\|_* < \infty$, $\|(D, h)\|_{**} < \infty$, and*

$$\begin{cases} \mathbb{L}(B, \eta) = (D, h), & (23) \end{cases}$$

$$\begin{cases} \int_{r,\phi} \zeta_1 \langle B, \partial_3 A - \nabla A_3 \rangle + \zeta_1 \text{Re}[\bar{\eta}(\partial_3 \psi - iA_3 \psi)] = 0, & (24) \end{cases}$$

$$\begin{cases} \int_{r,\phi} \zeta_1 \langle B, \partial_4 A - \nabla A_4 \rangle + \zeta_1 \text{Re}[\bar{\eta}(\partial_4 \psi - iA_4 \psi)] = 0, & (25) \end{cases}$$

for any (s, θ) . Then $\|(B, \eta)\|_{**} \leq C\|(D, h)\|_*$, where C is independent of ϵ , D and h .

Proof. By L^p theory, it suffices to show that

$$\begin{aligned} & \sup_P \rho^2 e^{\delta r} (\|B\|_{L^q(B_1(P))} + \|\eta\|_{L^q(B_1(P))}) \\ & \leq C \sup_P \rho^2 e^{\delta r} (\|D\|_{L^q(B_2(P))} + \|h\|_{L^q(B_2(P))}). \end{aligned}$$

To prove this, we argue by contradiction and assume that there exists a sequence ϵ_j , $(B^{(j)}, \eta^{(j)})$, $(D^{(j)}, h^{(j)})$, such that:

$$\sup_P \rho^2 e^{\delta r} (\|B^{(j)}\|_{L^q(B_1(P))} + \|\eta^{(j)}\|_{L^q(B_1(P))}) = 1,$$

$$\sup_P \rho^2 e^{\delta r} (\|D^{(j)}\|_{L^q(B_2(P))} + \|h^{(j)}\|_{L^q(B_2(P))}) \rightarrow 0,$$

where $\mathbb{L}(B^{(j)}, \eta^{(j)}) = (D^{(j)}, h^{(j)})$.

By the decay of $(B^{(j)}, \eta^{(j)})$, we know that $(B^{(j)}, \eta^{(j)}) \in W^{2,q}(\mathbb{R}^4) \subset \subset H^1(\mathbb{R}^4)$. So there exists a subsequence, such that $(B^{(j)}, \eta^{(j)})$ converges to some $(B^{(0)}, \eta^{(0)})$ in $H^1(\mathbb{R}^4)$. For simplicity, we still denote them as $(B^{(j)}, \eta^{(j)})$.

So for any j , there exists P_j , such that:

$$\|B^{(j)}\|_{L^q(B_1(P_j))} + \|\eta^{(j)}\|_{L^q(B_1(P_j))} \geq \frac{1}{2}\rho^{-2}e^{-\delta r}(P_j).$$

As a result, there are three cases for the convergence of $r(P_j)$:

- 1. $r(P_j)$ will converge to some finite constant $r_0 > 0$, then we can always assume P_j converges to P_0 . At this case, we assume that $x_j \in \Gamma_{\epsilon_j}$ and $P_j = (x_j, y_j)$, $y_j \in \mathbb{R}^2$ and define:

$$\begin{aligned}\tilde{B}^{(j)}(x, y) &= \rho(P_j)^2 B^{(j)}(x + x_j, y), \\ \tilde{\eta}^{(j)}(x, y) &= \rho(P_j)^2 \eta^{(j)}(x + x_j, y),\end{aligned}$$

then we have

$$\|\tilde{B}^{(0)}\|_{L^q(B_1(0, y_0))} + \|\tilde{\eta}^{(0)}\|_{L^q(B_1(0, y_0))} \geq \frac{1}{4}e^{-\delta r_0}(0, y_0). \quad (26)$$

Then let $j \rightarrow \infty$, we get:

$$\lim_{j \rightarrow \infty} \mathbb{L}(\tilde{B}^{(j)}, \tilde{\eta}^{(j)}) = L^{(1)}(\tilde{B}^{(0)}, \tilde{\eta}^{(0)}) = 0, \text{ in } \mathbb{R}^2 \times \mathbb{R}^2.$$

And because of (24) and (25), as well as Corollary 6.1, we know that $(\tilde{B}^{(0)}, \tilde{\eta}^{(0)}) = 0$, which contradicts to (26).

- 2. $r(P_j)$ will tend to infinity. We assume that $x_j \in \Gamma_{\epsilon_j}$ and $P = (x_j, y_j)$, $y_j \in \mathbb{R}^2$ and define a sequence of pairs $(\tilde{B}^{(j)}, \tilde{\eta}^{(j)})$:

$$\begin{aligned}\tilde{B}^{(j)}(x, y) &= \rho(P_j)^2 e^{\delta r(P_j)} \tilde{B}^{(j)}(x + x_j, y + y_j), \\ \tilde{\eta}^{(j)}(x, y) &= \rho(P_j)^2 e^{\delta r(P_j)} \tilde{\eta}^{(j)}(x + x_j, y + y_j).\end{aligned}$$

Then $(\tilde{B}^{(j)}, \tilde{\eta}^{(j)})$ will converge to some $(\tilde{B}^{(0)}, \tilde{\eta}^{(0)})$, which satisfies:

$$\begin{cases} -\Delta B + B = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}^2, \\ -\Delta \eta + \frac{1}{2}(\lambda - 1)\tilde{\eta} + \frac{1}{2}(\lambda + 1)\eta = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}^2, \\ \sup_P e^{-\delta|y|}(\|B\|_{L^q(B_3(P))} + \|\eta\|_{L^q(B_3(P))}) \leq 1, & \text{in } \mathbb{R}^2 \times \mathbb{R}^2. \end{cases}$$

Thus $(\tilde{B}^{(0)}, \tilde{\eta}^{(0)}) = 0$. But we have:

$$(\|\tilde{B}^{(0)}\|_{L^q(B_3(0))} + \|\tilde{\eta}^{(0)}\|_{L^q(B_3(0))}) > \frac{1}{4}.$$

This is a contradiction. □

Denote $T_1 = \zeta_1(\partial_3 A - \nabla A_3, \partial_3 \psi - i A_3 \psi)$, $T_2 = \zeta_1(\partial_4 A - \nabla A_4, \partial_4 \psi - i A_4 \psi)$. Then we have:

$$\int_{r,\phi} \langle T_1, T_2 \rangle = O(\epsilon^2 \rho^{-2}).$$

Define orthogonal projections:

$$\begin{aligned} \pi &:= L^2 - \text{orthogonal projection onto } \text{span}\{T_1, T_2\}, \\ \pi^\perp &:= 1 - \pi. \end{aligned}$$

Then we will solve the projected linear problem:

Theorem 6.3. *If $\|(D, h)\|_{**} < \infty$ and $\pi(D, h) = 0$, then for ϵ sufficiently small there exists a unique solution (B, η) to*

$$\begin{cases} \pi^\perp \mathbb{L}(B, \eta) = (D, h), \\ \int_{r,\phi} \langle (B, \eta), T_j \rangle = 0, \text{ for } j = 1, 2, \end{cases} \quad (27)$$

and we have

$$\|(B, \eta)\|_* \leq C \|(D, h)\|_{**},$$

with C only depending on q, γ, δ .

Proof. The proof is standard. We first solve the equation in bounded domains.

$$\begin{cases} \pi^\perp \mathbb{L}(B, \eta) = (D, h) \text{ in } B_M(0), \\ B, \eta = 0 \text{ on } \partial B_M(0). \end{cases} \quad (28)$$

And define the space

$$\mathcal{H} := \{w = (B, \eta) : B, \eta \in H_0^1(B_M), \pi^\perp(B, \eta) = 0\}.$$

We equip \mathcal{H} with the inner product:

$$[w_1, w_2] := \int_{B_M(0)} \langle \nabla B_1, \nabla B_2 \rangle + \text{Re} \int_{B_M(0)} \nabla \eta_1 \cdot \nabla \eta_2.$$

Then \mathcal{H} is a Hilbert space.

The equation can be rewritten in its variational form. Namely, for any $\tilde{w} = (\tilde{B}, \tilde{\eta}) \in \mathcal{H}$:

$$\begin{aligned}
& \int_{B_M(0)} \langle \nabla B, \nabla \tilde{B} \rangle + \operatorname{Re} \int_{B_M(0)} \overline{\nabla_A \eta} \cdot \nabla_A \tilde{\eta} \\
& + \int_{B_M(0)} \langle 2\operatorname{Im}(\overline{\nabla_A \psi} \cdot \eta), \tilde{B} \rangle + \int_{B_M(0)} \langle B|\psi|^2, \tilde{B} \rangle - \operatorname{Re} \int_{B_M(0)} 2i \langle B, \overline{\nabla_A \psi} \rangle \tilde{\eta} \\
& + \operatorname{Re} \int_{B_M(0)} \frac{1}{2}(\lambda - 1)\bar{\psi}^2 \eta \tilde{\eta} + \operatorname{Re} \int_{B_M(0)} (\lambda + \frac{1}{2})|\psi|^2 \eta \tilde{\eta} - \operatorname{Re} \int_{B_M(0)} \frac{\lambda}{2} \bar{\eta} \tilde{\eta} \\
& = \int_{B_M(0)} \langle D, \tilde{B} \rangle + \operatorname{Re} \int_{B_M(0)} \bar{h} \tilde{\eta}.
\end{aligned}$$

Then we define the linear form $\langle k(x)w, \cdot \rangle$ on \mathcal{H} :

$$\begin{aligned}
\langle k(x)w, \tilde{w} \rangle := & \int_{B_M(0)} \langle 2\operatorname{Im}(\overline{\nabla_A \psi} \cdot \eta), \tilde{B} \rangle + \int_{B_M(0)} \langle B|\psi|^2, \tilde{B} \rangle \\
& - \operatorname{Re} \int_{B_M(0)} 2i \langle B, \overline{\nabla_A \psi} \rangle \tilde{\eta} \\
& + \operatorname{Re} \int_{B_M(0)} \frac{1}{2}(\lambda - 1)\bar{\psi}^2 \eta \tilde{\eta} + \operatorname{Re} \int_{B_M(0)} (\lambda + \frac{1}{2})|\psi|^2 \eta \tilde{\eta} - \operatorname{Re} \int_{B_M(0)} \frac{\lambda}{2} \bar{\eta} \tilde{\eta}.
\end{aligned}$$

Similarly, we denote $z := (D, h)$ and $\langle z, \cdot \rangle$ the linear form defined by:

$$\langle z, \tilde{w} \rangle := \int_{B_M(0)} \langle D, \tilde{B} \rangle + \operatorname{Re} \int_{B_M(0)} \bar{h} \tilde{\eta}.$$

Then the equation can be written as

$$[w, \tilde{w}] + \langle k(x)w, \tilde{w} \rangle = \langle z, \tilde{w} \rangle, \forall \tilde{w} \in \mathcal{H}.$$

Using the Riesz representation theorem, we can find a bounded linear operator K on \mathcal{H} , and an element Z of \mathcal{H} depending linearly on z . Thus the equation has the following operational form:

$$w + K(w) = z. \tag{29}$$

Besides because of the compact Sobolev injection $H_0^1(B_M(0)) \subset\subset L^2(B_M(0))$, we get that K is compact. Then by Fredholm's alternative, we deduce the existence of w satisfying (29) if the homogeneous equation only has the trivial solution. In this case, we have $z = 0$.

Now let us rewrite the equation (27):

$$\mathbb{L}(B, \eta) = c_1(s, \theta)T_1 + c_2(s, \theta)T_2.$$

We will estimate c_1 and c_2 . Here we always "modify" $B_M(0)$, such that $\{(s, \theta, r, \phi) : r < 3\} \subset B_M(0)$ for all $(s, \theta, 0, 0) \in B_M(0)$. (For example, we can use the Fermi coordinates, and use the region $\{\rho < M\}$). Then multiply the equation with T_1 and integral on $\{(s, \theta, r, \phi) : r < 3\}$ for any $(s, \theta) \in \Gamma_\epsilon \cap B_M(0)$.

$$\begin{aligned} c_1 &= \frac{1}{c_1^*} \int_{r, \phi} \langle \mathbb{L}(B, \eta), T_1 \rangle + O(\epsilon^2 \rho^{-2})c_2, \\ c_2 &= \frac{1}{c_2^*} \int_{r, \phi} \langle \mathbb{L}(B, \eta), T_2 \rangle + O(\epsilon^2 \rho^{-2})c_1, \end{aligned}$$

where

$$c_j^* = \int_{r, \phi} \langle T_j, T_j \rangle, j = 1, 2.$$

So we only need to estimate $\mathbb{L}T_1$. Note that there are no first order differentiation items. So we can always divide Δ and Δ_A into two parts individually.

$$\mathbb{L} := \mathbb{L}_1 + \mathbb{L}_2,$$

where

$$\mathbb{L}_1(B, \eta) = \begin{pmatrix} -\Delta B + \Delta_{\Gamma_\epsilon} B \\ -\Delta_A \eta + \Delta_{A, \Gamma_\epsilon} \eta \end{pmatrix}.$$

Then for any fixed (s, θ)

$$\begin{aligned} \int_{r, \phi} \langle \mathbb{L}_2(B, \eta), T_1 \rangle &= \int_{r, \phi} \langle (B, \eta), \mathbb{L}_2 T_1 \rangle \\ &\quad + O(\epsilon^2 \rho^{-2}) \int_{r < \frac{1}{2\epsilon\sqrt{\sin 2s}}} |(B, \eta)|^q + |\nabla(B, \eta)|^q. \end{aligned}$$

By the condition (27), we know

$$\left\| \int_{r, \phi} \langle \mathbb{L}_1(B, \eta), T_1 \rangle \right\|_{0,2,q} = O(\epsilon^2) \|(B, \eta)\|_*.$$

Note that when ϵ tends to zero, $\mathbb{L}_2 T_1$ also tends to zero. So we only need to consider the $O(\epsilon)$ items in $\mathbb{L}_2 T_1$. Recalling the calculation in Section 1, we finally get:

$$\mathbb{L}_2 T_1 = O(\epsilon^2 \rho^{-2} e^{-\delta r}).$$

Therefore

$$\|c_j\|_{0,2,q} = O(\epsilon^2)\|(B, \eta)\|_*.$$

By Theorem 6.2, we have:

$$\|(B, \eta)\|_* \leq C\epsilon^2\|(B, \eta)\|_*.$$

So we get that $(B, \eta) = 0$ for the homogeneous equation. Then for any large enough M , we obtain the existence of a solution of (28) satisfying:

$$\|(B, \eta)\|_* \leq C\|(D, h)\|_{**}.$$

with C independent of M . Note that in the previous arguments the norm $\|\cdot\|_*$, $\|\cdot\|_{**}$ are adapted to deal with the bounded domain situation. Similarly in the proof of Theorem 6.2, we can get a subsequence such that $(B, \eta) \rightharpoonup (B, \eta)$ in $H_{loc}^1(\mathbb{R}^4)$ with (B, η) solving (27) in \mathbb{R}^4 . □

7 A projected nonlinear problem

In this section we will solve the equation:

$$\pi^\perp[F(u) + \mathbb{L}w + N(w)] = 0. \quad (30)$$

Theorem 7.1. *If ϵ is small enough, for any f, g satisfying (20), there exists one $w = (B, \eta)$ such that (30) is true and $\pi(w) = 0$ for any (s, θ) . And we have*

$$\|w\|_* \leq C\epsilon^2.$$

Proof. The proof is standard. Here we always choose $1 < q < 2$ is close to 2. For example, we can let $q = \frac{19}{10}$. Different from the cases in [19], where Holder norm is widely used, we need to check that $\|N(w)\|_{**} \leq C\delta_0^2$ if $\|w\|_* \leq \delta_0$.

By (22), we know that we only need to show:

$$|\eta|^3, |\nabla\eta \cdot \eta| \leq C\delta_0^2. \quad (31)$$

Then by Sobolev embedding, we know $W^{2,q} \hookrightarrow W^{1, \frac{4q}{4-q}} \hookrightarrow L^{\frac{4q}{4-2q}}$. Thus by Holder inequality, as long as $\frac{4}{3} \leq q < 2$ and ϵ is small enough, (31) holds.

When ϵ is small enough, by Theorem 6.3, \mathbb{L} is invertible. Then we can rewrite the equation (30) as a fixed point equation:

$$\begin{aligned} \pi^\perp\mathbb{L}w &= -\pi^\perp F(u) - \pi^\perp N(w) \\ \Rightarrow w &= S(w), \end{aligned}$$

where $S(w) := \mathbb{L}^{-1}[-\pi^\perp F(u) - \pi^\perp N(w)]$.

Denote the ball $B_{\delta_0}^\perp = \{w : \|w\|_* < \delta_0\} \cap \{w : \pi(w) = 0\}$, where $\delta_0 > 0$ is small. Then for ϵ small enough and any $w \in B_{\delta_0}^\perp$, we have:

$$\begin{aligned} \|S(w)\|_* &\leq C \|-\pi^\perp F(u) - \pi^\perp N(w)\|_{**}, \\ &\leq C \|F(u)\|_{**} + C\delta_0^2 \\ &\leq C\epsilon^2 + C\delta_0^2 \leq \delta_0. \end{aligned}$$

Therefore $S(w)$ is also in $B_{\delta_0}^\perp$. Hence we get that for $w, w' \in B_{\delta_0}^\perp$

$$\begin{aligned} \|S(w) - S(w')\|_* &= \|\mathbb{L}^{-1}\pi^\perp N(w) - \mathbb{L}^{-1}\pi^\perp N(w')\|_{**} \\ &\leq \frac{1}{2} \|w - w'\|_*. \end{aligned}$$

Thus the map S is a contraction map and has a unique fixed point in $B_{\delta_0}^\perp$. Finally we will estimate the fixed point w :

$$\begin{aligned} \|w\|_* &\leq \|S(0)\|_* + \|S(0) - S(w)\|_* \\ &\leq C \|F(u)\|_{**} + \frac{1}{2} \|w\|_*, \\ \Rightarrow \|w\|_* &\leq C\epsilon^2. \end{aligned}$$

□

In the last of the section, we show the Lipschitz continuity of w with respect to f and g .

Theorem 7.2. *If $(f_j, g_j), j = 1, 2$ are two pairs of functions satisfying*

$$\|(f_j, g_j)\|_{2,1,q} \leq C,$$

then the two solutions w_j to (30) with respect to (f_j, g_j) in Theorem 7.1 satisfy:

$$\|w_1 - w_2\|_* \leq C\epsilon^2 \|(f_1, g_1) - (f_2, g_2)\|_{2,1,q}.$$

Proof. Denote u_j is the approximate solution with respect to (f_j, g_j) and rewrite (30):

$$\pi^\perp [F(u_j) + \mathbb{L}_j w_j + N_j(w_j)] = 0.$$

So we denote $S_j(w) : \mathbb{L}_j^{-1}[-\pi^\perp F(u_j) - \pi^\perp N_j(w)]$. Then we only need to estimate $F(u_1) - F(u_2)$.

In fact, by the results in section 1, we get :

$$\|F(u_1) - F(u_2)\|_{**} \leq C\epsilon^2 \|(f_1, g_1) - (f_2, g_2)\|_{2,1,q}.$$

Thus we have:

$$\begin{aligned} \|w_1 - w_2\|_* &= \|S_1(w_1) - S_2(w_2)\|_* \\ &\leq \|S_1(w_1) - S_1(w_2)\|_* + \|S_1(w_2) - S_2(w_2)\|_* \\ &\leq \frac{1}{2} \|w_1 - w_2\|_* + C \|F(u_1) - F(u_2)\|_{**}. \end{aligned}$$

Therefore we get:

$$\|w_1 - w_2\|_* \leq C\epsilon^2 \|(f_1, g_1) - (f_2, g_2)\|_{2,1,q}.$$

□

8 From Jacobi operator to the true solution

In the section, we hope to find appropriate f and g , which satisfies (20), such that the equation holds:

$$F(u) + \mathbb{L}w + N(w) = 0.$$

This is involved with Jacobi operator, which is discussed in Arezzo-Pacard [1]. Because of the decay of f and g , we can get similar results for the Jacobi operator in dimension 4.

Multiply T_j and integral on $\{a, b\}$, then using the results in section 1 we get:

$$\begin{aligned} \rho^{-6} \partial_s^2 f + 2\rho^{-4} \cos 2s \cdot \partial_s f + \rho^{-2} \cdot \partial_\theta^2 f - 2\rho^{-2} \cos 2s \cdot \partial_\theta g + 2\rho^{-6} \cdot f \\ - \rho^{-2} (\cos 2s)^2 \cdot f = F_1, \\ \rho^{-6} \partial_s^2 g + 2\rho^{-4} \cos 2s \cdot \partial_s g + \rho^{-2} \cdot \partial_\theta^2 g + 2\rho^{-2} \cos 2s \cdot \partial_\theta f + 2\rho^{-6} \cdot g \\ - \rho^{-2} (\cos 2s)^2 \cdot g = F_2, \end{aligned} \tag{32}$$

where

$$\|F_j\|_{0,2,q} \leq C.$$

If we denote $N = ie^{-is} f \cdot \Theta + ie^{is} g \cdot \Theta^\perp$, then we can rewrite (32):

$$L_H N = ie^{-is} F_1 \cdot \Theta + ie^{is} F_2 \cdot \Theta^\perp.$$

Next we will recall the results of Arezzo-Pacard in [1]. Define the variable t by:

$$e^{-2t} = \frac{\sin 2s}{1 - \cos 2s} = \frac{\cos s}{\sin s}.$$

Then we have:

$$dt = \frac{1}{\sin 2s} ds, \quad \sin 2s = (\cosh 2t)^{-1}, \\ \cos 2s = -\tanh 2t.$$

We define the conjugate operator:

$$\mathcal{L}_H := (\sin 2s)^{-1} L_H,$$

then we have:

$$\mathcal{L}_H N = \partial_t^2(f, g) + \partial_\theta^2(f, g) + 2 \tanh 2t \cdot \partial_\theta(g, -f) + \left[\frac{3}{(\cosh 2t)^2} - 1 \right] (f, g).$$

Let t tend to $-\infty$, we get that \mathcal{L}_H is asymptotic to the following differential operator:

$$\mathcal{L}_0 N := \partial_t^2(f, g) + \partial_\theta^2(f, g) - 2\partial_\theta(g, -f) - (f, g).$$

On S^1 , the spectrum of Δ^0 and Δ^1 are given by:

$$\sigma(\Delta^0) = \{k^2 : k \geq 0\}, \\ \sigma(\Delta^1) = \{k^2 : k \geq 0\}.$$

We remark that we consider $g \cdot \Theta^\perp$ as the tangent vector field on S^1 . Then the corresponding 1-form is $gd\theta$.

We define

$$\mathcal{V}^0 := \{(f, g) : df = 0, dg = 0\}.$$

For all $k \geq 1$, we define:

$$\mathcal{V}^k := \{(f, g) : \Delta(f, g) = -k^2(f, g)\}.$$

In fact, we know the eigenfunctions of Δ^0 corresponding to eigenvalue k^2 are $\cos k\theta$ and $\sin k\theta$. So we can always assume that:

$$f = f_0 + \sum_{k \geq 1} f_{k,1} \cos k\theta + \sum_{k \geq 1} f_{k,2} \sin k\theta, \\ g = g_0 + \sum_{k \geq 1} g_{k,1} \cos k\theta + \sum_{k \geq 1} g_{k,2} \sin k\theta.$$

This time the equation $\mathcal{L}_H(f, g) = 0$ yields the couple system:

$$\begin{cases} \ddot{f}_{k,1} - k^2 f_{k,1} + 2k \tanh 2t \cdot g_{k,2} + [\frac{3}{(\cosh 2t)^2} - 1]f_{k,1} = 0, & (33) \\ \ddot{g}_{k,2} - k^2 g_{k,2} + 2k \tanh 2t \cdot f_{k,1} + [\frac{3}{(\cosh 2t)^2} - 1]g_{k,2} = 0, & (34) \end{cases}$$

when $k \neq 0$. If we denote $h_{k,1} = f_{k,1} + g_{k,2}$, $h_{k,2} = f_{k,1} - g_{k,2}$, then we have:

$$\begin{cases} \ddot{h}_{k,1} - k^2 h_{k,1} + 2k \tanh 2t \cdot h_{k,1} + [\frac{3}{(\cosh 2t)^2} - 1]h_{k,1} = 0, & (35) \\ \ddot{h}_{k,2} - k^2 h_{k,2} - 2k \tanh 2t \cdot h_{k,2} + [\frac{3}{(\cosh 2t)^2} - 1]h_{k,2} = 0. & (36) \end{cases}$$

And for $k = 0$, we get:

$$\begin{cases} \ddot{f}_0 - [1 - \frac{3}{(\cosh 2t)^2}]f_0 = 0, \\ \ddot{g}_0 - [1 - \frac{3}{(\cosh 2t)^2}]g_0 = 0. \end{cases}$$

We find that when $k = 0$, the asymptotic behavior of f_0, g_0 at both $\pm\infty$ is governed by the exponents

$$\mu_0^\pm = \pm 1.$$

And when $k > 0$, the asymptotic behavior of $f_{k,j}, g_{k,j}$ at both $\pm\infty$ is governed by the following sets of indicial roots:

$$\mu_k^\pm = \pm(k+1), \nu_k^\pm = \pm(k-1).$$

In [1], Arezzo-Pacard introduced all the geometric Jacobi fields as well as their explicit expression.

- 1. Jacobi fields corresponding to translation:

$$f = (a \cdot \Theta) \sin(\alpha + s), g = (a \cdot \Theta^\perp) \sin(\alpha - s),$$

where $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}^2$

- 2. Jacobi fields corresponding to dilations:

$$f = (\sin 2s)^{\frac{1}{2}}, g = 0.$$

- 3. Jacobi fields corresponding to the action of $SU(2)$:

$$\begin{aligned} f &= (\sin 2s)^{-\frac{1}{2}} \cos 2s (A\Theta \cdot \Theta), \\ g &= (\sin 2s)^{-\frac{1}{2}} (A\Theta \cdot \Theta^\perp), \end{aligned}$$

where A is a 2×2 symmetric matrix.

- 4. Jacobi fields corresponding to the action of $O(4)/SU(2)$

$$f = 0, g = (\sin 2s)^{\frac{1}{2}} (A\Theta \cdot \Theta^\perp),$$

or

$$f = 0, g = (\sin 2s)^{-\frac{1}{2}} \cos 2s \cdot (A\Theta \cdot \Theta^\perp),$$

where A is a 2×2 skew-symmetric matrix.

We observed that the Jacobi field 2 and 4 are in \mathcal{V}^0 , and 1 is in \mathcal{V}^1 . As for the Jacobi field 3, its decay is bad. Besides, similarly in Proposition 11.2 and 11.3 in [1], we get:

Proposition 8.1. *There exists $\hat{t}_0 > 0$ such that if (f, g) is a solution of $\mathcal{L}(f, g) = 0$ in $(t_1, t_2) \times \mathbb{S}^1$, for some $t_1 < -\hat{t}_0$ and $t_2 > \hat{t}_0$ (with $U = 0$ on the boundary if either $t_1 > -\infty$ or $t_2 < \infty$, satisfying*

$$\|(f, g)\|_{0, \delta, q} < 1,$$

for some $0 < \delta < \frac{1}{4}$ and with the property that for any $t \in (t_1, t_2)$, $(f, g)(t, \cdot)$ is orthogonal to \mathcal{V}^0 in the L^2 sense on \mathbb{S}^1 , then $(f, g) = 0$.

The proof is the same as Proposition 11.3 in [1]. In fact we can drop the condition for \hat{t}_0 , since the equations can be reduced to (35) and (36). Finally we can get the following two results:

Proposition 8.2. *Assume $0 < \delta < \frac{1}{4}$ is fixed. Then for all $t_0 \in \mathbb{R}$, there exists an operator*

$$\mathcal{G}_{t_0} : W_\delta^{0, q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R}) \rightarrow W_\delta^{2, q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R}),$$

such that for all $V := (F_1, F_2) \in W_\delta^{0, q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R})$, if for all $t > t_0$, $V(t, \cdot)$ is orthogonal to \mathcal{V}^0 in the L^2 sense on \mathbb{S}^1 , then $U = \mathcal{G}_{t_0} V$ is the unique solution of

$$\begin{cases} \mathcal{L}_H U = V, & \text{in } [t_0, +\infty) \times \mathbb{S}^1, \\ V = 0, & \text{on } \{t_0\} \times \mathbb{S}^1, \end{cases}$$

which belongs to $W_\delta^{2, q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R})$. Furthermore, $\|U\|_{2, \delta, q} \leq C \|V\|_{0, \delta, q}$.

Proposition 8.3. *Assume that $\delta > 1$ is fixed. There exists some const $c > 0$ and an operator*

$$\mathcal{G}_{t_0} : W_\delta^{0,q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R}) \rightarrow W_\delta^{2,q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R}),$$

such that for all $V := (F_1, F_2) \in W_\delta^{0,q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R})$, if for all $t > t_0$, V is constant on \mathbb{S}^1 , then $U = \mathcal{G}_{t_0}V$ is the unique solution of

$$\begin{cases} \mathcal{L}_H U = V, & \text{in } [t_0, +\infty) \times \mathbb{S}^1, \\ V = 0, & \text{on } \{t_0\} \times \mathbb{S}^1, \end{cases}$$

which belongs to $W_\delta^{2,q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R})$. Furthermore, $\|U\|_{2,\delta,q} \leq C\|V\|_{0,\delta,q}$.

With these results, we can solve the last Jacobi operator equations (32):

Theorem 8.1. *When ϵ is small enough, there exists a solution to (32), satisfying:*

$$\|(f, g)\|_{2,1,q} \leq C.$$

Proof. By the previous calculation, we can rewrite (F_1, F_2) :

$$(F_1, F_2) = (F'_1, F'_2) + (F''_1, F''_2),$$

$$\begin{aligned} \epsilon^3 F'_1 &= \int_{\tilde{r}} \pi \epsilon^3 \frac{(V')^2}{\tilde{r}} \cdot (6(\cos 2s)^2 \rho^{-5} \tilde{r}^2 - 4\rho^{-9} \tilde{r}^2) \\ &\quad + \int_{\tilde{r}} 4\pi \tilde{r} V (-V'' + \frac{V'}{\tilde{r}}) (\cos 2s)^2 \epsilon^3 \rho^{-5} - \int_{\tilde{r}} 4\pi \tilde{r} V \epsilon^3 \rho^{-9} (-V'' + \frac{V'}{\tilde{r}}) \\ &\quad + \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-5} \cdot 4\pi \tilde{r}^2 V U U' - \int_{\tilde{r}} (\cos 2s)^2 \epsilon^3 \rho^{-5} \cdot 2\pi \tilde{r}^2 V^2 U U', \end{aligned}$$

In fact (F'_1, F'_2) comes from the proof of Theorem 4.1 and does not depend on (f, g) . Besides, we know:

$$F''_1 = \epsilon^{-3} \int_{r,\phi} \langle \mathbb{L}w + N(w), T_1 \rangle.$$

If we suppose $\|(f, g)\|_{2,1,q} \leq \epsilon^{-\frac{1}{2}}$ and define the operator $J(f, g) := \mathcal{L}_H^{-1}(F_1, F_2)$, then $\|(F''_1, F''_2)\|_{0,1,q} \leq C\epsilon^{\frac{1}{2}}$ and by Proposition 8.2 and 8.3, we have:

$$\|J(f, g)\|_{2,1,q} \leq C\|(F_1, F_2)\|_{0,1,q} \leq C \leq \epsilon^{-\frac{1}{2}}.$$

So J maps from $B_{\epsilon^{-\frac{1}{2}}}$ to $B_{\epsilon^{-\frac{1}{2}}}$ in $W_\delta^{2,q}([t_0, +\infty) \times \mathbb{S}^1; \mathbb{R} \times \mathbb{R})$.

Similarly in the proof of Theorem 7.1, we get that J is a contraction map and has a fixed point. And we can also get:

$$\|(f, g)\|_{2,1,q} \leq C.$$

□

In the end, we will explain that we have got the desired solution when f, g are the ones in Theorem 8.1: (see [5])

Theorem 8.2. *If w satisfies that:*

$$F(u) + \mathbb{L}w + N(w) = 0,$$

then $F(u + w) = 0$. In other words, $v := u + w$ is the solution to Ginzburg Landau equations.

Proof. We only need to show that $TT^*(B, \eta) = 0$. For short, we define $\xi = T^*(B, \eta)$ and $v = (\tilde{A}, \tilde{\psi}) + u + w$. Recall that:

$$\begin{aligned} T\xi &= (d\xi, i\xi\tilde{\psi}), \\ T^*(B, \eta) &= d^*B - \text{Im}(\bar{\eta}\tilde{\psi}). \end{aligned}$$

Then by the definition of $F(v) = 0$, we get:

$$\begin{aligned} -\Delta_{\tilde{A}}\tilde{\psi} + \frac{\lambda}{2}(|\tilde{\psi}|^2 - 1)\tilde{\psi} + i\xi\tilde{\psi} &= 0, \\ d^*d\tilde{A} - \text{Im}(\nabla_{\tilde{A}}\tilde{\psi} \cdot \bar{\tilde{\psi}}) + d\xi &= 0. \end{aligned}$$

Therefore we have:

$$\begin{aligned} -\Delta\xi &= d^*d\xi = d^*\text{Im}(\nabla_{\tilde{A}}\tilde{\psi} \cdot \bar{\tilde{\psi}}) \\ &= \text{Im}(\nabla_{\tilde{A}}^*\nabla_{\tilde{A}}\tilde{\psi} \cdot \bar{\tilde{\psi}}) \\ &= \text{Im}(-i\xi\tilde{\psi} \cdot \bar{\tilde{\psi}}) = -\xi|\tilde{\psi}|^2. \end{aligned}$$

So we conclude that $\xi = 0$. □

A The proof in section 3

A.1 The proof of Lemma 3.2

Proof. $d^2 = \inf_{\rho} \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + 2\rho_1^2 - \sqrt{2}\rho_1\sqrt{\rho^2 + \rho^{-2} + 2\cos\alpha}$.

Suppose $t = \sqrt{\rho^2 + \rho^{-2} + 2 \cos \alpha}$, then we only need to consider the function $h(t) = \frac{1}{2}t^2 - \sqrt{2}\rho_1 t + 2\rho_1^2 - \cos \alpha$, $t \geq \sqrt{2 + 2 \cos \alpha}$.

So when $\rho_1 > \sqrt{1 + \cos \alpha}$, we have

$$\inf_t h(t) = h(\sqrt{2}\rho_1) \Rightarrow 2\rho_1^2 = \rho^2 + \rho^{-2} + 2 \cos \alpha.$$

At this case, there are two closest points ρ', ρ'' , satisfying $\rho' \rho'' = 1$.

When $\rho_1 \leq \sqrt{1 + \cos \alpha}$, we have $\inf_t h(t) = h(\sqrt{2 + 2 \cos \alpha})$, then $\rho = 1$. So there is only one closest point. This completes the proof. \square

A.2 The proof of Lemma 3.3

Proof. If $\rho_1 \neq \rho_2$, then $s \neq \frac{\pi}{4}$ and the conclusion naturally holds by Lemma 3.1. Now we have $a^2 + b^2 < \frac{1}{\epsilon^2 \sin 2s}$.

If $\rho_1 = \rho_2$ and there is only one closest point, then by symmetry we know $s = \frac{\pi}{4}$. As a result,

$$y = \left(\frac{\Theta}{\sqrt{2}\epsilon} + \frac{a\Theta}{\sqrt{2}} - \frac{b\Theta^\perp}{\sqrt{2}}, \frac{\Theta}{\sqrt{2}\epsilon} + \frac{a\Theta}{\sqrt{2}} + \frac{b\Theta^\perp}{\sqrt{2}} \right),$$

$$\rho_1^2 = \frac{(a + \epsilon^{-1})^2 + b^2}{2}, \quad e^{i(\theta_1 - \theta_2)} = \frac{\epsilon^{-1} + a - bi}{\epsilon^{-1} + a + bi}.$$

So by $\rho_1 < \frac{\sqrt{2}}{2\epsilon} \sqrt{2 + 2 \cos(\theta_1 - \theta_2)}$ from Lemma 3.2, we have

$$\left((a + \epsilon^{-1})^2 + b^2 \right)^2 < \frac{4}{\epsilon^2} (a + \epsilon^{-1})^2 \implies (a + \epsilon^{-1})^2 + b^2 < 2\epsilon^{-1} |a + \epsilon^{-1}|. \quad (37)$$

If $a > -\epsilon^{-1}$, then (1) $\implies a^2 + b^2 < \epsilon^{-2}$.

If $a < -\epsilon^{-1}$, then (1) $\implies (a + \epsilon^{-1})^2 + b^2 < -2\epsilon^{-1}(a + \epsilon^{-1})$

$\implies (a + 2\epsilon^{-1})^2 + b^2 < \epsilon^{-2}$. At this case, we observe that

$$y = \left(-\frac{\Theta}{\sqrt{2}\epsilon} + \frac{(a + \frac{2}{\epsilon})\Theta}{\sqrt{2}} - \frac{b\Theta^\perp}{\sqrt{2}}, -\frac{\Theta}{\sqrt{2}\epsilon} + \frac{(a + \frac{2}{\epsilon})\Theta}{\sqrt{2}} + \frac{b\Theta^\perp}{\sqrt{2}} \right).$$

This means the two different cases about a in fact are the same.

In conclusion, when $a^2 + b^2 < \frac{1}{\epsilon^2 \sin 2s}$, then $y = T(s, \theta, a, b)$ has a unique nearest point to Γ_ϵ . \square

A.3 The proof of Lemma 3.4

Proof. If $\rho_1 = \rho_2$, then from the previous arguments in Lemma 3.2, after rescaling, we know

$$\begin{aligned} \text{dist}(y, \Gamma_\epsilon) &= \epsilon^{-2} + 2\rho_1^2 - 2\rho_1\epsilon^{-1}\sqrt{1 + \cos \alpha} \\ &= a^2 + b^2. \end{aligned}$$

If $\rho_1 > \rho_2$, then $0 < s < \frac{\pi}{4}$. From Lemma 3.1, we know the closest point in Γ_ϵ to y is in $\{\rho_1 > \rho_2\}$. Next, we will show that there is only one extreme point of distance in $\{\rho_1 > \rho_2\} \cap \Gamma_\epsilon$. For simplicity, we only need to consider Γ rather than Γ_ϵ . Denote $d = \text{dist}(y, \Gamma)$.

$$\begin{aligned} f(\rho, \theta) &= \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + \rho_1^2 + \rho_2^2 - \sqrt{2}\rho_1\rho \cos(\theta - \theta_1) - \sqrt{2}\rho_2\rho^{-1} \cos(\theta - \theta_2), \\ d^2 &= \inf_{\rho, \theta} f(\rho, \theta). \end{aligned}$$

For $\forall \rho, \exists \theta_0$, we have

$$\begin{aligned} \frac{\partial f}{\partial \theta}(\rho, \theta_0) &= \frac{\partial f}{\partial \theta}(\rho, \theta_0 + \pi) = 0, \\ f(\rho, \theta_0) &= \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + \rho_1^2 + \rho_2^2 - \sqrt{2}\sqrt{\rho_1^2\rho^2 + \rho_2^2\rho^{-2} + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}, \\ f(\rho, \theta_0 + \pi) &= \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} + \rho_1^2 + \rho_2^2 + \sqrt{2}\sqrt{\rho_1^2\rho^2 + \rho_2^2\rho^{-2} + 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)}. \end{aligned}$$

So we know that $f(\rho, \theta_0 + \pi)$ is strictly increasing in $[1, +\infty)$, which means there is no extreme point.

On the other hand, by Lemma 3.1, if $r_2 > r_1 > 1$ satisfy $\frac{\partial f}{\partial \rho}(r_1, \theta_0) = \frac{\partial f}{\partial \rho}(r_2, \theta_0) = 0$, then $f(r_1, \theta_0) < f(r_2, \theta_0)$. So there is only one local minimum point in $\{\rho_1 > \rho_2\}$, and it is also a global minimum point. By rescaling, the conclusion also holds for Γ_ϵ . By calculation, we know $\left(\frac{\cos s}{\epsilon\sqrt{\sin 2s}}\Theta, \frac{\sin s}{\epsilon\sqrt{\sin 2s}}\Theta\right)$ is the minimum point. So we have $\text{dist}(y, X_\epsilon) = \sqrt{a^2 + b^2}$. This completes the proof. \square

B The proof in section 4

The metric matrix of Fermi coordinates is

$$(g_{ij}) = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & -b \cos 2s & a \cos 2s \\ 0 & -b \cos 2s & 1 & 0 \\ 0 & a \cos 2s & 0 & 1 \end{pmatrix},$$

and

$$\tilde{A} = (\tilde{a}_{ij}) = \begin{pmatrix} \left(a - \frac{1}{\epsilon(\sin 2s)^{\frac{3}{2}}} \right)^2 + b^2 & \frac{-2b}{\epsilon\sqrt{\sin 2s}} \\ \frac{-2b}{\epsilon\sqrt{\sin 2s}} & a^2 + b^2 + \frac{1}{\epsilon^2 \sin 2s} + \frac{2a\sqrt{\sin 2s}}{\epsilon} \end{pmatrix}.$$

The inverse matrix of (g_{ij}) is:

$$(g^{ij}) = \begin{pmatrix} \frac{1}{c} + \frac{4b^2}{c^2 e \epsilon^2 \sin 2s} & \frac{2b}{ce \epsilon \sqrt{\sin 2s}} & \frac{2b^2 \cos 2s}{ce \epsilon \sqrt{\sin 2s}} & \frac{-2ab \cos 2s}{ce \epsilon \sqrt{\sin 2s}} \\ \frac{2b}{ce \epsilon \sqrt{\sin 2s}} & \frac{1}{e} & \frac{b \cos 2s}{e} & \frac{-a \cos 2s}{e} \\ \frac{2b^2 \cos 2s}{ce \epsilon \sqrt{\sin 2s}} & \frac{b \cos 2s}{e} & 1 + \frac{b^2 \cos^2 2s}{e} & \frac{-ab \cos^2 2s}{e} \\ \frac{-2ab \cos 2s}{ce \epsilon \sqrt{\sin 2s}} & \frac{-a \cos 2s}{e} & \frac{-ab \cos^2 2s}{e} & 1 + \frac{a^2 \cos^2 2s}{e} \end{pmatrix}.$$

By direct calculation, we have:

$$\begin{aligned} \frac{\partial_s(\sqrt{G})}{\sqrt{G}} &= -4(\sin 2s)^{-1} \cos 2s - 6(a^2 + b^2)\epsilon^2 \cdot (\sin 2s)^2 \cos 2s + O(\epsilon^4 \rho^{-6}) \\ &= -4(\sin 2s)^{-1} \cos 2s + o(\epsilon^{2-\sigma} \rho^{-2}), \\ \frac{\partial_a(\sqrt{G})}{\sqrt{G}} &= -2a\epsilon^2(\sin 2s)^3 + O(\epsilon^4 \rho^{-9}), \\ \frac{\partial_b(\sqrt{G})}{\sqrt{G}} &= -2b\epsilon^2(\sin 2s)^3 + O(\epsilon^4 \rho^{-9}), \\ \frac{1}{c} &= \epsilon^2 \sin^3 2s + 2a\epsilon^3(\sin 2s)^{\frac{9}{2}} + (3a^2 - b^2)\epsilon^4(\sin 2s)^6 + o(\epsilon^{5-\sigma} \rho^{-12}), \\ \partial_s \left(\frac{1}{c} \right) &= 6\epsilon^2 \sin^2 2s \cos 2s + 18a\epsilon^3(\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4 \rho^{-10}) \\ \partial_a \left(\frac{1}{c} \right) &= 2\epsilon^3(\sin 2s)^{\frac{9}{2}} + O(\epsilon^4 \rho^{-12}), \\ \partial_b \left(\frac{1}{c} \right) &= O(\epsilon^4 \rho^{-12}), \\ \frac{1}{e} &= \epsilon^2 \sin 2s - 2a\epsilon^3(\sin 2s)^{\frac{5}{2}} + 3(a^2 + b^2)\epsilon^4(\sin 2s)^4 + o(\epsilon^{5-\sigma} \rho^{-8}), \\ \partial_s \left(\frac{1}{e} \right) &= 2\epsilon^2 \cos 2s - 10a\epsilon^3(\sin 2s)^{\frac{3}{2}} \cos 2s + O(\epsilon^4 \rho^{-6}), \\ \partial_a \left(\frac{1}{e} \right) &= -2\epsilon^3(\sin 2s)^{\frac{5}{2}} + O(\epsilon^4 \rho^{-8}), \\ \partial_b \left(\frac{1}{e} \right) &= O(\epsilon^4 \rho^{-8}). \end{aligned}$$

Using the formula: if $\alpha = \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$, then

$$(d^* \alpha)_{i_1 \dots i_{p-1}} = -g^{kl} \left(\frac{\partial \alpha_{k i_1 \dots i_{p-1}}}{\partial x^l} - \Gamma_{kl}^j \alpha_{j i_1 \dots i_{p-1}} \right),$$

with

$$\Gamma_{kl}^j = \frac{1}{2} g^{ji} (g_{ki,l} + g_{li,k} - g_{kl,i}),$$

$$A = A_\tau dy^\tau,$$

with $y^1 = s, y^2 = \theta, y^3 = a, y^4 = b$

So we have:

$$d^* A = -g^{kl} \left(\frac{\partial A_k}{\partial y^l} - \Gamma_{kl}^j A_j \right).$$

If we define $d^* dA = (d^* dA)_\tau dy^\tau$, then

$$(d^* dA)_\tau = -g^{kl} \left(\frac{\partial A_{k\tau}}{\partial y^l} - \Gamma_{kl}^j A_{j\tau} \right).$$

The most complicated item is $B^j := g^{kl} \Gamma_{kl}^j$.

On the other hand, we have another definition for d^* :

If $\alpha = \alpha_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$, then

$$\begin{aligned} * \alpha &= \eta_{i_1 \dots i_r j_{r+1} \dots j_n} \alpha^{i_1 \dots i_r} dx^{j_{r+1}} \wedge \dots \wedge dx^{j_n}, \\ d^* \alpha &= (-1)^{n(r+1)+1} * d*, \end{aligned}$$

with

$$\begin{aligned} \eta_{i_1 \dots i_n} &= \sqrt{G} \delta_{i_1 \dots i_n}^{1 \dots n}, \\ \alpha^{i_1 \dots i_r} &= g^{i_1 k_1} \dots g^{i_r k_r} \alpha_{k_1 \dots k_r}. \end{aligned}$$

Therefore, for $A = A_l dy^l$ we have:

$$\begin{aligned} d^* A &= -\partial_j A^j - \frac{\partial_j(\sqrt{G})}{\sqrt{G}} A^j \\ &= (-\partial_j g^{jl} - \frac{\partial_j(\sqrt{G})}{\sqrt{G}} \cdot g^{jl}) A_l - g^{kl} \partial_k A_l. \end{aligned}$$

As a result, we get:

$$B^j = -\partial_k g^{kj} - \frac{\partial_k(\sqrt{G})}{\sqrt{G}} \cdot g^{jk}.$$

First, we have the following estimates for (g^{ij}) when it is close to the minimal surface:

$$\begin{aligned}
g^{11} &= \epsilon^2 \rho^{-6} + 2a\epsilon^3 \rho^{-9} + 3(a^2 + b^2)\epsilon^4 \rho^{-12} + o(\epsilon^{5-\sigma} \rho^{-12}), \\
g^{12} &= 2b\epsilon^3 \rho^{-7} + o(\epsilon^{5-\sigma} \rho^{-10}), \\
g^{13} &= 2b^2 \cos 2s \cdot \epsilon^3 \rho^{-7} + o(\epsilon^4 \rho^{-9}), \\
g^{14} &= -2ab \cos 2s \cdot \epsilon^3 \rho^{-7} + o(\epsilon^4 \rho^{-9}), \\
g^{22} &= \epsilon^2 \rho^{-2} - 2a\epsilon^3 \rho^{-5} + 3(a^2 + b^2)\epsilon^4 \rho^{-8} + o(\epsilon^{5-\sigma} \rho^{-8}), \\
g^{23} &= b \cos 2s \cdot \epsilon^2 \rho^{-2} - 2ab \cos 2s \cdot \epsilon^3 \rho^{-5} + o(\epsilon^{4-\sigma} \rho^{-7}), \\
g^{24} &= -a \cos 2s \cdot \epsilon^2 \rho^{-2} + 2a^2 \cos 2s \cdot \epsilon^3 \rho^{-5} + o(\epsilon^{4-\sigma} \rho^{-7}), \\
g^{33} &= 1 + b^2(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5} + o(\epsilon^{4-\sigma} \rho^{-4}), \\
g^{34} &= -ab(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5} + o(\epsilon^{4-\sigma} \rho^{-4}), \\
g^{44} &= 1 + a^2(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2a^3(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5} + o(\epsilon^{4-\sigma} \rho^{-4}).
\end{aligned}$$

Besides we also have:

$$\begin{aligned}
B^1 &= -2\epsilon^2 \rho^{-4} \cos 2s - 8a\epsilon^3 \rho^{-7} \cos 2s + O(\epsilon^4 \rho^{-8}), \\
B^2 &= -4b\epsilon^3 \rho^{-5} \cos 2s + O(\epsilon^4 \rho^{-7}), \\
B^3 &= 2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2)(\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9} + O(\epsilon^4 \rho^{-7}), \\
B^4 &= 2b\epsilon^2 \rho^{-6} + b(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2ab(\cos 2s)^2 \epsilon^3 \rho^{-5} - 4ab\epsilon^3 \rho^{-9} + O(\epsilon^4 \rho^{-7}).
\end{aligned}$$

B.1 The proof of Lemma 4.1

Proof. Denote $w_{ij} = \frac{1}{\sqrt{G}} \partial_i (\sqrt{G} g^{ij} \partial_j)$ and we only consider the $O(\epsilon^3)$ items because of the exponential decay.

$$\begin{aligned}
w_{1,1} &= \left[2\epsilon^2 (\sin 2s)^2 \cos 2s + 10a\epsilon^3 (\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s + \\
&\quad \left[\epsilon^2 (\sin 2s)^3 + 2a\epsilon^3 (\sin 2s)^{\frac{5}{2}} + O(\epsilon^4) \right] \partial_s \partial_s \\
w_{2,1} &= \left[2b\epsilon^3 (\sin 2s)^{\frac{7}{2}} + O(\epsilon^4) \right] \partial_s \partial_\theta, \\
w_{3,1} &= O(\epsilon^4) \partial_s + \left[2b^2 \epsilon^3 (\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s \partial_a, \\
w_{4,1} &= \left[-2a\epsilon^3 (\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s + \left[-2ab\epsilon^3 (\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s \partial_b.
\end{aligned}$$

$$\begin{aligned}
w_{1,2} &= \left[6b\epsilon^3(\sin 2s)^{\frac{5}{2}} \cos 2s + O(\epsilon^4) \right] \partial_\theta + \left[2b\epsilon^3(\sin 2s)^{\frac{7}{2}} + O(\epsilon^4) \right] \partial_s \partial_\theta, \\
w_{2,2} &= \left[\epsilon^2 \sin 2s - 2a\epsilon^3(\sin 2s)^{\frac{5}{2}} + O(\epsilon^4) \right] \partial_\theta \partial_\theta, \\
w_{3,2} &= \left[-2b\epsilon^3(\sin 2s)^{\frac{5}{2}} \cos 2s + O(\epsilon^4) \right] \partial_\theta + \\
&\quad \left[b\epsilon^2 \sin 2s \cos 2s - 2ab\epsilon^3(\sin 2s)^{\frac{5}{2}} \cos 2s + O(\epsilon^4) \right] \partial_a \partial_\theta, \\
w_{4,2} &= O(\epsilon^4) \partial_\theta + \left[-a\epsilon^2 \sin 2s \cos 2s + 2a^2\epsilon^3(\sin 2s)^{\frac{5}{2}} \cos 2s + O(\epsilon^4) \right] \partial_b \partial_\theta.
\end{aligned}$$

$$\begin{aligned}
w_{1,3} &= \left[6b^2\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 - 4b^2\epsilon^3(\sin 2s)^{\frac{9}{2}} + O(\epsilon^4) \right] \partial_a \\
&\quad + \left[2b^2\epsilon^3(\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s \partial_a, \\
w_{2,3} &= b \cos 2s \left[\epsilon^2 \sin 2s - 2a\epsilon^3(\sin 2s)^{\frac{5}{2}} + O(\epsilon^4) \right] \partial_\theta \partial_a, \\
w_{3,3} &= \left[-2a\epsilon^2(\sin 2s)^3 - 2b^2\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_a + \\
&\quad \left[1 + b^2\epsilon^2 \sin 2s(\cos 2s)^2 - 2ab^2\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_a \partial_a, \\
w_{4,3} &= \left[-a\epsilon^2 \sin 2s(\cos 2s)^2 + 2a^2\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_a + \\
&\quad \left[-ab\epsilon^2 \sin 2s(\cos 2s)^2 + 2a^2b\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_a \partial_b.
\end{aligned}$$

$$\begin{aligned}
w_{1,4} &= \left[-6ab\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + 4ab\epsilon^3(\sin 2s)^{\frac{9}{2}} + O(\epsilon^4) \right] \partial_b \\
&\quad + \left[-2ab\epsilon^3(\sin 2s)^{\frac{7}{2}} \cos 2s + O(\epsilon^4) \right] \partial_s \partial_b, \\
w_{2,4} &= -a \cos 2s \left[\epsilon^2 \sin 2s - 2a\epsilon^3(\sin 2s)^{\frac{5}{2}} + O(\epsilon^4) \right] \partial_\theta \partial_b, \\
w_{3,4} &= \left[-b\epsilon^2 \sin 2s(\cos 2s)^2 + 4ab\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_b + \\
&\quad \left[-ab\epsilon^2 \sin 2s(\cos 2s)^2 + 2a^2b\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_a \partial_b, \\
w_{4,4} &= \left[-2b\epsilon^2(\sin 2s)^3 + O(\epsilon^4) \right] \partial_b + \\
&\quad \left[1 + a^2\epsilon^2 \sin 2s(\cos 2s)^2 - 2a^3\epsilon^3(\sin 2s)^{\frac{5}{2}}(\cos 2s)^2 + O(\epsilon^4) \right] \partial_b \partial_b.
\end{aligned}$$

The we can get the conclusion. □

B.2 The proof of Lemma 4.2

Proof.

$$\begin{aligned}
d^* A &= -g^{kl} \frac{\partial A_k}{\partial y^l} + B^j A_j \\
&= -g^{1l} \frac{\partial A_1}{\partial y^l} - g^{2l} \frac{\partial A_2}{\partial y^l} - g^{3l} \frac{\partial A_3}{\partial y^l} - g^{4l} \frac{\partial A_4}{\partial y^l} + B^1 A_1 + B^3 A_3 + B^4 A_4 \\
&= \epsilon^2 \rho^{-6} \cdot (\epsilon f_{ss} \cdot A_3 + \epsilon g_{ss} \cdot A_4) + \epsilon^2 \rho^{-2} \cdot (\epsilon f_{\theta\theta} A_3 + \epsilon g_{\theta\theta} A_4) \\
&\quad + 2b \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta \partial_3 A_3 + b \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta (\partial_4 A_3 + \partial_3 A_4) \\
&\quad - a \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta (\partial_3 A_4 + \partial_4 A_3) - 2a \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta \partial_4 A_4 \\
&\quad - (-ab(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2a^2 b(\cos 2s)^2 \epsilon^3 \rho^{-5}) (\partial_4 A_3 + \partial_3 A_4) \\
&\quad - (1 + b^2(\cos 2s)^2 \epsilon^2 \rho^{-2} - 2ab^2(\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_3 A_3 \\
&\quad - (1 + a^2(\cos 2s)^2 \epsilon^2 \rho^{-2} - 2a^3(\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_4 A_4 \\
&\quad + 2\epsilon^2 \rho^{-4} \cos 2s \cdot (\epsilon f_s A_3 + \epsilon g_s A_4) \\
&\quad + (2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2)(\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_3 \\
&\quad + (2b\epsilon^2 \rho^{-6} + b(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2ab(\cos 2s)^2 \epsilon^3 \rho^{-5} - 4ab\epsilon^3 \rho^{-9}) A_4 + o(\epsilon^4).
\end{aligned}$$

□

B.3 The proof of Lemma 4.3

Proof.

$$\begin{aligned}
\langle A, d\psi \rangle &= -2b \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta A_3 \partial_3 \psi - b \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta (A_3 \partial_4 \psi + A_4 \partial_3 \psi) \\
&\quad + a \cos 2s \cdot \epsilon^3 \rho^{-2} f_\theta (A_4 \partial_3 \psi + A_3 \partial_4 \psi) + 2a \cos 2s \cdot \epsilon^3 \rho^{-2} g_\theta A_4 \partial_4 \psi \\
&\quad + (1 + b^2(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) A_3 \cdot \partial_3 \psi \\
&\quad + (-ab(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) (A_3 \partial_4 \psi + A_4 \partial_3 \psi) + \\
&\quad + (1 + a^2(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2a^3(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) A_4 \partial_4 \psi \\
&\quad + o(\epsilon^4).
\end{aligned}$$

□

B.4 The proof of Lemma 4.4

Proof.

$$\begin{aligned}
(d^*dA)_1 &= -\partial_3 A_{31} - \partial_4 A_{41} + O(\epsilon^2) \\
&= \cos \tilde{\phi} \cdot \epsilon g_s \left(\frac{V''}{\tilde{r}} - \frac{V'}{\tilde{r}^2} \right) - \sin \tilde{\phi} \cdot \epsilon f_s \left(\frac{V''}{\tilde{r}} - \frac{V'}{\tilde{r}^2} \right) \\
&\quad + O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
(d^*dA)_2 &= -\partial_3 A_{32} - \partial_4 A_{42} + O(\epsilon^2) \\
&= \cos \tilde{\phi} \cdot \epsilon g_\theta \left(\frac{V''}{r} - \frac{V'}{r^2} \right) - \sin \tilde{\phi} \cdot \epsilon f_\theta \left(\frac{V''}{r} - \frac{V'}{r^2} \right) \\
&\quad + O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
(d^*dA)_3 &= -\epsilon^2 \rho^{-6} \partial_1 A_{13} - \epsilon^2 \rho^{-2} \partial_2 A_{23} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_3 A_{23} \\
&\quad + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_4 A_{23} + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_2 A_{43} \\
&\quad - (-ab(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2a^2 b(\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_3 A_{43} \\
&\quad - (1 + a^2(\cos 2s)^2 \epsilon^2 \rho^{-2} - 2a^3(\cos 2s)^2 \epsilon^3 \rho^{-5}) \partial_4 A_{43} \\
&\quad - 2\epsilon^2 \rho^{-4} \cos 2s \cdot A_{13} + O(\epsilon^4) \\
&\quad + (2b\epsilon^2 \rho^{-6} + b(\cos 2s)^2 \epsilon^2 \rho^{-2} + 2ab(\cos 2s)^2 \epsilon^3 \rho^{-5} - 4ab\epsilon^3 \rho^{-9}) A_{43},
\end{aligned}$$

$$\begin{aligned}
(d^*dA)_4 &= -\epsilon^2 \rho^{-6} \cdot \partial_1 A_{14} - \epsilon^2 \rho^{-2} \partial_2 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_3 A_{24} \\
&\quad + a \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_4 A_{24} - b \cos 2s \cdot \epsilon^2 \rho^{-2} \partial_2 A_{34} \\
&\quad - (1 + b^2(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} - 2ab^2(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_3 A_{34} \\
&\quad - (-ab(\cos 2s)^2 \cdot \epsilon^2 \rho^{-2} + 2a^2 b(\cos 2s)^2 \cdot \epsilon^3 \rho^{-5}) \partial_4 A_{34} \\
&\quad + (-2\epsilon^2 \rho^{-4} \cos 2s) A_{14} + O(\epsilon^4) \\
&\quad + (2a\epsilon^2 \rho^{-6} + a(\cos 2s)^2 \epsilon^2 \rho^{-2} - (2a^2 + 4b^2)(\cos 2s)^2 \epsilon^3 \rho^{-5} + 4b^2 \epsilon^3 \rho^{-9}) A_{34}.
\end{aligned}$$

□

B.5 The proof of Lemma 4.5

Proof.

$$\begin{aligned}
\operatorname{Im}(\nabla_A \psi \cdot \bar{\psi}) &= (\psi_1 \partial_1 \psi_2 - \psi_2 \partial_1 \psi_1 - A_1 |\psi|^2) dy^1 \\
&\quad + (\psi_1 \partial_2 \psi_2 - \psi_2 \partial_2 \psi_1 - A_2 |\psi|^2) dy^2 \\
&\quad + \left(-\frac{U^2}{\tilde{r}} \sin \tilde{\phi} - A_3 |\psi|^2\right) dy^3 + \left(\frac{U^2}{\tilde{r}} \cos \tilde{\phi} - A_4 |\psi|^2\right) dy^4 \\
&= (\epsilon f_s \cdot \frac{U^2 - VU^2}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_s \cdot \frac{U^2 - VU^2}{\tilde{r}} \cos \tilde{\phi}) dy^1 \\
&\quad + (\epsilon f_\theta \cdot \frac{U^2 - VU^2}{\tilde{r}} \sin \tilde{\phi} - \epsilon g_\theta \cdot \frac{U^2 - VU^2}{\tilde{r}} \cos \tilde{\phi}) dy^2 \\
&\quad + \left(-\frac{U^2}{\tilde{r}} \sin \tilde{\phi} - A_3 |\psi|^2\right) dy^3 + \left(\frac{U^2}{\tilde{r}} \cos \tilde{\phi} - A_4 |\psi|^2\right) dy^4.
\end{aligned}$$

□

C The proof of lemma 5.1

Proof. Recall that

$$\begin{aligned}
\omega(x) &= -2\pi \int_{\Gamma} * [d_x (G(x-y)) \wedge \chi(y)] dH^2(y) \\
&= \frac{\pi}{4\omega_4} \int_{\Gamma} i \frac{d_x |x-y|^2}{|x-y|^4} (*\chi(y)) dH^2(y).
\end{aligned}$$

Denote $x = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})$, $y = (\frac{\sqrt{2}}{2} \rho e^{i\theta}, \frac{\sqrt{2}}{2} \rho^{-1} e^{i\theta})$, $\alpha = \theta_2 - \theta_1$. Let $e_1 = \frac{1}{\sqrt{1+\rho^{-4}}}(\partial_{\rho_1} - \rho^{-2} \partial_{\rho_2})$, $e_2 = \frac{\sqrt{2}}{\sqrt{\rho^2 + \rho^{-2}}}(\partial_{\theta_1} + \partial_{\theta_2})$, which are induced by ∂_ρ and ∂_θ . Then by direct calculation, when $\rho_1 = \rho_2$, we have:

1) If $\sqrt{1 + \cos \alpha} \leq \rho_1$, then

$\tilde{\rho}(x)^2 + \tilde{\rho}(x)^{-2} + 2 \cos \alpha = 2\rho_1^2$, where $(\frac{\sqrt{2}}{2} \tilde{\rho} e^{i\tilde{\theta}}, \frac{\sqrt{2}}{2} \tilde{\rho}^{-1} e^{i\tilde{\theta}})$ is the closest points in Γ to x .

$$a(x) = \cos \alpha \sqrt{\rho_1^2 - \cos \alpha}, \quad b(x) = \sin \alpha \sqrt{\rho_1^2 - \cos \alpha},$$

$$\operatorname{dist}(x, \Gamma) = \sqrt{\rho_1^2 - \cos \alpha},$$

$$d\phi = -d\theta_1 + d\theta_2.$$

2) If $\sqrt{1 + \cos \alpha} > \rho_1$, then

$$a(x) = \rho_1 \sqrt{1 + \cos \alpha} - 1, \quad b(x) = \frac{\rho_1 \sin \alpha}{\sqrt{1 + \cos \alpha}}.$$

so $d\phi = -j(\alpha)d\theta_1 + j(\alpha)d\theta_2$, where $j(\alpha)$ is a smooth function.

$$e^1 = \frac{1}{\sqrt{1 + \rho^{-4}}}(d\rho_1 - \rho^{-2}d\rho_2), \quad e^2 = \frac{1}{\sqrt{2\rho^2 + 2\rho^{-2}}}(\rho^2 d\theta_1 + \rho^{-2}d\theta_2),$$

$$(1 + \rho^{-4})\chi = \frac{1}{\sqrt{2}}\rho d\rho_1 \wedge d\theta_1 + \frac{1}{\sqrt{2}}\rho^{-3}d\rho_1 \wedge d\theta_2 - \frac{1}{\sqrt{2}}\rho^{-1}d\rho_2 \wedge d\theta_1 - \frac{1}{\sqrt{2}}\rho^{-5}d\rho_2 \wedge d\theta_2.$$

The metric matrix with respect to $(\rho_1, \theta_1, \rho_2, \theta_2)$ is :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho_1^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \rho_2^2 \end{pmatrix},$$

$$*(d\rho_1 \wedge d\theta_1) = \rho_1^{-1}\rho_2 d\rho_2 \wedge d\theta_2,$$

$$*(d\rho_1 \wedge d\theta_2) = -\rho_1\rho_2^{-1}d\rho_2 \wedge d\theta_1,$$

$$*(d\rho_2 \wedge d\theta_1) = -\rho_1^{-1}\rho_2 d\rho_1 \wedge d\theta_2,$$

$$*(d\rho_2 \wedge d\theta_2) = \rho_1\rho_2^{-1}d\rho_1 \wedge d\theta_1.$$

So we can get :

$$(1 + \rho^{-4}) * \chi = \frac{1}{\sqrt{2}}\rho^{-1}d\rho_2 \wedge d\theta_2 - \frac{1}{\sqrt{2}}\rho^{-1}d\rho_2 \wedge d\theta_1$$

$$+ \frac{1}{\sqrt{2}}\rho^{-3}d\rho_1 \wedge d\theta_2 - \frac{1}{\sqrt{2}}\rho^{-3}d\rho_1 \wedge d\theta_1,$$

$$|x - y|^2 = \rho_1^2 + \rho_2^2 + \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} - \sqrt{2}\rho_1\rho \cos(\theta - \theta_1) - \sqrt{2}\rho_2\rho^{-1} \cos(\theta - \theta_2).$$

In Γ , we know:

$$\partial_\rho = \partial_{\rho_1} - \rho^{-2}\partial_{\rho_2},$$

$$\partial_\theta = \partial_{\theta_1} + \partial_{\theta_2}.$$

We remark that, we denote $d\rho_1$ as $d\rho_1(y)$ for simplicity. Similarly $d\theta_1, d\rho_2, d\theta_2$ are the same.

$$d\rho_1(x) = \cos(\theta - \theta_1)d\rho_1 - \sin(\theta - \theta_1)\frac{\sqrt{2}}{2}\rho d\theta_1,$$

$$\rho_1(x)d\theta_1(x) = \sin(\theta - \theta_1)d\rho_1 + \cos(\theta - \theta_1)\frac{\sqrt{2}}{2}\rho d\theta_1,$$

$$d\rho_2(x) = \cos(\theta - \theta_2)d\rho_2 - \sin(\theta - \theta_2)\frac{\sqrt{2}}{2}\rho^{-1}d\theta_2,$$

$$\rho_2(x)d\theta_2(x) = \sin(\theta - \theta_2)d\rho_2 + \cos(\theta - \theta_2)\frac{\sqrt{2}}{2}\rho^{-1}d\theta_2.$$

Conversely,

$$\begin{aligned}
d\rho_1 &= \cos(\theta - \theta_1)d\rho_1(x) + \rho_1 \sin(\theta - \theta_1)d\theta_1(x), \\
\rho d\theta_1 &= \sqrt{2}\rho_1 \cos(\theta - \theta_1)d\theta_1(x) - \sqrt{2}\sin(\theta - \theta_1)d\rho_1(x), \\
d\rho_2 &= \cos(\theta - \theta_2)d\rho_2(x) + \rho_2 \sin(\theta - \theta_2)d\theta_2(x), \\
\rho^{-1}d\theta_2 &= \sqrt{2}\rho_2 \cos(\theta - \theta_2)d\theta_2(x) - \sqrt{2}\sin(\theta - \theta_2)d\rho_2(x),
\end{aligned}$$

$$\begin{aligned}
d_x |x - y|^2 &= [2\rho_1 - \sqrt{2}\rho \cos(\theta - \theta_1)]d\rho_1(x) - \sqrt{2}\rho_1\rho \sin(\theta - \theta_1)d\theta_1(x) \\
&\quad + [2\rho_2 - \sqrt{2}\rho^{-1} \cos(\theta - \theta_2)]d\rho_2(x) - \sqrt{2}\rho_2\rho^{-1} \sin(\theta - \theta_2)d\theta_2(x) \\
&= [2\rho_1 - \sqrt{2}\rho \cos(\theta - \theta_1)] \cdot [\cos(\theta - \theta_1)d\rho_1 - \sin(\theta - \theta_1) \cdot \frac{\sqrt{2}}{2}\rho d\theta_1] \\
&\quad - \sqrt{2}\rho \sin(\theta - \theta_1) \cdot [\sin(\theta - \theta_1)d\rho_1 + \cos(\theta - \theta_1) \cdot \frac{\sqrt{2}}{2}\rho d\theta_1] \\
&\quad + [2\rho_2 - \sqrt{2}\rho^{-1} \cos(\theta - \theta_2)] \cdot [\cos(\theta - \theta_2)d\rho_2 - \sin(\theta - \theta_2) \cdot \frac{\sqrt{2}}{2}\rho^{-1}d\theta_2] \\
&\quad - \sqrt{2}\rho^{-1} \sin(\theta - \theta_2) \cdot [\sin(\theta - \theta_2)d\rho_2 + \cos(\theta - \theta_2) \cdot \frac{\sqrt{2}}{2}\rho^{-1}d\theta_2] \\
&= 2\rho_1 \cos(\theta - \theta_1)d\rho_1 - \sin(\theta - \theta_1) \cdot \sqrt{2}\rho\rho_1 d\theta_1 - \sqrt{2}\rho d\rho_1 \\
&\quad + 2\rho_2 \cos(\theta - \theta_2)d\rho_2 - \sin(\theta - \theta_2) \cdot \sqrt{2}\rho^{-1}\rho_2 d\theta_2 - \sqrt{2}\rho^{-1}d\rho_2,
\end{aligned}$$

$$\begin{aligned}
(1 + \rho^{-4})i_{d_x|x-y|^2}(*\chi) &= [2\rho_1 \cos(\theta - \theta_1) - \sqrt{2}\rho] \cdot \left(\frac{1}{\sqrt{2}}\rho^{-3}d\theta_2 - \frac{1}{\sqrt{2}}\rho^{-3}d\theta_1\right) \\
&\quad - \sin(\theta - \theta_1) \cdot \sqrt{2}\rho\rho_1[\sqrt{2}\rho^{-3}d\rho_2 + \sqrt{2}\rho^{-5}d\rho_1] \\
&\quad + [2\rho_2 \cos(\theta - \theta_2) - \sqrt{2}\rho^{-1}] \cdot \left(\frac{1}{\sqrt{2}}\rho^{-1}d\theta_2 - \frac{1}{\sqrt{2}}\rho^{-1}d\theta_1\right) \\
&\quad - \sin(\theta - \theta_2) \cdot \sqrt{2}\rho^{-1}\rho_2 \cdot [-\sqrt{2}\rho d\rho_2 - \sqrt{2}\rho^{-1}d\rho_1] \\
&= [2\rho^{-2}\rho_2 \sin(\theta - \theta_2) - 2\rho^{-4}\rho_1 \sin(\theta - \theta_1)]d\rho_1 \\
&\quad + [2\rho_2 \sin(\theta - \theta_2) - 2\rho^{-2}\rho_1 \sin(\theta - \theta_1)]d\rho_2 \\
&\quad + [2\rho^{-2} - \sqrt{2}\rho_1\rho^{-3} \cos(\theta - \theta_1) - \sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2)]d\theta_1 \\
&\quad + [-2\rho^{-2} + \sqrt{2}\rho^{-3}\rho_1 \cos(\theta - \theta_1) + \sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2)]d\theta_2 \\
&= [2\rho^{-2}\rho_2 \sin(\theta - \theta_2) - 2\rho^{-4}\rho_1 \sin(\theta - \theta_1)] \cdot [\cos(\theta - \theta_1)d\rho_1(x) + \rho_1 \sin(\theta - \theta_1)d\theta_1(x)] \\
&\quad + [2\rho_2 \sin(\theta - \theta_2) - 2\rho^{-2}\rho_1 \sin(\theta - \theta_1)] \cdot [\cos(\theta - \theta_2)d\rho_2(x) + \rho_2 \sin(\theta - \theta_2)d\theta_2(x)] \\
&\quad + [2\rho^{-2} - \sqrt{2}\rho_1\rho^{-3} \cos(\theta - \theta_1) - \sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2)] \cdot \\
&\quad [\sqrt{2}\rho^{-1}\rho_1 \cos(\theta - \theta_1)d\theta_1(x) - \sqrt{2}\rho^{-1} \sin(\theta - \theta_1)d\rho_1(x)] \\
&\quad + [-2\rho^{-2} + \sqrt{2}\rho^{-3}\rho_1 \cos(\theta - \theta_1) + \sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2)] \cdot \\
&\quad [\sqrt{2}\rho\rho_2 \cos(\theta - \theta_2)d\theta_2(x) - \sqrt{2}\rho \sin(\theta - \theta_2)d\rho_2(x)] \\
&= [-2\sqrt{2}\rho^{-3} \sin(\theta - \theta_1) + 2\rho^{-2}\rho_2 \sin(2\theta - \theta_1 - \theta_2)]d\rho_1(x) \\
&\quad + [2\sqrt{2}\rho^{-1} \sin(\theta - \theta_2) - 2\rho^{-2}\rho_1 \sin(2\theta - \theta_1 - \theta_2)]d\rho_2(x) \\
&\quad + [-2\rho^{-4}\rho_1^2 + 2\sqrt{2}\rho^{-3}\rho_1 \cos(\theta - \theta_1) - 2\rho^{-2}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_1(x) \\
&\quad + [2\rho_2^2 - 2\sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2) + 2\rho^{-2}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_2(x).
\end{aligned}$$

Besides, for the manifold Γ , the metric matrix with respect to (ρ, θ) is:

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{2}\rho^{-4} & 0 \\ 0 & \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} \end{pmatrix}.$$

Therefore, we finally get:

$$\begin{aligned}
\omega(x) &= \frac{\pi}{4\omega_4} \int_{\Gamma} i \frac{d_x |x-y|^2}{|x-y|^4} (*\chi(y)) dH^2(y) \\
&= \frac{\pi}{8\omega_4} \int_{\rho, \theta} [\rho_1^2 + \rho_2^2 + \frac{1}{2}\rho^2 + \frac{1}{2}\rho^{-2} - \sqrt{2}\rho_1\rho \cos(\theta - \theta_1) - \sqrt{2}\rho_2\rho^{-1} \cos(\theta - \theta_2)]^{-2} \cdot \\
&\quad \{[-2\sqrt{2}\rho^{-3} \sin(\theta - \theta_1) + 2\rho^{-2}\rho_2 \sin(2\theta - \theta_1 - \theta_2)]d\rho_1(x) \\
&\quad + [2\sqrt{2}\rho^{-1} \sin(\theta - \theta_2) - 2\rho^{-2}\rho_1 \sin(2\theta - \theta_1 - \theta_2)]d\rho_2(x) \\
&\quad + [-2\rho^{-4}\rho_1^2 + 2\sqrt{2}\rho^{-3}\rho_1 \cos(\theta - \theta_1) - 2\rho^{-2}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_1(x) \\
&\quad + [2\rho_2^2 - 2\sqrt{2}\rho^{-1}\rho_2 \cos(\theta - \theta_2) + 2\rho^{-2}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_2(x)\} \cdot \rho d\rho d\theta \\
&= \frac{\pi}{8\omega_4} \int_{\rho, \theta} |x-y|^{-4} \cdot \{[-2\sqrt{2}\rho^{-2} \sin(\theta - \theta_1) + 2\rho^{-1}\rho_2 \sin(2\theta - \theta_1 - \theta_2)]d\rho_1(x) \\
&\quad + [2\sqrt{2} \sin(\theta - \theta_2) - 2\rho^{-1}\rho_1 \sin(2\theta - \theta_1 - \theta_2)]d\rho_2(x) \\
&\quad + [-2\rho^{-3}\rho_1^2 + 2\sqrt{2}\rho^{-2}\rho_1 \cos(\theta - \theta_1) - 2\rho^{-1}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_1(x) \\
&\quad + [2\rho_2^2\rho - 2\sqrt{2}\rho_2 \cos(\theta - \theta_2) + 2\rho^{-1}\rho_1\rho_2 \cos(2\theta - \theta_1 - \theta_2)]d\theta_2(x)\} d\rho d\theta.
\end{aligned}$$

If we change θ into $-\theta + \theta_1 + \theta_2$, then:

$$\begin{aligned}
\theta - \theta_1 &\rightarrow -\theta + \theta_2, \\
\theta - \theta_2 &\rightarrow -\theta + \theta_1.
\end{aligned}$$

Thus, if $\rho_1 = \rho_2$, i.e. $x \in \{(\rho_1, \theta_1, \rho_2, \theta_2) : \rho_1 = \rho_2\}$, we have

$$\begin{aligned}
\omega(x) &= \frac{\pi}{8\omega_4} \int_{\rho \geq 1, 0 \leq \theta < 2\pi} |x-y|^{-4} \cdot \{[-2\sqrt{2}\rho^{-2} \sin(\theta - \theta_1) + 4\rho^{-1}\rho_1 \sin(2\theta - \theta_1 - \theta_2) \\
&\quad - 2\sqrt{2} \sin(\theta - \theta_2)]d\rho_1(x) \\
&\quad + [2\sqrt{2} \sin(\theta - \theta_2) - 4\rho^{-1}\rho_1 \sin(2\theta - \theta_1 - \theta_2) + 2\sqrt{2}\rho^{-2} \sin(\theta - \theta_1)]d\rho_2(x) \\
&\quad + [-2\rho^{-3}\rho_1^2 + 2\sqrt{2}\rho^{-2}\rho_1 \cos(\theta - \theta_1) + 2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2)]d\theta_1(x) \\
&\quad + [2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2) - 2\rho_1^2\rho^{-3} + 2\sqrt{2}\rho_1\rho^{-2} \cos(\theta - \theta_1)]d\theta_2(x)\} d\rho d\theta.
\end{aligned}$$

Note that the tangent space of $\{\rho_1 = \rho_2\}$ is spanned by $\partial_{\rho_1} + \partial_{\rho_2}$, ∂_{θ_1} , ∂_{θ_2} . So we get that:

$$\begin{aligned}
\omega_{\top}(x) &= \frac{\pi}{8\omega_4} \int_{\rho \geq 1, 0 \leq \theta < 2\pi} |x-y|^{-4} \cdot \{[-2\rho^{-3}\rho_1^2 + 2\sqrt{2}\rho^{-2}\rho_1 \cos(\theta - \theta_1) \\
&\quad + 2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2)]d\theta_1(x) \\
&\quad + [2\rho_1^2\rho - 2\sqrt{2}\rho_1 \cos(\theta - \theta_2) - 2\rho_1^2\rho^{-3} + 2\sqrt{2}\rho_1\rho^{-2} \cos(\theta - \theta_1)]d\theta_2(x)\} d\rho d\theta.
\end{aligned}$$

□

References

- [1] C. Arezzo and F. Pacard. Complete, embedded, minimal n -dimensional submanifolds in C^n . *Communications on pure and applied mathematics* 56.3 (2003): 283-327.
- [2] M. S. Berger, Y.Y. Chen. Symmetric vortices for the Ginzberg-Landau equations of superconductivity and the nonlinear desingularization phenomenon. *Journal of functional analysis* 82.2 (1989): 259-295.
- [3] F. Bethuel, H. Brezis, and F Hlein. *Ginzburg-landau vortices*. Vol. 13. Boston: Birkhuser, 1994.
- [4] F. Bethuel, H. Brezis, and G. Orlandi. Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions. *Journal of Functional Analysis* 186.2 (2001): 432-520.
- [5] S. Brendle, On solutions to the Ginzburg-Landau equations in higher dimensions, arXiv:math/0302070v2 [math. DG].
- [6] X. Cabré. Uniqueness and stability of saddle-shaped solutions to the AllenCahn equation. *Journal de mathmatiques pures et appliques* 98.3 (2012): 239-256.
- [7] J. Dávila, M. Del Pino, M. Medina, and R. Rodiac. Interacting helical vortex filaments in the 3-dimensional Ginzburg-Landau equation. arXiv preprint arXiv:1901.02807 (2019).
- [8] M. Del. Pino, and J. Wei. Solutions to the Allen Cahn equation and minimal surfaces. *Milan Journal of Mathematics* 79.1 (2011): 39-65.
- [9] R. Frank, C. Hainzl, R. Seiringer, and J. Solovej . Microscopic derivation of Ginzburg-Landau theory. *Journal of the American Mathematical Society* 25.3 (2012): 667-713.
- [10] V. L. Ginzburg, and L. D. Landau. *On the theory of superconductivity. On Superconductivity and Superfluidity*. Springer, Berlin, Heidelberg, 2009. 113-137.
- [11] L. P. Gorkov. Microscopic derivation of the Ginzburg-Landau equations in the theory of superconductivity. *Sov. Phys. JETP* 9.6 (1959): 1364-1367.

- [12] S. Gustafson, I. M. Sigal. The stability of magnetic vortices. *Comm. Math. Phys.* 212 (2000), no. 2, 257–275.f
- [13] S. Gustafson, I. M. Sigal, and T. Tzaneteas. Statics and dynamics of magnetic vortices and of Nielsen-Olesen (Nambu) strings. *Journal of mathematical physics* 51.1 (2010): 015217.
- [14] R. L. Jerrard, and H. M. Soner. The jacobian and the Ginzburg-Landau energy. *Calculus of Variations and Partial Differential Equations* 14.2 (2002): 151-191.
- [15] G. Lawlor, The angle criterion, *Invent. Math.* 95 (1989), no. 2, 437-446.
- [16] F. H. Lin. Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension 2 submanifolds. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 51.4 (1998): 385-441.
- [17] F. H. Lin, and T. Riviere. Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents. *Journal of the European Mathematical Society* 1.3 (1999): 237-311.
- [18] Y. Liu, K. Wang, and J. Wei. Stability of the saddle solutions for the Allen-Cahn equation. *arXiv:2001.07356* (2020).
- [19] Y. Liu, and J. Wei. Multivortex Traveling Waves for the Gross-Pitaevskii Equation and the Adler-Moser Polynomials. *SIAM Journal on Mathematical Analysis* 52.4 (2020): 3546-3579.
- [20] A. Pakylak, F. Ting, and J. Wei. Multi-vortex solutions to Ginzburg-Landau equations with external potential. *Archive for Rational Mechanics and Analysis* 204.1 (2012): 313-354.
- [21] A. Pigati, and D. Stern. Minimal submanifolds from the abelian Higgs model. *Inventiones mathematicae* 223.3 (2021): 1027-1095.
- [22] B.J. Plohr. The existence, regularity, and behavior of isotropic solutions of classical gauge field theories. Princeton University, 1980.
- [23] O. Savin. Regularity of flat level sets in phase transitions. *Annals of Mathematics* (2009): 41-78.
- [24] F. Ting, and J. Wei. Multi-vortex non-radial solutions to the magnetic Ginzburg-Landau equations. *Communications in Mathematical Physics* 317.1 (2013): 69-97.