INFINITELY MANY VORTEX SOLUTIONS OF THE MAGNETIC GINZBURG-LANDAU EQUATION WITH EXTERNAL POTENTIALS IN $\mathbb{R}^2$

JUNCHENG WEI AND YUANZE WU

ABSTRACT. In this paper, we consider the magnetic Ginzburg-Landau equation with external potentials:

$$\begin{cases} -\Delta_A \psi + \dfrac{\lambda}{2}(|\psi|^2 - 1)\psi + \mu V(x)\psi = 0 & \text{in } \mathbb{R}^2, \\ \nabla \times \nabla \times A + \text{Im}(\overline{\psi}\nabla_A \psi) = 0 & \text{in } \mathbb{R}^2, \\ |\psi| \to 1 & \text{as } |x| \to +\infty, \end{cases}$$

where $\lambda > 1$ is a coupling constant, $\mu > 0$ is a parameter, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A \cdot \nabla_A$ are, respectively, the covariant gradient and Laplacian, $\nabla \times$ is the curl operator in $\mathbb{R}^2$ and $V(x)$ is a potential of impurities. We prove, by secondary Liapunov-Schmidt reduction method, that under suitable conditions on $V(x)$ and a smallness condition on $\mu > 0$, the magnetic Ginzburg-Landau equation with external potentials in $\mathbb{R}^2$ has infinitely many multi-vortex solutions.

Keywords: Magnetic Ginzburg-Landau equation; External potentials; Multi-vortex solutions; Infinitely many solutions.

AMS Subject Classification 2010: 35B20; 35J47; 35Q56.

1. INTRODUCTION

The Ginzburg-Landau theory [20] is a central part of theory of superconductivity. It gives a macroscopic description of a superconducting material in terms of a complex-valued function $\psi(x)$ (named order parameters) and the vector field $A(x)$, so that $|\psi(x)|^2$ gives the local density of (Cooper pairs of) superconducting electrons and $B(x) = \nabla \times A(x)$ is the magnetic field. Here, $\nabla \times$ is the curl operator. In this theory, equilibrium configurations of superconductors are described by a system of nonlinear PDE called the Ginzburg-Landau equations. Since in the idealized situation of a superconductor occupying, all space are homogeneous in one direction, the Ginzburg-Landau equations can be written down as follows:

$$\begin{cases} -\Delta_A \psi + \dfrac{\lambda}{2}(|\psi|^2 - 1)\psi = 0 & \text{in } \mathbb{R}^2, \\ \nabla \times \nabla \times A + \text{Im}(\overline{\psi}\nabla_A \psi) = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where for a vector function A, $\nabla \times A = \partial_1 A_2 - \partial_2 A_1$ and for a scalar function A, $\nabla \times A = (-\partial_2 A, \partial_1 A)$.

1

It is well known that (1.1) is the Euler-Lagrange equation of the following Ginzburg-Landau energy functional:

$$\mathcal{E}_\lambda(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla_A \psi|^2 + |\nabla \times A|^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2, \qquad (1.2)$$

which models the difference in free energy between the superconducting and normal states near the transition temperature in the Ginzburg-Landau theory. As above, in the Ginzburg-Landau energy functional (1.2), ψ indicates the local state of the material: If $|\psi| \approx 1$ then the material is in the superconducting phase while if $|\psi| \approx 0$ then the material is in the normal phase. A is the vector potential where $\nabla \times A$ is the induced magnetic field. The parameter λ is a material constant, corresponding to the ratio between characteristic lengthscales of the material: If $\lambda < 1$ then the material is of type I superconductor while if $\lambda > 1$ then the material is of type II superconductor. $\lambda = 1$ is the critical case of these two types. Experimentally, type I and type II materials differ in their magnetic behavior. In type I superconductors, magnetic fields are excluded from the bulk of the material except for a very thin layer near the surface. In type II superconductors, magnetic fields penetrate the material in vortex structures. In general, type II superconductors can sustain magnetic fields much higher than type I superconductors without losing their superconducting state. The existence of magnetic vortices and of type II superconductors was predicted in 1957 by Abrikosov [1]. We remark that the Ginzburg-Landau energy functional (1.2) (and also the Ginzburg-Landau equations (1.1)) also arise in particle physics, as the energy of a static configuration in the Yang-Mills-Higgs classical gauge theory on the plane, with abelian gauge group $U(1)$. For more details of the physical backgrounds of the Ginzburg-Landau equations (1.1) and the Ginzburg-Landau energy functional (1.2), we would like refer the readers to [23, 25, 38, 43] and the references therein.

It has been proved in [25] that finite energy solutions of the Ginzburg-Landau equations (1.1) satisfy the boundary condition:

$$(|\psi|, |\nabla_A \psi|, |\nabla \times A|) \to (1, 0, 0) \quad \text{as } |x| \to +\infty.$$

Thus, their topological degrees which are also called winding numbers or vortex numbers, are well defined in the following way:

$$deg(\psi) = deg\left(\frac{\psi}{|\psi|}|_{|x|=R}\right) = \frac{1}{2\pi} \int_{|x|=R} d(arg(\psi)) \quad \text{for } R \text{ sufficiently large.}$$

It is worth pointing out that by the Stokes theorem, this degree of ψ satisfies

$$2\pi deg(\psi) = \int_{\mathbb{R}^2} \nabla \times A,$$

so that $deg(\psi)$ is also related to the flux quantization of the magnetic field $B = \nabla \times A$. Except global minimizers of (1.2), other finite energy solutions of (1.1) must satisfy $\deg(\psi) \neq 0$, so that ψ must have zeros. These zeros are often called vortices of ψ and the presence of vortices in solutions is one of the most interesting mathematical and physical phenomenon connected with Ginzburg-Landau equations (1.1), which make it to be a hot topic in the community of nonlinear PDEs in the past thirty years or so. The first non-trivial, finite energy, rigorously known solutions of the Ginzburg-Landau equations (1.1) are the radially symmetric

solutions, which are given by

$$\phi_{\lambda,N}(x) = f_\lambda(r)e^{iN\theta}; \quad B_{\lambda,N}(x) = Na_\lambda(r)\nabla\theta,$$

where $|N| \geq 1$ is its degree $\deg(\phi)$. We remark that the discrete symmetry $\psi \to \overline{\psi}$ and $A \to -A$ of (1.1) interchanges the negative degrees to the positive degrees. Thus, we can assume the degrees of solutions to be nonnegative in what follows. The existence of these radial solutions is established in [13] by variational arguments (see also [30]). The uniqueness of these radial solutions is proved in [5] and [15], respectively for $\lambda > 0$ sufficiently large and λ sufficiently close to 1 (including $\lambda = 1$). The stability of these radial solutions is also studied in the literature. It has been proved in [21] that $(\phi_{\lambda,N}, B_{\lambda,N})$ are all stable for $\lambda < 1$ while for $\lambda > 1$, $(\phi_{\lambda,1}, B_{\lambda,1})$ is stable and $(\phi_{\lambda,N}, B_{\lambda,N})$ are unstable for $N \geq 2$. By considering the singular limit ("extreme type II") $\lambda \to +\infty$, which seems to be the major direction in studying Ginzburg-Landau equations, non-radial vortex solutions of Ginzburg-Landau equations were first established in bounded domains for λ sufficiently large. A significant finding in these studies is that for λ sufficiently large, the locations of vortices is determined by some reduced finite-dimensional problem under some suitable assumptions. Since it seems almost impossible for us to provide a complete list of references for these studies, we refer the readers only to the books [11, 31, 37, 38] for their detailed introductions and references. The existence of non-radial vortex solution of Ginzburg-Landau equations (1.1) for any value of λ, which seems to be another major direction in studying Ginzburg-Landau equations nowadays, is not very clear except the critical case $\lambda = 1$. In this case, all solutions can be classified by its vortices according to Taubes' work [42, 43]. A review of this theory can be found in the book of Jaffe and Taubes [25]. For other cases $\lambda \neq 1$, it is conjectured in [29] by numerical evidence that for the non-magnetic Ginzburg-Landau equations on the whole plane, non-radial solutions do exist, while the studies in [23] suggest that for magnetic vortices, stationary multi-vortex configurations of degrees ± 1 occur with discrete symmetry group. The later conjecture was proved in [46] by reduction arguments for large degrees and large number vortices. We also world like to refer the readers to the paper [23], which reviews some mathematical aspects of the Ginzburg-Landau equations of superconductivity and of particle physics, and the very recent work [40], which considers the Abrikosov lattices, also for their detailed introductions and references.

As mentioned above, type II superconductors can sustain very large magnetic fields (over 10^5 Gauss). However, a major obstacle in the attempt to produce large magnetic fields is the dissipation of energy due to the creeping or flow of vortices [48]. One way to overcome this problem is to pin down the vortices to particular locations in the material. Since as pointed out in [17], the pinning down of vortices is achieved by the presence of point defects, impurities, or inhomogeneities, or by a variation in the thickness of the sample of superconducting material, to adapt this idea, the Ginzburg-Landau equations (1.1) will be modified by external potentials as follows:

$$\begin{cases} -\Delta_A\psi + \dfrac{\lambda}{2}(|\psi|^2 - 1)\psi + \mu V(x)\psi = 0 & \text{in } \mathbb{R}^2, \\ \qquad\qquad \nabla \times \nabla \times A + \text{Im}(\overline{\psi}\nabla_A\psi) = 0 & \text{in } \mathbb{R}^2, \\ \qquad\qquad\qquad\qquad |\psi| \to 1 & \text{as } |x| \to +\infty, \end{cases} \quad (1.3)$$

where $\mu > 0$ is a parameter and $V(x)$ is a potential of impurities. The pinning phenomenon was first observed from numerical evidence, in [16, 18], that fundamental magnetic vortices(degrees of ± 1) of the same degree are attracted to maxima of $V(x)$. Such pinning phenomenon is rigorously proved in [39] by proving the existence and uniqueness of single-vortex solution of (1.3) for $\mu > 0$ sufficiently small under some suitable conditions on $V(x)$. Moreover, it is also shown in [39] that within the standard macroscopic theory of superconductivity, a single-vortex solution will localize near a critical point of $V(x)$ for $\mu > 0$ sufficiently small. The effective dynamics of the parabolic version of (1.3) and dynamics stability of the single-vortex solution obtained in [39] were studied in [41] and [24], respectively. Recently, more pinning phenomenon has been observed in [33] by proving that under suitable assumptions on the potential $V(x)$, multi-vortex solutions of (1.3) exist. Moreover, these multi-vortex configurations can either be localized near multiple critical points of the impurity potential, one to one, or one can pin an arbitrary number of vortices to one critical point and near infinity, respectively. Effective dynamics of multi-vortices of (1.3) was also considered in [44]. We remark that the pinning phenomenon is also a hot topic for other nonlinear PDEs, see, for example, [2–4, 6, 7, 10, 12, 19, 28, 32, 34, 36, 45, 47, 51] and the references therein.

In this paper, we shall find out more pinning phenomenon of (1.3). Before we state our main results, we need first introduce some necessary notations. Let $(\phi, B) = (f(r)e^{i\theta}, b(r)\nabla\theta)$ be the fundamental vortex solution of the magnetic Ginzburg-Landau equation:

$$
\begin{cases}
-\Delta_A \psi + \dfrac{\lambda}{2}(|\psi|^2 - 1)\psi = 0 & \text{in } \mathbb{R}^2, \\
\nabla \times \nabla \times A + \text{Im}(\overline{\psi}\nabla_A \psi) = 0 & \text{in } \mathbb{R}^2, \\
\hspace{4.5cm} |\psi| \to 1 & \text{as } |x| \to +\infty,
\end{cases}
\tag{1.4}
$$

$m \geq 2$ be an integer and $\underline{z} = (z_1, z_2, \cdots, z_m) \in \mathbb{R}^m$. Then our main result can be stated as follows.

Theorem 1.1. *Suppose that the potential $V(x)$ satisfies*

(V_1) $V(x) \in L^\infty(\mathbb{R}^2)$ *such that* $\|V\|_{L^\infty(\mathbb{R}^2)} = 1$ *and* $V(x) \sim e^{-(1-\sigma_0)|x|}$ *as* $|x| \to +\infty$ *for some* $\sigma_0 \in (0, 1)$.

Then there exists $\mu_0 > 0$ sufficiently small such that for all $0 < \mu < \mu_0$, (1.3) has a sequence of solutions $\{(\psi_m, A_m)\}$ where

$$
(\psi_m, A_m) = \left(\prod_{j=1}^m \phi(x - z_j) + \xi, \sum_{j=1}^m B(x - z_j) + D \right)
$$

with $\|(\xi, D)\|_{L^\infty(\mathbb{C}\times\mathbb{R}^2)} \ll 1$ and $\min_{l \neq j} |z_l - z_j| \gg 1$.

Remark 1.1.
(a) *By Theorem 1.1, one can pin down infinitely many vortices of (1.3) near infinity under the slow decay assumption (V_1).*
(b) *Comparing with the pinning phenomenon of (1.3) near infinity which is observed in [33], in Theorem 1.1, we do not need the the impurity potential $V(x)$ to be radially symmetric. Thus, we enlarge the class of impurities which can be used to pin down vortices of (1.3) near infinity.*

For the convenience of the readers, let us now sketch our proof of Theorem 1.1. Our plan in proving Theorem 1.1 is mainly to adapt the strategies in [33,46], which can be traced back to [39], to pin down infinitely many vortices of (1.3) near infinity, that is, solving (1.3) in the orthogonal direction and the tangential direction of (1.3), respectively, by reduction arguments. Since we will deal with the non-radial potentials which are not considered in [33,46], some nontrivial modifications are needed. We start the reduction arguments by constructing approximate solutions. As in [22,33,39,46], we define approximate solutions by $v_{\underline{z},\chi} = (\psi_{\underline{z},\chi}, A_{\underline{z},\chi})$, where

$$\psi_{\underline{z},\chi} = e^{i(F_{\underline{z}}(x)+\chi(x))} \prod_{j=1}^{m} \phi(x - z_j)$$

and

$$A_{\underline{z},\chi} = \sum_{j=1}^{m} B(x - z_j) + \nabla(F_{\underline{z}}(x) + \chi(x)),$$

with $F_{\underline{z}}(x) = \sum_{j=1}^{m} z_j \cdot B(x - z_j)$ and $\chi \in H^2(\mathbb{R}^2)$. Here, $\cdot$ is the usual inner product in $\mathbb{R}^2$. We also set

$$\mathcal{M}_\varepsilon = \{v_{\underline{z},\chi} \mid (\underline{z}, \chi) \in \Sigma_\varepsilon\} \tag{1.5}$$

with

$$\Sigma_\varepsilon = \{(\underline{z}, \chi) \mid \mathcal{Q}(\underline{z}) = \min_{i \neq j} |z_i - z_j| > \varepsilon^{-1} \quad \text{and} \quad \chi \in H^2(\mathbb{R}^2)\}.$$

For the sake of simplicity, we denote $f_j(x) = f(x - z_j)$, $\phi_j(x) = \phi(x - z_j)$ and $B_j(x) = B(x - z_j)$. For every $v_{\underline{z},\chi} \in \mathcal{M}_\varepsilon$, we choose the tangent space of $\mathcal{M}_\varepsilon$ at $v_{\underline{z},\chi}$ as in [22,33,46], which is given by

$$\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon = span\left\{G_\gamma^{\underline{z},\chi}, T_{j,k}^{\underline{z},\chi} \mid j = 1, 2, \cdots, m; k = 1, 2; \gamma \in H^2(\mathbb{R}^2)\right\}.$$

where

$$G_\gamma^{\underline{z},\chi} := \langle \gamma, \partial_\chi \rangle v_{\underline{z},\chi} = \partial_\chi v_{\underline{z},\chi} \mid_\gamma = (i\gamma\psi_{\underline{z},\chi}, \nabla\gamma)$$

and

$$\begin{aligned} T_{j,k}^{\underline{z},\chi} : &= \partial_{z_{j,k}}^{B_k} v_{\underline{z},\chi} = -\partial_{z_{j,k}} v_{\underline{z},\chi} + \langle z_j \cdot \partial_{z_{j,k}} B_j, \partial_\chi \rangle v_{\underline{z},\chi} \\ &= (e^{i(F_{\underline{z}}(x)+\chi(x))} \prod_{l \neq j} \phi_l (\nabla_{B_j}\phi_j)_k, \nabla \times B_j e_k^\perp), \end{aligned} \tag{1.6}$$

with $\nabla_{B_j}\phi_j = ((\nabla_{B_j}\phi_j)_1, (\nabla_{B_j}\phi_j)_2)$ and $e_1^\perp = (0, 1)$ and $e_2^\perp = (-1, 0)$. It is known in [21] that $T_{j,k}^{\underline{z},\chi} \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. By our choice, we can roughly divide the equation (1.3) near $v_{\underline{z},\chi}$ into two parts: the orthogonal direction $\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon^\perp$ and the tangential direction $\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon$, where the tangential direction $\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon$ can be further divided into the gauge-transformational direction

$$\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon^g = span\left\{G_\gamma^{\underline{z},\chi} \mid \gamma \in H^2(\mathbb{R}^2)\right\}$$

and the translational direction

$$\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon^T = span\left\{T_{j,k}^{\underline{z},\chi} \mid j = 1, 2, \cdots, m; k = 1, 2\right\}.$$

Since we want to pin down infinitely many vortices of (1.3) near infinity, we need to keep the estimates in reduction arguments to be independent of the number of

vortices. However, because the impurity potential $V(x)$ may be non-radial now, the functional framework in [33, 39, 46], that is, solving (1.3) in $L^2(\mathbb{R}^2; \mathbb{R}^2 \times \mathbb{C})$, is not very suitable. Indeed, if we work in $L^2(\mathbb{R}^2; \mathbb{R}^2 \times \mathbb{C})$ then by the well-known estimates (cf. [25, 30]),

$$
\begin{aligned}
&|f(r) - 1| \lesssim e^{-m_\lambda r} \quad \text{and} \quad |b(r) - 1| \lesssim e^{-r}; \\
&|f'(r)| \lesssim e^{-m_\lambda r}; \\
&Im(\overline{\phi}\nabla_B \phi) = \beta K(|x|)[1 + o(e^{-m_\lambda r})]x_0^\perp; \\
&\nabla \times B = \beta K(r)[1 - \frac{1}{2r} + O(\frac{1}{r^2})],
\end{aligned}
\tag{1.7}
$$

where $m_\lambda = \min\{\sqrt{\lambda}, 2\}$, $x_0^\perp = (-\sin\theta, \cos\theta)$ and $K(x)$ is the modified Bessel function of order 1 of the second kind such that $K(r) \sim \frac{1}{\sqrt{r}}e^{-r}$ as $r \to +\infty$, the error of $v_{\underline{z},\chi}$ to be a solution of (1.3) will be of the form

$$
\sum_{l \neq j} c_{l,j} \frac{e^{-|z_j - z_l|}}{\sqrt{|z_j - z_l|}} \left(1 + O\left(\frac{1}{|z_l - z_j|}\right)\right).
\tag{1.8}
$$

It is easy to see that such kind of error can not be uniformly for the number of vortices. Thus, to keep the error to be sufficiently small, we need to enlarge the distance of all z_j and z_l, which will make $\mu \to 0$ in construction, in adding the number of vortices, and thus, only arbitrary number of multi-vortex solutions of (1.3) can be pinned down near infinity for fixed $\mu > 0$ sufficiently small if we use this framework. Hence, our nontrivial modifications should begin with the functional setting. We remark that the norm $\|u\|_\sharp = \sup_{x \in \mathbb{R}^2} \frac{|u|}{W_{\underline{z},\sigma}}$, where $W_{\underline{z},\sigma} = \sum_{j=1}^m e^{-(1-\sigma)|x-z_j|}$ for a small $\sigma > 0$, has been used to construct infinitely many solutions of the scalar field equations in $\mathbb{R}^N$ (cf. [8, 27]) and optimal number of solutions of the Lin-Ni-Takagi problem (cf. [9]). Therefore, we shall adapt the ideas in [8, 9, 27] to modify the functional setting in our study on (1.3). We also remark that since the fundamental vortex solution (ϕ, B) do not decay at infinity, we need to slightly strengthen the slow decay assumption used in [8] (see also [14]) to (V_1) and modify the norm introduced in [8, 27] to $\|u\|_\sharp = \sup_{x \in \mathbb{R}^2} \frac{|u|}{W_{\underline{z},\sigma}^*}$, where $W_{\underline{z},\sigma}^*(x) = W_{\underline{z},\sigma}(x) + e^{-(1-\sigma_0)|x|}$, so that we can obtain a good error estimate of $v_{\underline{z},\chi}$ in the space generated by the norm $\|u\|_\sharp$. To continue the reduction argument, we need to establish a good linear theory. Our ideas to prove the linear theory is standard (cf. [8,9,27]), based on blow-up arguments. However, since we want to find infinitely many multi-vortex solutions, the main difficulty in establishing the linear theory is to keep the estimates in the linear theory to be uniformly for the number of vortices. To achieve this goal, we establish a basic lemma (Lemma 2.1), choose good gauges for the approximate solution $v_{\underline{z},\chi}$ and apply some cutoff technique to control the decay property at infinity to be uniformly for the number of vortices. When a good linear theory is established, the corresponding nonlinear problem $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = 0$, which can be expanded as

$$
\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = \mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{L}_{\underline{z},\chi}(\eta) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta),
$$

can be solved in the orthogonal direction $\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon^\perp$ directly by applying the contraction mapping theorem in the Banach space generated by the norm $\|u\|_\sharp$. After doing these, we are in the position to solve (1.3) in the tangential direction $\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}_\varepsilon$.

As pointed out in [33, 46], since (1.3) is gauge invariant and the perturbations η are in the orthogonal direction of (1.3), the gauge-transformational direction $\mathcal{T}_{v_{\underline{z},\chi}} \mathcal{M}_\varepsilon^g$ can be solved automatically. For the translational direction $\mathcal{T}_{v_{\underline{z},\chi}} \mathcal{M}_\varepsilon^T$, we will use variational arguments to solve it by considering a minimizing problem of the reduced energy functional about the locations of vortices, as that in [33]. However, the crucial energy estimates used in [33] (see [33, Lemma 6.1]), which is essentially established in [22], is also not very suitable to deal with the non-radial cases of pinning vortices at infinity, since it still contains the term (1.8) so that it is still not uniformly for the number of vortices. Thus, we shall establish some other energy estimates to continue our reduction arguments, which is the crucial in our proof. We start our energy estimates by expanding the reduced energy functional, which is slightly different from that in [22, 33, 46]. We then use the secondary reduction arguments, as that in [8, 9], to obtain a good upper bound of the reduced energy functional when the configuration $\underline{z}$ is splitting in minimizing the reduced energy functional in a suitable configuration space. We remark that since the magnetic Ginzburg-Landau equation (1.3) is actually a four-coupled system and it is gauge invariant, the estimates are very complicated and we need to analyze (1.3) very carefully and choose good gauges to control the errors very well in this secondary reduction argument. When a good upper bound of the reduced energy functional is established, then the slow decay assumption (V_1) will help us to exclude the case that the configuration $\underline{z}$ will split in minimizing the reduced energy functional in a suitable configuration space, as that in [8, 9, 14]. To finish the reduction arguments, we also need to drive a good lower bound of the reduced energy functional to kill the chance that the configuration $\underline{z}$ will move to the boundary of the configuration space in minimizing the reduced energy functional in this space, which need us to find out the leading order term in expanding the reduced energy functional. This leading order term in expanding the reduced energy functional is given by

$$\int_{\mathbb{R}^2} (\sum_{j=1}^{m-1} \nabla \times B_j) \nabla \times B_m + \prod_{j=1}^m f_j^2 (\sum_{j=1}^{m-1} (B_j - \nabla \theta_j))(B_m - \nabla \theta_m)$$

and it looks like $\sum_{j=1}^m d_j' |z_j - z_m|^{\frac{1}{2}} e^{-|z_j - z_m|}$ as $\varepsilon \to 0^+$, which is good enough to kill the chance that the configuration $\underline{z}$ will move to the boundary of the configuration space in minimizing the reduced energy functional in this space by taking $\mu > 0$ sufficiently small. After doing these, the translational direction can be solved by adapting the standard variational arguments.

This paper is organized as follows. In section 2, we estimate the approximate solution to know how far it to be a true solution in the Banach space generated by the norm $\|u\|_\sharp$. We then establish the linear theory in section 3 and solve the nonlinear problem in the orthogonal direction in section 4. The section 5 is devoted to the property of the reduced energy functional, while in section 6, we will drive the crucial energy estimates by the secondary reduction. In section 7, we finish our reduction argument by solving a minimizing problem of the reduced energy functional.

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$.

2. Approximate solution

Clearly, (1.3) is variational in $H^1_{loc}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ and its corresponding functional is given by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla_A \psi|^2 + |\nabla \times A|^2 + \frac{\lambda}{4}(|\psi|^2 - 1)^2 + \mu V(x)(|\psi|^2 - 1),$$

where $u = (\psi, A) \in H^1_{loc}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Let us write $\mathcal{F}_{\lambda,\mu}(u) = \mathcal{E}'_{\lambda,\mu}(u)$, that is,

$$\mathcal{F}_{\lambda,\mu}(u) = \left(-\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi + \mu V(x)\psi, -\nabla \times \nabla \times A - \mathrm{Im}(\overline{\psi}\nabla_A \psi) \right).$$

Then solving (1.3) is equivalent to solving $\mathcal{F}_{\lambda,\mu}(u) = 0$ in $\mathcal{D}^{-1}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, where $\mathcal{D}^{-1}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ is the dual space of $\mathcal{C}_0^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. As in [22, (92)], by a direct calculation, we have $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) = ([\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})]_\psi, [\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})]_A)$, where

$$
\begin{aligned}
[\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})]_\psi &= e^{i(F_{\underline{z}}+\chi)}\left(\frac{\lambda}{2}\prod_{l=1}^m \phi_l(\sum_{j=1}^m (1 - f_j^2) - (1 - \prod_{j=1}^m f_j^2)) \right. \\
&\left. -2\sum_{j<l}\prod_{k\neq j,l}\phi_k \nabla_{B_j}\phi_j \cdot \nabla_{B_l}\phi_l + \mu V(x)\prod_{j=1}^m \phi_j \right)
\end{aligned}
\tag{2.1}
$$

and

$$[\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})]_A = \sum_{j=1}^m (1 - \prod_{l\neq j} f_l^2) Im(\overline{\phi}_j \nabla_{B_j}\phi_j). \tag{2.2}$$

To carry on the reduction arguments, we will start by the estimate of approximate solutions $v_{\underline{z},\chi} = (\psi_{\underline{z},\chi}, A_{\underline{z},\chi})$ in a suitable sense. Let

$$W_{\underline{z},\sigma}(x) = \sum_{j=1}^m e^{-(1-\sigma)|x-z_j|}, \tag{2.3}$$

where $\sigma < \min\{\sigma_0, \frac{1}{100}, \frac{m_\lambda - 1}{100m_\lambda}\}$. Then we have the following lemma, which will be useful below.

Lemma 2.1. *We have* $\|W_{\underline{z},\sigma}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$ *uniformly for* $\underline{z} \in \mathcal{M}_\varepsilon$ *and* $m \in \mathbb{N}$.

Proof. We re-denote $\underline{z}$ by $\underline{z}_m$ to emphasize its dependence on m. Let us fix $x \in \mathbb{R}^2$ and without loss of generality, we assume $|x - z_1| \leq |x - z_j|$ for all other j. Then $W_{\underline{z}_m,\sigma}(x) \leq W_{\underline{z}_m^*,\sigma}(x)$, where $\underline{z}_m^* = \{x, z_2, \cdots, z_m\} \in \mathcal{M}_{\frac{\varepsilon}{2}}$. Since $\underline{z}_m^* \in \mathcal{M}_{\frac{\varepsilon}{2}}$, $W_{\underline{z}_m^*,\sigma}(x) \leq W_{\underline{z}^{**},\sigma}(x)$, where $\underline{z}^{**} = \{x, z_2^{**}, \cdots, z_m^{**}, \cdots\} \in \mathcal{M}_{\frac{\varepsilon}{2}}$ has infinitely many points such that x is the center, every 7 points in $\underline{z}^{**}$ will form a regular hexagon and every side in regular hexagons is equal to $\frac{1}{2\varepsilon}$. Clearly, $W_{\underline{z}^{**},\sigma}(x) = \max_{\underline{z}\in\mathcal{M}_{\frac{\varepsilon}{2}}} W_{\underline{z},\sigma}(x)$. We choose one line, which cross x and is parallel with one side of the regular hexagon centered at x, to re-label it as the zero line, and we re-label the line which is orthogonal to the zero line to be the zero row. We next re-label every z_l^{**} in $\overrightarrow{z}^{**}$ by $z_{t,s}^{**}$ to denote the distance of z_l^{**} and x in regular hexagons in this new coordinate. Clearly, $s \in \mathbb{Z}$. Moreover, when s is odd then $t = \pm(j - \frac{1}{2})$ and when s is even then $t = \pm j$ where $j \in \mathbb{N}$. Thus, it is easy to see that $|x - z_{t,s}^{**}|^2 \geq \frac{3}{4}|s|^2(\frac{1}{2\varepsilon})^2 + \min\{(j - \frac{1}{2})^2, j^2\}(\frac{1}{2\varepsilon})^2$ for all (t,s). It follows that

$$W_{\underline{z}^{**},\sigma}(x) \lesssim \sum_{j=0}^{+\infty} e^{-(1-\sigma)j\frac{1}{2\varepsilon}} \sum_{s=0}^{+\infty} e^{-(1-\sigma)s\frac{1}{2\varepsilon}} \lesssim 1.$$

Thus, for every $x \in \mathbb{R}^2$, we have $W_{\underline{z},\sigma}(x) \lesssim 1$, which is uniformly for $\underline{z} \in \mathcal{M}_\varepsilon$ and $m \in \mathbb{N}$. It completes the proof. $\qquad \square$

Based on Lemma 2.1, we introduce the following norm:

$$\|u\|_\sharp = \sup_{x \in \mathbb{R}^2} \frac{|u|}{W_{\underline{z},\sigma}^*},$$

where $W_{\underline{z},\sigma}^*(x) = W_{\underline{z},\sigma}(x) + e^{-(1-\sigma_0)|x|}$ with $W_{\underline{z},\sigma}$ given by (2.3). Since

$$\sum_{j=1}^m (1 - f_j^2) - (1 - \prod_{j=1}^m f_j^2) = f_j^2 \left(\prod_{l \neq j}(f_l^2 - 1 + 1) - 1 \right) - \sum_{l \neq j}(f_l^2 - 1)$$

$$= (f_j^2 - 1)(\sum_{l \neq j}(f_l^2 - 1) + h.o.t.)$$

and

$$e^{-(1-\sigma)|x-z_j|}e^{-(1-\sigma)|x-z_l|} \leq e^{-\frac{1-\sigma}{2\varepsilon}}(e^{-(1-\sigma)|x-z_j|} + e^{-(1-\sigma)|x-z_l|}),$$

by (1.7) and Lemma 2.1, it is easy to see that

$$\|\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})\|_\sharp \lesssim e^{-\frac{1-\sigma}{2\varepsilon}} + \mu. \tag{2.4}$$

Therefore, if μ, ε is sufficiently small, then $\mathcal{F}_{\lambda,\mu}(u) = 0$ in $\mathcal{D}^{-1}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ has an almost solution $v_{\underline{z},\chi} = (\psi_{\underline{z},\chi}, A_{\underline{z},\chi})$ in $H_{loc}^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Thus, to solve $\mathcal{F}_{\lambda,\mu}(u) = 0$ in $\mathcal{D}^{-1}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, it is sufficient to write $u = v_{\underline{z},\chi} + \eta$ and find $\eta_{\underline{z},\chi}$ sufficiently small in a suitable sense such that $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta_{\underline{z},\chi}) = 0$ in $\mathcal{D}^{-1}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ for $\mu, \varepsilon > 0$ all sufficiently small.

3. Linear theory

If we write $\mathcal{L}_{\underline{z},\chi} = \mathcal{E}''_{\lambda,0}(v_{\underline{z},\chi})$, then for $\eta = (\xi, D) \in H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, it is known in [21] that $\mathcal{L}_{\underline{z},\chi}(\eta) = ([\mathcal{L}_{\underline{z},\chi}(\eta)]_\psi, [\mathcal{L}_{\underline{z},\chi}(\eta)]_A)$, where

$$[\mathcal{L}_{\underline{z},\chi}(\eta)]_\psi = -\Delta_{A_{\underline{z},\chi}}\xi + \frac{\lambda}{2}(2|\psi_{\underline{z},\chi}|^2 - 1)\xi + \frac{\lambda}{2}\psi_{\underline{z},\chi}^2\overline{\xi}$$
$$+ 2i\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi} \cdot D + i\psi_{\underline{z},\chi}\,div(D)$$

and

$$[\mathcal{L}_{\underline{z},\chi}(\eta)]_A = -\nabla \times \nabla \times D + |\psi_{\underline{z},\chi}|^2 D + Im(\overline{\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi}}\xi - \overline{\psi_{\underline{z},\chi}}\nabla_{A_{\underline{z},\chi}}\xi).$$

As in [21, 35], we define the space

$$\mathbb{X}_{\underline{z},\chi} = \left\{ (\xi, D) \in H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \mid \langle (\xi, D), (i\gamma\psi_{\underline{z},\chi}, \nabla\gamma) \rangle = 0, \forall \gamma \in H^2(\mathbb{R}^2) \right\}$$

$$= \left\{ (\xi, D) \in H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \mid Im(\overline{\psi_{\underline{z},\chi}}\xi) - div(D) = 0 \right\}.$$

For $\eta = (\xi, D) \in H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, $div(D) - Im(\overline{\psi_{\underline{z},\chi}}\xi) \in H^1(\mathbb{R}^2)$. Thus, by the classical regularity theorem, the following equation,

$$-\Delta\gamma + |\psi_{\underline{z},\chi}|^2\gamma = div(D) - Im(\overline{\psi_{\underline{z},\chi}}\xi) \quad \text{in } \mathbb{R}^2,$$

has a unique solution $\gamma_{\underline{z},\chi}^\eta$ in $H^3(\mathbb{R}^2)$. It follows that

$$\widetilde{\eta} = \eta + (i\psi_{\underline{z},\chi}\gamma_{\underline{z},\chi}^\eta, \nabla\gamma_{\underline{z},\chi}^\eta) \in \mathbb{X}_{\underline{z},\chi}.$$

Because $(i\psi_{\underline{z},\chi}\gamma^{\eta}_{\underline{z},\chi}, \nabla\gamma^{\eta}_{\underline{z},\chi}) \in \mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}^g_\varepsilon$, where

$$\mathcal{T}_{v_{\underline{z},\chi}}\mathcal{M}^g_\varepsilon = span\left\{ G^{\underline{z},\chi}_\gamma \mid \gamma \in H^2(\mathbb{R}^2) \right\},$$

we have $\mathcal{L}_{\underline{z},\chi}(\eta) = \mathcal{L}_{\underline{z},\chi}(\widetilde{\eta})$. On the other hand, it is also known in [21] that for $\eta = (\xi, D) \in \mathbb{X}_{\underline{z},\chi}$, we have

$$\mathcal{L}_{\underline{z},\chi}(\eta) = \widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = ([\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta)]_\psi, [\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta)]_A), \tag{3.1}$$

where

$$
\begin{aligned}
[\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta)]_\psi &= -\Delta_{A_{\underline{z},\chi}}\xi + (\frac{\lambda}{2} + \frac{1}{2})|\psi_{\underline{z},\chi}|^2\xi + \frac{\lambda-1}{2}\psi^2_{\underline{z},\chi}\overline{\xi} \\
&\quad + \frac{\lambda}{2}(|\psi_{\underline{z},\chi}|^2 - 1)\xi + 2i\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi}D
\end{aligned}
$$

and

$$[\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta)]_A = -\Delta D + |\psi_{\underline{z},\chi}|^2 D + 2Im(\overline{\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi}}\xi).$$

We define

$$\widetilde{T}^{\underline{z},\chi}_{j,k} = T^{\underline{z},\chi}_{j,k}\zeta_j, \tag{3.2}$$

where $\zeta_j(x)$ is a smooth cutoff function such that $\zeta_j(x) = 1$ for $|x - z_j| \leq \frac{\mathcal{Q}(\underline{z})-1}{2}$ and $\zeta_j(x) = 0$ for $|x - z_j| \geq \frac{\mathcal{Q}(\underline{z})+1}{2}$, and $T^{\underline{z},\chi}_{j,k}$ is given by (1.6). Let

$$\mathbb{X}^{\perp}_{\underline{z},\chi} = \left\{ (\xi, D) \in \mathbb{X}_{\underline{z},\chi} \mid \langle (\xi, D), \widetilde{T}^{\underline{z},\chi}_{j,k} \rangle = 0, \forall j, k \right\}$$

and

$$\mathbb{Y}^{\perp}_{\underline{z},\chi} = \left\{ (\xi, D) \in L^\infty \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \mid \langle (\xi, D), \widetilde{T}^{\underline{z},\chi}_{j,k} \rangle = 0, \forall j, k \right\},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Then we have the following linear theory.

Proposition 3.1. *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the linear problem $\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = g$ has a unique solution in $\mathbb{X}^{\perp}_{\underline{z},\chi}$ for all $g \in \mathbb{Y}^{\perp}_{\underline{z},\chi}$ with all $m \geq 2$, all $\chi \in H^2(\mathbb{R}^2)$ and all $\underline{z} \in \mathcal{M}_\varepsilon$. Moreover, $\|\eta\|_* \lesssim \|g\|_\sharp$, where $\|\eta\|_* = \|\eta\|_\sharp + \|\nabla_{A_{\underline{z},\chi}}\xi\|_{L^\infty(\mathbb{R}^2)}$.*

Proof. The ideas of the proof mainly come from [27]. We first prove the a-prior estimate $\|\eta\|_\sharp \lesssim \|g\|_\sharp$ for $\varepsilon > 0$ sufficiently small which is uniformly for all $m \geq 2$, all $\chi \in H^2(\mathbb{R}^2)$ and all $\underline{z} \in \mathcal{M}_\varepsilon$. Assume the contrary that there exist $\varepsilon_n \to 0$ as $n \to \infty$, $m_n \in \mathbb{N}$, $\chi_n \in H^2(\mathbb{R}^2)$, $\underline{z}^n \in \mathcal{M}_{\varepsilon_n}$ and $g_n \in \mathbb{Y}^{\perp}_{\underline{z}^n,\chi^n}$ such that $\|g_n\|_\sharp \to 0$ as $n \to \infty$ and $\|\eta_n\|_\sharp = 1$, where $\widetilde{\mathcal{L}}_{\underline{z}^n,\chi_n}(\eta_n) = g_n$, that is,

$$
\begin{aligned}
g_{n,1} &= -\Delta_{A_{\underline{z}^n,\chi_n}}\xi_n + (\frac{\lambda}{2} + \frac{1}{2})|\psi_{\underline{z}^n,\chi_n}|^2\xi_n + \frac{\lambda-1}{2}\psi^2_{\underline{z}^n,\chi_n}\overline{\xi}_n \\
&\quad + \frac{\lambda}{2}(|\psi_{\underline{z}^n,\chi_n}|^2 - 1)\xi_n + 2i\nabla_{A_{\underline{z}^n,\chi_n}}\psi_{\underline{z}^n,\chi_n} \cdot D_n
\end{aligned} \tag{3.3}
$$

and

$$g_{n,2} = -\Delta D_n + |\psi_{\underline{z}^n,\chi_n}|^2 D_n + 2Im(\overline{\nabla_{A_{\underline{z}^n,\chi_n}}\psi_{\underline{z}^n,\chi_n}}\xi_n). \tag{3.4}$$

Let $\widetilde{\gamma}_{j,n}$ be a smooth cutoff function such that $\widetilde{\gamma}_{j,n} = 0$ in $B_1(z_j^n)$ and $\widetilde{\gamma}_{j,n} = 1$ in $\mathbb{R}^2 \backslash B_2(z_j^n)$. Since $\theta_{j,n}$ are all smooth functions in $\mathbb{R}^2 \backslash B_{\frac{1}{2}}(z_j^n)$, $\widetilde{\omega}_{j,n} = \widetilde{\gamma}_{j,n}\theta_{j,n}$ are also smooth in $\mathbb{R}^2$. Moreover,

$$\|\nabla\widetilde{\omega}_{j,n}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\theta_{j,n}\|_{C^1(\mathbb{R}^2 \backslash B_{\frac{1}{2}}(z_j^n))} \lesssim 1. \tag{3.5}$$

We define

$$\widetilde{\xi}_n = e^{-i(F_{\underline{z}^n} + \chi_n + \sum_{j=1}^{m_n} \widetilde{\omega}_{j,n})}\xi_n \quad \text{and} \quad \widetilde{A}_{\underline{z}^n,0} = \sum_{j=1}^{m_n} \widetilde{B}_{j,n}, \tag{3.6}$$

where $\widetilde{B}_{j,n} = b_{j,n}\nabla\theta_{j,n} - \nabla\widetilde{\omega}_{j,n}$. Then by the gauge invariance, (3.3) is equivalent to

$$
\begin{aligned}
\widetilde{g}_{n,1} =\ & -\Delta\widetilde{\xi}_n + 2i\widetilde{A}_{\underline{z}^n,0} \cdot \nabla\widetilde{\xi}_n + idiv(\widetilde{A}_{\underline{z}^n,0})\widetilde{\xi}_n + |\widetilde{A}_{\underline{z}^n,0}|^2\widetilde{\xi}_n \\
& + (\frac{\lambda}{2} + \frac{1}{2})\prod_{j=1}^{m_n} e^{2i(\theta_{j,n} - \widetilde{\omega}_{j,n})}f_{j,n}^2\widetilde{\xi}_n + \frac{\lambda-1}{2}\prod_{j=1}^{m_n} e^{2i(\theta_{j,n} - \widetilde{\omega}_{j,n})}f_{j,n}^2\overline{\widetilde{\xi}}_n \\
& + \frac{\lambda}{2}(\prod_{j=1}^{m_n} f_{j,n}^2 - 1)\widetilde{\xi}_n + 2i\sum_{j=1}^{m_n}\prod_{j=1}^{m_n} e^{i(\theta_{j,n} - \widetilde{\omega}_{j,n})}\prod_{l \neq j} f_{l,n}f_{j,n}' x_{j,n}^0 \cdot D_n \\
& + 2\sum_{j=1}^{m_n}\prod_{j=1}^{m_n} e^{i(\theta_{j,n} - \widetilde{\omega}_{j,n})}\prod_{l \neq j} f_{l,n}f_{j,n}(b_{j,n}\nabla\theta_{j,n} - \nabla\theta_{j,n}) \cdot D_n, \tag{3.7}
\end{aligned}
$$

where $\widetilde{g}_{n,1} = g_{n,1}e^{-i(F_{\underline{z}^n} + \chi_n + \sum_{j=1}^{m_n}\widetilde{\omega}_{j,n})}$ and $x_{j,n}^0 = (\cos\theta_{j,n}, \sin\theta_{j,n})$. By Lemma 2.1, (1.7) and (3.5),

$$|\widetilde{A}_{\underline{z}^n,0}| \lesssim W_{\underline{z}^n,\sigma}^*(x) \lesssim 1 \tag{3.8}$$

and

$$|\sum_{j=1}^{m_n}\prod_{l \neq j} f_{l,n}(f_{j,n}' x_{j,n}^0 - if_{j,n}(b_{j,n}\nabla\theta_{j,n} - \nabla\theta_{j,n})| \lesssim W_{\underline{z}^n,\sigma}^*(x) \lesssim 1. \tag{3.9}$$

Let $\varrho_{y,R}$ be a smooth cutoff function in $B_{R+1}(y)$ for all $y \in \mathbb{R}^2$ and $R > 0$ such that $\varrho_{y,R} = 1$ in $B_R(y)$ and $\varrho_{y,R} = 0$ in $\mathbb{R}^2 \backslash B_{R+\frac{1}{2}}(y)$. Then, by multiplying (3.7) with $\overline{\widetilde{\xi}}_n \varrho_{y,R}$ on both sides and integrating by parts,

$$
\begin{aligned}
\int_{B_R(y)} |\nabla\widetilde{\xi}_n|^2 \lesssim\ & |\int_{B_{R+1}(y)} Re(\nabla\widetilde{\xi}_n\overline{\widetilde{\xi}}_n) \cdot \nabla\varrho_{y,R}| + C(\|\eta_n\|_\sharp^2 + \|g_n\|_\sharp^2)R^2 \\
& + |\int_{B_{R+1}(y)} \varrho_{y,R}\widetilde{A}_{\underline{z}^n,0} \cdot Re(\nabla\widetilde{\xi}_n\overline{\widetilde{\xi}}_n)| \\
\lesssim\ & \frac{1}{2}|\int_{B_{R+1}(y)} \nabla|\widetilde{\xi}_n|^2 \cdot \nabla\varrho_{y,R}| + C(\|\eta_n\|_\sharp^2 + \|g_n\|_\sharp^2)R^2 \\
& + \frac{1}{2}|\int_{B_{R+1}(y)} \varrho_{y,R}\widetilde{A}_{\underline{z}^n,0} \cdot \nabla|\widetilde{\xi}_n|^2| \\
\lesssim\ & (\|\eta_n\|_\sharp^2 + \|g_n\|_\sharp^2)R^2. \tag{3.10}
\end{aligned}
$$

Here, we have used the fact that

$$|div(\widetilde{A}_{\underline{z}^n,0})| \leq \sum_{j=1}^{m_n} |(\Delta\widetilde{\gamma}_{j,n})\theta_{j,n}| \lesssim 1, \tag{3.11}$$

which comes from the well-known facts that $div(B) = 0$ in $\mathbb{R}^2$ and $\Delta\theta = 0$ in $\mathbb{R}^2\backslash\{0\}$, and the assumption $\varepsilon_n \to 0$ which leads to $\Delta\widetilde{\gamma}_{j,n}\Delta\widetilde{\gamma}_{k,n} = 0$ if $j \neq k$ for n sufficiently large. Now, by (3.10), (3.11) and multiplying (3.7) with $\Delta\overline{\widetilde{\xi}_n}$ on both sides and integrating over $B_R(y)$ for all $y \in \mathbb{R}^2$ and $R > 0$,

$$
\begin{aligned}
\int_{B_R(y)} |\Delta\widetilde{\xi}_n|^2 &\lesssim \int_{B_R(y)} Re(i\widetilde{A}_{\underline{z}^n,0} \cdot \nabla\widetilde{\xi}_n\Delta\overline{\widetilde{\xi}_n}) + (\|\eta_n\|_\sharp^2 + \|g_n\|_\sharp^2)R^2 \\
&\quad + \int_{B_R(y)} Re|div(\widetilde{A}_{\underline{z}^n,0})\widetilde{\xi}_n\Delta\overline{\widetilde{\xi}_n}| \\
&\leq \frac{1}{2}\int_{B_R(y)} |\Delta\widetilde{\xi}_n|^2 + C(\|\eta_n\|_\sharp^2 + \|g_n\|_\sharp^2)R^2. \quad (3.12)
\end{aligned}
$$

Thus, $\{\widetilde{\xi}_n\}$ is bounded in $H^2_{loc}(\mathbb{R}^2; \mathbb{C})$. On the other hand, by (3.4), it is easy to see that $\{D_n\}$ is also uniformly bounded in $H^2_{loc}(\mathbb{R}^2; \mathbb{R}^2)$. By the Sobolev embedding theorem, we may assume, without loss of generality, that $\widetilde{\eta}_n = (\widetilde{\xi}_n, D_n) \rightharpoonup \eta_0 = (\widetilde{\xi}_0, D_0)$ weakly in $H^2_{loc}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ and $\widetilde{\eta}_n = (\widetilde{\xi}_n, D_n) \to \eta_0 = (\widetilde{\xi}_0, D_0)$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ as $n \to \infty$ for some $\alpha \in (0,1)$. Moreover, since the estimates in (3.10) and (3.12) is independent of y, $\{\nabla\widetilde{\xi}_n\}$ is bounded in $L^\infty(\mathbb{R}^2)$. Let us now go back to (3.4) and (3.7). By multiplying (3.4) and (3.7) with $\overline{\widetilde{\xi}_n}$ and D_n, respectively, these two equations in $\mathbb{R}^2\backslash(\cup_{j=1}^{m_n} B_R(z_j^n))$ can be re-written as

$$-\Delta|D_n|^2 + 2(1 - \frac{\sigma}{2})|D_n|^2 \lesssim (\|g_n\|_\sharp^2 + e^{-2\sigma R})(W^*_{\underline{z}^n,\sigma}(x))^2$$

and

$$-\Delta|\widetilde{\xi}_n|^2 + 2(1 - \frac{\sigma}{2})|\widetilde{\xi}_n|^2 \lesssim (\|g_n\|_\sharp^2 + e^{-2\sigma R})(W^*_{\underline{z}^n,\sigma}(x))^2.$$

Here, we have used (1.7) and Lemma 2.1. Now, by the maximum principle,

$$
\begin{aligned}
|\widetilde{\xi}_n| + |D_n| &\lesssim (\|g_n\|_\sharp + e^{-\sigma R} + \max_{1\leq j\leq m_n}\|\widetilde{\xi}_n\|_{L^\infty(\partial B_R(z_j^n))} \\
&\quad + \max_{1\leq j\leq m_n}\|D_n\|_{L^\infty(\partial B_R(z_j^n))})W^*_{\underline{z}^n,\sigma}, \quad (3.13)
\end{aligned}
$$

in $\mathbb{R}^2\backslash(\cup_{j=1}^{m_n} B_R(z_j^n))$. Since $\|\eta_n\|_\sharp = 1$, without loss of generality, we must have

$$\max_{1\leq j\leq m_n}(\|\widetilde{\xi}_n\|_{\sharp,B_R(z_{j,n})} + \|D_n\|_{\sharp,B_R(z_{j,n})}) \gtrsim 1,$$

which implies that there exists $z_{j_n,n}$ such that

$$\|\widetilde{\xi}_n\|_{\sharp,B_R(z_{j_n,n})} + \|D_n\|_{\sharp,B_R(z_{j_n,n})} \gtrsim 1. \quad (3.14)$$

Here, $R > 0$ is a sufficiently large constant and the norm $\|\cdot\|_{\sharp,B_R(z_{j,n})}$ is defined in $B_R(z_{j,n})$ which is similar to that of $\|\cdot\|_\sharp$. Clearly,

$$\widehat{\eta}_n = (\widehat{\xi}_n, \widehat{D}_n) = (\widetilde{\xi}_n(\cdot + z_{j_n}^n), D_n((\cdot + z_{j_n}^n))) \rightharpoonup \widehat{\eta}_0 = (\widehat{\xi}_0, \widehat{D}_0)$$

weakly in $H^2_{loc}(\mathbb{R}^2; \mathbb{C}\times\mathbb{R}^2)$, $\widehat{\eta}_n \to \widehat{\eta}_0$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^2; \mathbb{C}\times\mathbb{R}^2)$ for some $\alpha \in (0,1)$ and $\widehat{\eta}_n \to \widehat{\eta}_0$ in $\mathbb{R}^2$ as $n \to \infty$. Because $\varepsilon_n \to 0$ as $n \to \infty$, we have $\mathcal{Q}(\underline{z}^n) \to +\infty$ as $n \to \infty$. Thus, by (1.7) and lemma 2.1,

$$\widetilde{A}_{\underline{z}^n,0}(\cdot + z_{j_n}^n) \to b(r)\nabla\theta - \nabla(\widetilde{\gamma}\theta) \text{ and } \nabla_{A_{\underline{z}^n,\chi_n}}\psi_{\underline{z}^n,\chi_n}(\cdot + z_{j_n}^n) \to \nabla_B\phi$$

for all $x \in \mathbb{R}^2$, where $\widetilde{\gamma}$ is a cutoff function such that $\widetilde{\gamma} = 0$ in $B_1(0)$ and $\widetilde{\gamma} = 1$ in $\mathbb{R}^2 \backslash B_2(0)$. It follows from (3.8), (3.9) and (3.10) and the Lebesgue dominated convergence theorem that $\widehat{\eta}_0 = (e^{i\widetilde{\gamma}\theta}\widehat{\xi}_0, \widehat{D}_0)$ satisfies the following equation:

$$\begin{cases} -\Delta_B\widehat{\xi}_0 + (\frac{\lambda}{2} + \frac{1}{2})|\phi|^2\widehat{\xi}_0 + \frac{\lambda-1}{2}\phi^2\overline{\widehat{\xi}_0} + \frac{\lambda}{2}(|\phi|^2 - 1)\widehat{\xi}_0 + 2i\nabla_B\phi\widehat{D}_0 = 0, \\ \qquad\qquad\qquad\qquad -\Delta\widehat{D}_0 + |\phi|^2\widehat{D}_0 + 2Im(\overline{\nabla_B\phi}\widehat{\xi}_0) = 0 \end{cases}$$

We claim that $(\widehat{\xi}_0, \widehat{D}_0) \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Indeed, for every $x \in \mathbb{R}^2 \backslash B_R(z_{j_n}^n)$, since $\mathcal{Q}(\underline{z}^n) \to +\infty$ as $n \to \infty$, $x \in \mathbb{R}^2 \backslash (\cup_{j=1}^{m_n} B_R(z_j^n)))$ for n sufficiently large. It follows from $\widehat{\eta}_n \to \widehat{\eta}_0$ strongly in $C_{loc}^{1,\alpha}(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ as $n \to \infty$ and (3.13) that in $\mathbb{R}^2 \backslash B_R(0)$, either

$$|\widehat{\xi}_0| + |\widehat{D}_0| \lesssim e^{-(1-\sigma)|x|}$$

if $|z_j^n| \to +\infty$ as $n \to \infty$ or

$$|\widehat{\xi}_0| + |\widehat{D}_0| \lesssim e^{-(1-\sigma)|x|} + e^{-(1-\sigma)|x-z_0|}$$

if $|z_j^n| \lesssim 1$. Thus, $(\widehat{\xi}_0, \widehat{D}_0) \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ and it is known in [35] that

$$(e^{i\widetilde{\gamma}\theta}\widehat{\xi}_0, \widehat{D}_0) = \alpha_1 T_1 + \alpha_2 T_2,$$

where

$$T_1 = ((\nabla_B\phi)_1, \nabla \times Be_1^{\perp}) \quad \text{and} \quad T_2 = ((\nabla_B\phi)_2, \nabla \times Be_2^{\perp}).$$

Since

$$T_{j_n,k}^{\underline{z}^n, \chi_n} = (e^{i(F_{\underline{z}^n} + \chi_n)} \prod_{l=1; l\neq j}^{m_n} \phi_l(\nabla_{B_{j_n}}\phi_{j_n})_k, \nabla \times B_{j_n}e_k^{\perp})$$

and $\langle (\xi_n, D_n), \widetilde{T}_{j_n,k}^{\underline{z}^n, \chi_n} \rangle = 0$ for all $k = 1, 2$, we have

$$\int_{\mathbb{R}^2} Re(\widetilde{\xi}_n(\cdot + z_{j_n}^n) \prod_{j=1}^{m_n} e^{-i(\theta_{j,n} - \widetilde{\omega}_{j,n})} \prod_{l=1; l\neq j}^{m_n} f_l(\cdot + z_{j_n}^n)\overline{(\nabla_B\phi)_k}\zeta_{j_n}(\cdot + z_{j_n}^n))$$

$$+ D_n((\cdot + z_{j_n}^n)) \cdot \nabla \times Be_k^{\perp}\zeta_{j_n}(\cdot + z_{j_n}^n) = 0.$$

Note that

$$|Re(\widetilde{\xi}_n(\cdot + z_{j_n}^n) \prod_{j=1}^{m_n} e^{-i(\theta_{j,n} - \widetilde{\omega}_{j,n})} \prod_{l\neq j} f_l(\cdot + z_{j_n}^n)\overline{(\nabla_B\phi)_k}\zeta_{j_n}(\cdot + z_{j_n}^n)|$$

$$\leq \quad |\widetilde{\xi}_n(\cdot + z_{j_n}^n)||(\nabla_B\phi)_k)|$$

and $f_l(\cdot + z_{j_n}^n) \to 1$ for $l \neq j_n$, $\zeta_{j_n}(\cdot + z_{j_n}^n) \to 1$ as $n \to \infty$, by the Lebesgue dominated convergence theorem,

$$\langle \alpha_1 T_1 + \alpha_2 T_2, T_k \rangle = 0 \quad \text{for all } k = 1, 2.$$

It follows from $\langle T_1, T_2 \rangle = 0$ that $\alpha_1 = \alpha_2 = 0$, which contradicts (3.14). Thus, we have proved the a-prior estimate $\|\eta\|_\sharp \lesssim \|g\|_\sharp$ for $\varepsilon > 0$ sufficiently small which is uniformly for all $m \geq 2$, all $\chi \in H^2(\mathbb{R}^2)$ and all $\underline{z} \in \mathcal{M}_\varepsilon$. We next to prove the desired a-prior estimate $\|\eta\|_* \lesssim \|g\|_\sharp$. For this, we only need to further prove the estimate $\|\nabla_{A_{\underline{z},\chi}}\xi\|_{L^\infty(\mathbb{R}^2)} \lesssim \|g\|_\sharp$ by $\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = g$. By the gauge invariance of the operator ∇_A, we only need to prove that $\|\nabla_{\widetilde{A}_{\underline{z},0}}\widetilde{\xi}\|_{L^\infty(\mathbb{R}^2)} \lesssim \|g\|_\sharp$, where $\widetilde{\xi}$ is defined similar to that of (3.6). Let us go back to (3.12). By $\|\eta\|_\sharp \lesssim \|g\|_\sharp$, we could

obtain $\|\widetilde{\xi}\|_{H^2(B_1(y))} \lesssim \|g\|_\sharp$ via the estimates in (3.12) for all $y \in \mathbb{R}^2$. Then, the desired estimate comes from the Sobolev embedding theorem and (3.8). It remains to prove the existence and the uniqueness of η to the linear problem $\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = g$. By the a-prior estimate, we know that the linear operator $\widetilde{\mathcal{L}}_{\underline{z},\chi}$ is injective from $\mathbb{X}_{\underline{z},\chi}^\perp \to L^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. On the other hand, we can rewrite $\widetilde{\mathcal{L}}_{\underline{z},\chi}$ to be

$$\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = \mathcal{T}_{\underline{z},\chi}\bigg(\xi + \mathcal{K}_{\underline{z},\chi}(\xi, D), D + \mathcal{Y}_{\underline{z},\chi}(\xi, D)\bigg),$$

where $\mathcal{T}_{\underline{z},\chi}$ is an operator from $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to $L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ given by

$$\begin{aligned}
\mathcal{T}_{\underline{z},\chi}(\xi, D) &= (\mathcal{T}_{\underline{z},\chi,1}(\xi), \mathcal{T}_{\underline{z},\chi,2}(D)) \\
&= \bigg(-\Delta_{A_{\underline{z},\chi}}\xi + (\frac{\lambda}{2} + \frac{1}{2})|\psi_{\underline{z},\chi}|^2\xi \\
&\quad + \frac{\lambda-1}{2}\psi_{\underline{z},\chi}^2\overline{\xi}, -\Delta D + |\psi_{\underline{z},\chi}|^2 D\bigg)
\end{aligned}$$

and,

$$\mathcal{K}_{\underline{z},\chi}(\xi, D) = \mathcal{T}_{\underline{z},\chi,1}^{-1}\bigg(\frac{\lambda}{2}(|\psi_{\underline{z},\chi}|^2 - 1)\xi + 2i\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi}D\bigg)$$

and

$$\mathcal{Y}_{\underline{z},\chi}(\xi, D) = \mathcal{T}_{\underline{z},\chi,2}^{-1}\bigg(2Im(\overline{\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi}}\xi)\bigg)$$

are two operators from $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Since $\nabla_{A_{\underline{z},\chi}}\psi_{\underline{z},\chi} \in L^2(\mathbb{R}^2; \mathbb{C})$, $|\psi_{\underline{z},\chi}|^2 - 1 \in H^1(\mathbb{R}^2)$, we can run the above arguments for the compactness to show that $\mathcal{K}_{\underline{z},\chi}$ and $\mathcal{Y}_{\underline{z},\chi}$ are all compact. Moreover, since $\lambda > 1$ and $|\psi_{\underline{z},\chi}| \to 1$ as $|x| \to +\infty$, we also know that $\mathcal{T}_{\underline{z},\chi}$ is a bijection by the Riesz representation theorem. Thus, $\mathcal{T}_{\underline{z},\chi}^{-1} \circ \widetilde{\mathcal{L}}_{\underline{z},\chi}$ is a Fredholm operator with index 0 from $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Since for all $\gamma \in H^2(\mathbb{R}^2)$, $\eta_\gamma = (i\gamma\psi_{\underline{z},\chi}, \nabla\gamma) \notin \mathbb{X}_{\underline{z},\chi}$, $\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta_\gamma) \neq \mathcal{L}_{\underline{z},\chi}(\eta_\gamma) = 0$ for all $\gamma \in H^2(\mathbb{R}^2)$. Thus, by $\mathbb{X}_{\underline{z},\chi}^\perp \subset H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, $\mathbb{Y}_{\underline{z},\chi}^\perp \subset L^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ and the Fredholm alternative, $\widetilde{\mathcal{L}}_{\underline{z},\chi}$ is also surjective from $\mathbb{X}_{\underline{z},\chi}^\perp \to \mathbb{Y}_{\underline{z},\chi}^\perp$. Therefore, the linear problem $\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = g$ has a unique solution in $\mathbb{X}_{\underline{z},\chi}^\perp$ for all $g \in \mathbb{Y}_{\underline{z},\chi}^\perp$ if $\varepsilon > 0$ sufficiently small, which is independent of $m \geq 2$, $\chi \in H^2(\mathbb{R}^2)$ and $\underline{z} \in \mathcal{M}_\varepsilon$. It completes the proof. $\square$

4. Nonlinear problem

So far, we have established a linear theory which looks good to continue the reduction arguments. Let us now consider the nonlinear problem $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = 0$, where $\mathcal{F}_{\lambda,\mu}(u) = \mathcal{E}_{\lambda,\mu}'(u)$. By expanding $\mathcal{F}_{\lambda,\mu}(u)$ at the approximate solution $v_{\underline{z},\chi}$, we can rewrite $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta)$ as

$$\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = \mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{L}_{\underline{z},\chi}(\eta) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta),$$

where $\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta)$ is the nonlinear part given by

$$\begin{aligned}\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta) &= \left(\frac{\lambda}{2}(2\psi_{\underline{z},\chi}\overline{\xi} + \overline{\psi_{\underline{z},\chi}}\xi + |\xi|^2)\xi + |D|^2(\psi_{\underline{z},\chi} + \xi)\right. \\ &\quad + i[(divD)\xi + 2D\nabla_{A_{\underline{z},\chi}}\xi] + \mu V(x)\xi, \\ &\quad \left. -\mathrm{Im}(\overline{\xi}\nabla_{A_{\underline{z},\chi}}\xi) + D(2\mathrm{Re}(\overline{\psi_{\underline{z},\chi}}\xi) + |\xi|^2)\right)\end{aligned}$$

with $\eta = (\xi, D)$ (cf. [46, (3.14)]). Thus, the nonlinear problem now reads as

$$\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{L}_{\underline{z},\chi}(\eta) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta) = 0.$$

Proposition 4.1. *Let $\varepsilon, \mu > 0$ sufficiently small which are independent of $m \geq 2$, $\chi \in H^2(\mathbb{R}^2)$ and $\underline{z} \in \mathcal{M}_\varepsilon$. Then the nonlinear problem $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = 0$ in $\mathbb{Y}_{\underline{z},\chi}^\perp$ has a unique solution $\eta_{\underline{z},\chi}$ in $\mathbb{X}_{\underline{z},\chi}^\perp$ such that $\|\eta_{\underline{z},\chi}\|_* \lesssim (e^{-\sigma\varepsilon^{-1}} + \mu)$. Moreover, the map: $\underline{z} \to \eta_{\underline{z},\chi}$ is smooth for $\underline{z} \in \mathcal{M}_\varepsilon$.*

Proof. The main ideas of the proof also come from [27]. Let us consider the equation $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) = 0$ in $\mathbb{Y}_{\underline{z},\chi}^\perp$ for $\eta \in \mathbb{X}_{\underline{z},\chi}^\perp$. As pointed out before, the equation now reads as

$$\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta) = -(\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta)) + \sum_{j,k}\alpha_{j,k}^{\underline{z},\chi}(\eta)\widetilde{T}_{j,k}^{\underline{z},\chi}, \tag{4.1}$$

where

$$\alpha_{j,k}^{\underline{z},\chi}(\eta) = \langle \mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta), \widetilde{T}_{j,k}^{\underline{z},\chi}\rangle. \tag{4.2}$$

By (2.4) and Lemma 2.1, we know that $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi})$, given by (2.1) and (2.2), belongs to $L^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. On the other hand, let us define the Banach space

$$\widehat{\mathbb{X}}_{\underline{z},\chi} = \left\{\eta = (\xi, D) \in \mathbb{X}_{\underline{z},\chi}^\perp \mid \|\eta\|_* < +\infty\right\}. \tag{4.3}$$

For $\eta = (\xi, D) \in \widehat{\mathbb{X}}_{\underline{z},\chi}$, we have $div(D) = Im(\overline{\psi}_{\underline{z},\chi}\xi)$. Thus, the nonlinear part now reads as

$$\begin{aligned}\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta) &= \left(\frac{\lambda}{2}(2\psi_{\underline{z},\chi}\overline{\xi} + \overline{\psi_{\underline{z},\chi}}\xi + |\xi|^2)\xi + |D|^2(\psi_{\underline{z},\chi} + \xi)\right. \\ &\quad + i[Im(\overline{\psi}_{\underline{z},\chi}\xi)\xi + 2D\nabla_{A_{\underline{z},\chi}}\xi] + \mu V(x)\xi, \\ &\quad \left. -\mathrm{Im}(\overline{\xi}\nabla_{A_{\underline{z},\chi}}\xi) + D(2\mathrm{Re}(\overline{\psi_{\underline{z},\chi}}\xi) + |\xi|^2)\right). \end{aligned} \tag{4.4}$$

It is easy to check that $\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta) \in L^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ for $\eta \in \widehat{\mathbb{X}}_{\underline{z},\chi}$. Since it is known in [13, 21] that $\widetilde{T}_{j,k}^{\underline{z},\chi} \in L^\infty(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$, by Proposition 3.1, we can rewrite the equation (4.1) to be the following fixed point problem in $\widehat{\mathbb{X}}_{\underline{z},\chi}$:

$$\eta = \widetilde{\mathcal{L}}_{\underline{z},\chi}^{-1}(-(\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta)) + \sum_{j,k}\alpha_{j,k}^{\underline{z},\chi}(\eta)\widetilde{T}_{j,k}^{\underline{z},\chi}). \tag{4.5}$$

Let us consider this fixed point problem in the ball

$$\mathcal{B}_{\underline{z},\chi,\mu} = \left\{\eta = (\xi, D) \in \widehat{\mathbb{X}}_{\underline{z},\chi} \mid \|\eta\|_* \leq M(e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\right\},$$

where $M > 0$ is a sufficiently large constant. By Lemma 2.1, it is easy to check that

$$\|\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi},\eta)\|_\sharp \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2$$

for $\eta \in \mathcal{B}_{\underline{z},\chi,\mu}$ with $\mu, \varepsilon > 0$ sufficiently small which are independent of $m \geq 2$, $\chi \in H^2(\mathbb{R}^2)$ and $\underline{z} \in \mathcal{M}_\varepsilon$. It follows from (1.7), (2.4) and Lemma 2.1 that

$$|\alpha_{j,k}^{z,\chi}(\eta)| \lesssim e^{-\frac{1}{2\varepsilon}} + \mu \tag{4.6}$$

for $\eta \in \mathcal{B}_{\underline{z},\chi,\mu}$. On the other hand, by Lemma 2.1, we also have

$$\|\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi},\eta_1) - \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi},\eta_2)\|_\sharp \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\eta_1 - \eta_2\|_* \tag{4.7}$$

for $\eta_1, \eta_2 \in \mathcal{B}_{\underline{z},\chi,\mu}$. Thus, applying the a-prior estimates in Proposition 3.1, it is standard to use the contraction mapping theorem to solve the fixed point problem (4.5). We remark that thanks to Proposition 3.1, $\mu, \varepsilon > 0$ sufficiently small are independent of $m \geq 2$, $\chi \in H^2(\mathbb{R}^2)$ and $\underline{z} \in \mathcal{M}_\varepsilon$. It remains to check the smoothness of the map: $\underline{z} \to \eta_{\underline{z},\chi}$. For this, we consider the map from $\mathbb{C}^m \times \widehat{\mathbb{X}}_{\underline{z},\chi}$ to $\widehat{\mathbb{Y}}_{\underline{z},\chi}$ given by

$$\mathcal{G}(\underline{z},\eta) = \mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) - \sum_{j,k} \alpha_{j,k}^{z,\chi}(\eta)\widetilde{T}_{j,k}^{z,\chi},$$

where

$$\widehat{\mathbb{Y}}_{\underline{z},\chi} = \left\{ \eta = (\xi, D) \in L^\infty \cap L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \mid \|\eta\|_\sharp < +\infty \right\}.$$

Then, $\mathcal{G}(\underline{z}, \eta_{\underline{z},\chi}) = 0$. A direct computation yields that

$$\partial_\eta \mathcal{G}(\underline{z}, \eta_{\underline{z},\chi})[\varrho] = \widetilde{\mathcal{L}}_{\underline{z},\chi}(\varrho) + Rem_{\underline{z},\chi}(\varrho) - \sum_{j,k} \widetilde{\alpha}_{j,k}^{z,\chi}(\eta_{\underline{z},\chi}, \varrho)\widetilde{T}_{j,k}^{z,\chi}$$

for all $\varrho \in \widehat{\mathbb{X}}_{\underline{z},\chi}$, where

$$\widetilde{\alpha}_{j,k}^{z,\chi}(\eta_{\underline{z},\chi}, \varrho) = \langle \partial_\eta \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi},\eta)[\varrho], \widetilde{T}_{j,k}^{z,\chi} \rangle$$

and $Rem_{\underline{z},\chi}(\varrho) = ((Rem_{\underline{z},\chi}(\varrho))_\psi, (Rem_{\underline{z},\chi}(\varrho))_A)$ with

$$
\begin{aligned}
(Rem_{\underline{z},\chi}(\varrho))_\psi &= iIm(\overline{\psi_{\underline{z},\chi}}\xi_{\underline{z},\chi})\varrho_1 + 2i\nabla_{A_{\underline{z},\chi}}\varrho_1 \cdot D_{\underline{z},\chi} + |D_{\underline{z},\chi}|^2\varrho_1 \\
&\quad + \frac{\lambda}{2}(2Re(\overline{\psi_{\underline{z},\chi}}\xi_{\underline{z},\chi}) + |\xi_{\underline{z},\chi}|^2)\varrho_1 + \frac{1}{2}\overline{\psi_{\underline{z},\chi}}\xi_{\underline{z},\chi}\varrho_1 \\
&\quad + \frac{\lambda}{2}(2\psi_{\underline{z},\chi}\xi_{\underline{z},\chi} + \xi_{\underline{z},\chi}^2)\overline{\varrho}_1 - \frac{1}{2}\psi_{\underline{z},\chi}\xi_{\underline{z},\chi}\overline{\varrho}_1 \\
&\quad + 2i(\nabla_{A_{\underline{z},\chi}}\xi_{\underline{z},\chi} - i\xi_{\underline{z},\chi}D_{\underline{z},\chi} - i\psi_{\underline{z},\chi}D_{\underline{z},\chi}) \cdot \varrho_2
\end{aligned}
$$

and

$$
\begin{aligned}
(Rem_{\underline{z},\chi}(\varrho))_A &= (2Re(\overline{\psi_{\underline{z},\chi}}\xi_{\underline{z},\chi}) + |\xi_{\underline{z},\chi}|^2)\varrho_2 \\
&\quad + 2Im((i\overline{\xi_{\underline{z},\chi}}A_{\underline{z},\chi} + i\overline{\xi_{\underline{z},\chi}}D_{\underline{z},\chi} + i\overline{\psi_{\underline{z},\chi}}D_{\underline{z},\chi})\varrho_1) \\
&\quad + Im(\nabla\overline{\xi_{\underline{z},\chi}}\varrho_1 - \nabla\varrho_1\overline{\xi_{\underline{z},\chi}}).
\end{aligned}
$$

Since $\eta_{\underline{z},\chi} \in \mathcal{B}_{\underline{z},\chi,\mu}$, it is easy to check that

$$\|Rem_{\underline{z},\chi}(\varrho)\|_\sharp \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\varrho\|_*$$

and

$$\|\partial_\eta \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi},\eta)[\varrho]\|_\sharp \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\varrho\|_*.$$

Thus, by choosing $\mu, \varepsilon > 0$ small enough if necessary and applying Proposition 3.1, we know that

$$\|\partial_\eta \mathcal{G}(\underline{z}, \eta_{\underline{z},\chi})[\varrho]\|_\sharp \gtrsim \|\varrho\|_*$$

for all $\varrho \in \widehat{\mathbb{X}}_{\underline{z},\chi}$. Since it is known in [13, 35] that (ϕ, B) is of class C^2 in $\mathbb{R}^2$, $\mathcal{G}(\underline{z}, \eta) = \mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta) - \sum_{j,k} \alpha_{j,k}^{\underline{z},\chi}(\eta) \widetilde{T}_{j,k}^{\underline{z},\chi}$ is of class C^1 for the parameter $\underline{z}$. Thus, applying the implicit function theorem to the equation $\mathcal{G}(\underline{z}, \eta) = 0$ yields that the map: $\underline{z} \to \eta_{\underline{z},\chi}$ is smooth for $\underline{z} \in \mathcal{M}_\varepsilon$. $\qquad\square$

5. The reduced functional

So far, we have solved the equation

$$\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta_{\underline{z},\chi}) = 0$$

in $\mathbb{Y}_{\underline{z},\chi}^\perp$ for a unique $\eta_{\underline{z},\chi} \in \mathcal{B}_{\underline{z},\chi,\mu}$, that is,

$$\widetilde{\mathcal{L}}_{\underline{z},\chi}(\eta_{\underline{z},\chi}) = -(\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}) + \mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \eta_{\underline{z},\chi})) + \sum_{j,k} \alpha_{j,k}^{\underline{z},\chi}(\eta_{\underline{z},\chi}) \widetilde{T}_{j,k}^{\underline{z},\chi}. \qquad (5.1)$$

Thus, to complete the reduction arguments, we need to solve the remaining problem $\alpha_{j,k}^{\underline{z},\chi}(\eta_{\underline{z},\chi}) = 0$ for all $j = 1, 2, \cdots, m$ and $k = 1, 2$, which is a nonlinear and nonlocal system. Since (1.3) is variational, we shall use variational arguments to solve this system. Let

$$\mathcal{I}_{\lambda,\mu}(\underline{z}, \chi) = \mathcal{E}_{\lambda,\mu}(v_{\underline{z},\chi} + \eta_{\underline{z},\chi}).$$

Proposition 5.1. *Let the same assumptions of Proposition 4.1 be satisfied. Then $\mathcal{I}_{\lambda,\mu}(\underline{z}, \chi)$ is independent of gauges $\chi \in H^2(\mathbb{R}^2)$.*

Proof. The ideas of the proof come from [33, 46]. Clearly, the conclusion follows immediately from $\partial_\chi \mathcal{I}_{\lambda,\mu}(\underline{z}, \chi) = 0$ in the dual space of $H^2(\mathbb{R}^2)$. To prove $\partial_\chi \mathcal{I}_{\lambda,\mu}(\underline{z}, \chi) = 0$, it is sufficiently to show that

$$\begin{aligned}
v_{\underline{z},\chi} + \eta_{\underline{z},\chi} &= (\psi_{\underline{z},\chi} + \xi_{\underline{z},\chi}, A_{\underline{z},\chi} + D_{\underline{z},\chi}) \\
&= (e^{i\chi}(\psi_{\underline{z},0} + \xi_{\underline{z},0}), A_{\underline{z},0} + D_{\underline{z},0} + \nabla\chi),
\end{aligned}$$

since $\mathcal{E}_{\lambda,\mu}(u)$ is gauge invariant. Let us define

$$\widehat{\eta}_{\underline{z},\chi} = (e^{i\chi}\xi_{\underline{z},0}, D_{\underline{z},0}).$$

Since $\chi \in H^2(\mathbb{R}^2)$ and $(\xi_{\underline{z},0}, D_{\underline{z},0}) \in \widehat{\mathbb{X}}_{\underline{z},0}$ which is given by (4.3), by the Sobolev embedding theorem, $\widehat{\eta}_{\underline{z},\chi} \in H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. By direct calculations,

$$Im(\overline{\psi}_{\underline{z},0}\xi_{\underline{z},0}) = Im(\overline{\psi}_{\underline{z},\chi}e^{i\chi}\xi_{\underline{z},0})$$

and

$$\langle \eta_{\underline{z},0}, \widetilde{T}_{j,k}^{\underline{z},0} \rangle = \langle \widehat{\eta}_{\underline{z},\chi}, \widetilde{T}_{j,k}^{\underline{z},\chi} \rangle,$$

where $\widetilde{T}_{j,k}^{z,\chi}$ is given by (3.2). It follows from $\eta_{\underline{z},0} \in \mathbb{X}_{\underline{z},0}^{\perp}$ that $\widehat{\eta}_{\underline{z},\chi} \in \mathbb{X}_{\underline{z},\chi}^{\perp}$. Moreover, by (2.1) and (4.4),

$$
\begin{aligned}
& \mathcal{F}_{\lambda,\mu}(v_{\underline{z},0}) + \mathcal{N}_{\lambda,\mu}(v_{z,\chi}, \eta_{\underline{z},0}) \\
= \ & ((\mathcal{F}_{\lambda,\mu}(v_{\underline{z},0}))_{\psi}, (\mathcal{F}_{\lambda,\mu}(v_{\underline{z},0}))_A) \\
& + ((\mathcal{N}_{\lambda,\mu}(v_{\underline{z},0}, \eta_{\underline{z},0}))_{\psi}, (\mathcal{N}_{\lambda,\mu}(v_{\underline{z},0}, \eta_{\underline{z},0}))_A) \\
= \ & (e^{-i\chi}(\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}))_{\psi}, (\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi}))_A) \\
& + (e^{-i\chi}(\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \widehat{\eta}_{\underline{z},\chi}))_{\psi}, (\mathcal{N}_{\lambda,\mu}(v_{\underline{z},\chi}, \widehat{\eta}_{\underline{z},\chi}))_A)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mathcal{L}}_{\underline{z},0}(\eta_{\underline{z},0}) & = ((\widetilde{\mathcal{L}}_{\underline{z},0}(\eta_{\underline{z},0}))_{\psi}, (\widetilde{\mathcal{L}}_{\underline{z},0}(\eta_{\underline{z},0}))_A) \\
& = (e^{-i\chi}(\widetilde{\mathcal{L}}_{\underline{z},\chi}(\widehat{\eta}_{\underline{z},\chi}))_{\psi}, (\widetilde{\mathcal{L}}_{\underline{z},\chi}(\widehat{\eta}_{\underline{z},\chi}))_A),
\end{aligned}
$$

where $\widetilde{\mathcal{L}}_{\underline{z},0}(\eta_{\underline{z},0})$ is given by (3.1). Thus, by (5.1) and $\eta_{\underline{z},0} \in \mathcal{B}_{\underline{z},0,\mu}$, we have $\mathcal{F}_{\lambda,\mu}(v_{\underline{z},\chi} + \widehat{\eta}_{\underline{z},\chi}) = 0$ in $\mathbb{Y}_{\underline{z},\chi}^{\perp}$ and $\widehat{\eta}_{\underline{z},\chi} \in \mathcal{B}_{\underline{z},\chi,\mu}$. By the uniqueness of $\eta_{\underline{z},\chi}$ in $\mathcal{B}_{\underline{z},\chi,\mu}$, we must have $\eta_{\underline{z},\chi} = \widehat{\eta}_{\underline{z},\chi}$. It completes the proof. $\qquad\square$

By Proposition 5.1, we can fix $\chi = 0$ in $\mathcal{I}_{\lambda,\mu}(\underline{z}, \chi)$. As in the proof of Proposition 3.1, we define $\widetilde{\omega}_j = \widetilde{\gamma}_j \theta_j$, where $\widetilde{\gamma}_j$ be a smooth cutoff function such that $\widetilde{\gamma}_j = 0$ in $B_{\frac{\varrho(\underline{z})}{4}}(z_j)$ and $\widetilde{\gamma}_j = 1$ in $\mathbb{R}^2 \backslash B_{\frac{\varrho(\underline{z})}{4}+1}(z_j)$. Since θ_j are all smooth functions in $\mathbb{R}^2 \backslash B_{\frac{1}{2}}(z_j)$. $\widetilde{\omega}_j$ are all smooth in $\mathbb{R}^2$ for $\varepsilon > 0$ sufficiently small. Let

$$
\widetilde{\eta}_{\underline{z},0} = (e^{i\chi^*} \xi_{\underline{z},0}, D_{\underline{z},0}),
$$

where $\chi^* = -F_{\underline{z}} - \sum_{j=1}^{m} \widetilde{\omega}_j$. Since it is known in [13, 35] that $B(x) = b(|x|)\nabla\theta$ is of class C^2 in $\mathbb{R}^2$, by (1.7) and the equation satisfied by $b(r)$ (cf. [21, (12)]), $\widetilde{\eta}_{\underline{z},0} \in H^2(\mathbb{R}^2)$ and by Proposition 4.1, $\widetilde{\eta}_{\underline{z},0}$ is of class C^1 for $\underline{z}$. We define

$$
\mathcal{J}_{\lambda,\mu}(\underline{z}) = \mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z},0} + \widetilde{\eta}_{\underline{z},0}),
$$

where $\widetilde{v}_{\underline{z},0} = (\widetilde{\psi}_{\underline{z},0}, \widetilde{A}_{\underline{z},0}) = (\prod_{j=1}^{m} e^{i(\theta_j - \widetilde{\omega}_j)} f_j, \sum_{j=1}^{m}(b_j \nabla\theta_j - \nabla\widetilde{\omega}_j))$. Then, $\mathcal{J}_{\lambda,\mu}(\underline{z})$ is of class C^1 for $\underline{z}$.

Proposition 5.2. *Let the same assumptions of Proposition 4.1 be satisfied. If* $\nabla\mathcal{J}_{\lambda,\mu}(\underline{z}^0) = 0$, *then* $\alpha_{j,k}^{\underline{z}^0,\chi}(\eta_{\underline{z}^0,\chi}) = 0$ *for all* $j = 1, 2, \cdots, m$ *and* $k = 1, 2$.

Proof. Let $\nabla\mathcal{J}_{\lambda,\mu}(\underline{z}^0) = 0$. Then

$$
0 = \partial_{z_{j,k}} \mathcal{J}_{\lambda,\mu}(\underline{z}^0) = \langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^0,0} + \widetilde{\eta}_{\underline{z}^0,0}), \partial_{z_{j,k}} \widetilde{v}_{\underline{z}^0,0} + \partial_{z_{j,k}} \widetilde{\eta}_{\underline{z}^0,0} \rangle
$$

for all j, k. As pointed out in the proof of Proposition 5.1, we have $\widetilde{\eta}_{\underline{z},0} = \eta_{\underline{z},\chi^*}$. Thus, by Proposition 4.1,

$$
\begin{aligned}
0 & = \langle \mathcal{F}_{\lambda,\mu}(v_{\underline{z}^0,\chi^*} + \eta_{\underline{z}^0,\chi^*}), \partial_{z_{j,k}} \widetilde{v}_{\underline{z}^0,0} + \partial_{z_{j,k}} \widetilde{\eta}_{\underline{z}^0,0} \rangle \\
& = \sum_{l,s} \alpha_{l,s}^{\underline{z}^0,\chi^*}(\eta_{\underline{z}^0,\chi^*}) \langle \widetilde{T}_{l,s}^{\underline{z}^0,\chi^*}, \partial_{z_{j,k}} \widetilde{v}_{\underline{z}^0,0} + \partial_{z_{j,k}} \widetilde{\eta}_{\underline{z}^0,0} \rangle
\end{aligned}
\tag{5.2}
$$

for all j, k. By direct calculations, we also have

$$
\partial_{z_{j,1}} \widetilde{v}_{\underline{z}^0,0} = \left((\partial_{z_{j,1}} \widetilde{v}_{\underline{z}^0,0})_1, (\partial_{z_{j,1}} \widetilde{v}_{\underline{z}^0,0})_2 \right)
$$

and

$$\partial_{z_{j,2}}\widetilde{v}_{\underline{z}^0,0} \;=\; \left((\partial_{z_{j,2}}\widetilde{v}_{\underline{z}^0,0})_1, (\partial_{z_{j,2}}\widetilde{v}_{\underline{z}^0,0})_2\right),$$

where

$$
\begin{aligned}
(\partial_{z_{j,1}}\widetilde{v}_{\underline{z}^0,0})_1 \;=\;& e^{i(\theta_j-\widetilde{\omega}_j)}\prod_{l\neq j}^{m} e^{i(\theta_l-\widetilde{\omega}_l)} f_{l,0}[f'_{j,0}\cos\theta_{j,0} \\
&+ if_{j,0}(\frac{-\sin\theta_{j,0}(1-\widetilde{\gamma}_{j,0})}{r_{j,0}} - \theta_{j,0}\widetilde{\gamma}'_{j,0}\cos\theta_{j,0})], \\
(\partial_{z_{j,1}}\widetilde{v}_{\underline{z}^0,0})_2 \;=\;& (\frac{b_{j,0}-\widetilde{\gamma}_{j,0}}{r_{j,0}})'(-\sin\theta_{j,0}\cos\theta_{j,0}, \cos^2\theta_{j,0}) \\
&+ \frac{b_{j,0}-\widetilde{\gamma}_{j,0}}{r_{j,0}^2}(\sin\theta_{j,0}\cos\theta_{j,0}, \sin^2\theta_{j,0}) \\
&- \theta_{j,0}\widetilde{\gamma}''_{j,0}(\cos^2\theta_{j,0}, \sin\theta_{j,0}\cos\theta_{j,0}) \\
&+ \frac{\widetilde{\gamma}'_{j,0}}{r_{j,0}}(\sin\theta_{j,0}\cos\theta_{j,0}, \sin^2\theta_{j,0}) \\
&- \frac{\theta_{j,0}\widetilde{\gamma}'_{j,0}}{r_{j,0}}(\sin^2\theta_{j,0}, -\sin\theta_{j,0}\cos\theta_{j,0})
\end{aligned}
$$

and

$$
\begin{aligned}
(\partial_{z_{j,2}}\widetilde{v}_{\underline{z}^0,0})_1 \;=\;& e^{i(\theta_j-\widetilde{\omega}_j)}\prod_{l\neq j}^{m} e^{i(\theta_l-\widetilde{\omega}_l)} f_{l,0}[f'_{j,0}\sin\theta_{j,0} \\
&+ if_{j,0}(\frac{\cos\theta_{j,0}(1-\widetilde{\gamma}_{j,0})}{r_{j,0}} - \theta_{j,0}\widetilde{\gamma}'_{j,0}\sin\theta_{j,0})], \\
(\partial_{z_{j,2}}\widetilde{v}_{\underline{z}^0,0})_2 \;=\;& (\frac{b_{j,0}-\widetilde{\gamma}_{j,0}}{r_{j,0}})'(-\sin^2\theta_{j,0}, \sin\theta_{j,0}\cos\theta_{j,0}) \\
&+ \frac{b_{j,0}-\widetilde{\gamma}_{j,0}}{r_{j,0}^2}(-\cos^2\theta_{j,0}, -\sin\theta_{j,0}\cos\theta_{j,0}) \\
&- \theta_{j,0}\widetilde{\gamma}''_{j,0}(\sin\theta_{j,0}\cos\theta_{j,0}, \sin^2\theta_{j,0}) \\
&- \frac{\widetilde{\gamma}'_{j,0}}{r_{j,0}}(\cos^2\theta_{j,0}, \sin\theta_{j,0}\cos\theta_{j,0}) \\
&- \frac{\theta_{j,0}\widetilde{\gamma}'_{j,0}}{r_{j,0}}(-\sin\theta_{j,0}\cos\theta_{j,0}, \cos^2\theta_{j,0})
\end{aligned}
$$

Recall that

$$\widetilde{T}_{l,s}^{\underline{z}^0,\chi^*} \;=\; \left((\widetilde{T}_{l,s}^{\underline{z}^0,\chi^*})_1, (\widetilde{T}_{l,s}^{\underline{z}^0,\chi^*})_2\right), \tag{5.3}$$

where

$$(\widetilde{T}_{l,s}^{\underline{z}^0,\chi^*})_1 = e^{i(\theta_l-\widetilde{\omega}_l)}\prod_{t\neq l} e^{i(\theta_t-\widetilde{\omega}_t)} f_t(f'_l x_l^0 - if_l(b_l\nabla\theta_l - \nabla\theta_l))_s \zeta_l$$

and $(\widetilde{T}_{l,s}^{z^0,\chi^*})_2 = \nabla \times B_l e_s^{\perp} \zeta_l$. Thus, by symmetry, $\langle \widetilde{T}_{j,s}^{z^0,\chi^*}, \partial_{z_{j,k}} \widetilde{v}_{\underline{z}^0,0} \rangle = 0$ for $s \neq k$.
Moreover, by (1.7), Lemma 2.1 and the choices of cutoff functions,

$$\left| \sum_{l,s;l \neq j} \langle \widetilde{T}_{l,s}^{z^0,\chi^*}, \partial_{z_{j,k}} \widetilde{v}_{\underline{z}^0,0} \rangle \right|$$

$$\lesssim \int_{\mathbb{R}^2} \sum_{l \neq j} \zeta_l (f'_{l,0} f'_{j,0} + |\nabla \times B_{l,0}|| \frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}^2} + (\frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}})'|)$$

$$\lesssim \sum_{l \neq j} e^{-(1-\sigma)|z_l^0 - z_j^0|}$$

$$\lesssim e^{-\sigma \varepsilon^{-1}}.$$

On the other hand,

$$\langle \widetilde{T}_{j,1}^{z^0,\chi^*}, \partial_{z_{j,1}} \widetilde{v}_{\underline{z}^0,0} \rangle$$

$$= \int_{\mathbb{R}^2} \prod_{t \neq j} f_{t,0}^2 [(f'_{j,0})^2 \cos^2 \theta_{j,0} + \sin^2 \theta_{j,0} \frac{(1 - \widetilde{\gamma}_{j,0}) f_{j,0}^2 (1 - b_{j,0})}{r_{j,0}^2}$$

$$- \theta_{j,0} \sin \theta_{j,0} \cos \theta_{j,0} \frac{\widetilde{\gamma}'_{j,0} f_{j,0}^2 (b_{j,0} - 1)}{r_{j,0}}] \zeta_{j,0}$$

$$+ \int_{\mathbb{R}^2} [(\frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}})' \cos^2 \theta_{j,0} + \frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}^2} \sin^2 \theta_{j,0}] \nabla \times B_{j,0} \zeta_{j,0}$$

$$+ \int_{\mathbb{R}^2} [-\theta_{j,0} \widetilde{\gamma}''_{j,0} \sin \theta_{j,0} \cos \theta_{j,0} + \frac{\widetilde{\gamma}'_{j,0}}{r_{j,0}} \sin^2 \theta_{j,0}$$

$$+ \frac{\theta_{j,0} \widetilde{\gamma}'_{j,0}}{r_{j,0}} \sin \theta_{j,0} \cos \theta_{j,0}] \nabla \times B_{j,0} \zeta_{j,0}.$$

It is known (cf. [21]) that $1 - b(x) > 0$, thus,

$$\sin^2 \theta_{j,0} \frac{(1 - \widetilde{\gamma}_{j,0}) f_{j,0}^2 (1 - b_{j,0})}{r_{j,0}^2} \geq 0.$$

On the other hand, by a direct calculation, we know that

$$\nabla \times B_{j,0} = \frac{b_{j,0}}{r_{j,0}^2} + (\frac{b_{j,0}}{r_{j,0}})' = \frac{b'_{j,0}}{r_{j,0}}.$$

It follows from the choices of cutoff functions and (1.7) that

$$\int_{\mathbb{R}^2} [(\frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}})' \cos^2 \theta_{j,0} + \frac{b_{j,0} - \widetilde{\gamma}_{j,0}}{r_{j,0}^2} \sin^2 \theta_{j,0}] \nabla \times B_{j,0} \zeta_{j,0}$$

$$= \int_{\mathbb{R}^2} \frac{1}{2} [(\frac{b_{j,0}}{r_{j,0}})' + \frac{b_{j,0}}{r_{j,0}^2}]^2 \zeta_{j,0} + O(e^{-\frac{1-\sigma}{4\varepsilon}}).$$

Thus, by the choices of cutoff functions and (1.7) once more,

$$\langle \widetilde{T}_{j,1}^{z^0,\chi^*}, \partial_{z_{j,1}} \widetilde{v}_{\underline{z}^0,0} \rangle = \frac{1}{2} \int_{\mathbb{R}^2} [(f')^2 + [(\frac{b}{r})' + \frac{b}{r^2}]^2 + \frac{f^2(1-b)}{r^2}] + O(e^{-\frac{1-\sigma}{4\varepsilon}}).$$

Similarly, we also have

$$\langle \widetilde{T}_{j,2}^{z^0,\chi^*}, \partial_{z_{j,2}} \widetilde{v}_{\underline{z}^0,0} \rangle = \frac{1}{2} \int_{\mathbb{R}^2} [(f')^2 + [(\frac{b}{r})' + \frac{b}{r^2}]^2 + \frac{f^2(1-b)}{r^2}] + O(e^{-\frac{1-\sigma}{4\varepsilon}}).$$

Since $\langle \widetilde{T}_{l,s}^{z,0}, \eta_{\underline{z},0} \rangle = 0$ for all l, s, $\langle \widetilde{T}_{j,k}^{z^0, \chi^*}, \widetilde{\eta}_{\underline{z},0} \rangle = 0$ for all l, s. Thus,

$$\langle \widetilde{T}_{l,s}^{z^0, \chi^*}, \partial_{z_{j,k}} \widetilde{\eta}_{\underline{z}^0, 0} \rangle = -\langle \partial_{z_{j,k}} \widetilde{T}_{l,s}^{z^0, \chi^*}, \widetilde{\eta}_{\underline{z}^0, 0} \rangle$$

for all l, s. Since $\eta_{\underline{z}, \chi}$ in $\mathcal{B}_{\underline{z}, \chi, \mu}$, by (5.3) and the choices of cutoff functions, we can compute as before and obtain

$$| \sum_{l,s; l \neq j} \langle \partial_{z_{j,k}} \widetilde{T}_{l,s}^{z^0, \chi^*}, \widetilde{\eta}_{\underline{z}^0, 0} \rangle | \lesssim \int_{\mathbb{R}^2} \sum_{l \neq j} \zeta_{l,0} f'_{j,0} (f'_{l,0} + \frac{1 - b_{l,0}}{r_{l,0}}) \lesssim e^{-\sigma \varepsilon^{-1}}$$

and

$$|\langle \partial_{z_{j,k}} \widetilde{T}_{j,s}^{z^0, \chi^*}, \widetilde{\eta}_{\underline{z}^0, 0} \rangle| \lesssim \|\widetilde{\eta}_{\underline{z}^0, 0}\|_{L^\infty(\mathbb{R}^2)} \lesssim e^{-\frac{1-\sigma}{2\varepsilon}} + \mu$$

for all j, k, s, where we have used (1.7). Therefore, by choosing $\varepsilon, \mu > 0$ sufficiently small if necessary, we can see that the system (5.2) is diagonally dominant, so that it is uniquely solved by $\alpha_{l,s}^{z^0, \chi^*}(\eta_{\underline{z}^0, \chi^*}) = 0$ for all l, s. Note that by gauge invariance, we have $\alpha_{l,s}^{z^0, \chi^*}(\eta_{\underline{z}^0, \chi^*}) = \alpha_{l,s}^{z^0, \chi}(\eta_{\underline{z}^0, \chi})$ for all j, k. Thus, we also have $\alpha_{l,s}^{z^0, \chi}(\eta_{\underline{z}^0, \chi}) = 0$ for all l, s, which completes the proof. $\qquad \square$

6. Secondary reduction and crucial estimates

So far, by Proposition 5.2, we have reduced the problem (1.3) to find critical points of $\mathcal{J}_{\lambda,\mu}(\underline{z})$. Thus, let us now consider the following minimizing problem:

$$e_m = \min_{\underline{z}^m \in \mathcal{M}_\varepsilon} \mathcal{J}_{\lambda,\mu}(\underline{z}^m).$$

Then the equation (1.3) can be solved if the above minimizing problem has a solution in the interior of $\mathcal{M}_\varepsilon$, where $\mathcal{M}_\varepsilon$ is given by (1.5). We shall drive some energy estimates to prove that $\mathcal{J}_{\lambda,\mu}(\underline{z})$ has critical points in $\mathcal{M}_\varepsilon$ for $\varepsilon, \mu > 0$ sufficiently small. We recall that

$$\mathcal{J}_{\lambda,\mu}(\underline{z}) = \mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z},0} + \widetilde{\eta}_{\underline{z},0}).$$

In what follows, we shall establish estimates of e_m, as that in [8]. Since e_m is related to m, the number of vortices, we re-denote $\widetilde{v}_{\underline{z},0} + \widetilde{\eta}_{\underline{z},0}$ by $\widetilde{v}_{\underline{z}^m} + \widetilde{\eta}_{\underline{z}^m}$ for the sake of clarity.

Proposition 6.1. *Suppose that $\underline{z}^m \in \mathcal{M}_\varepsilon$ such that*

$$\min_{j=1,2,\cdots,m-1} |z_m - z_j| \gg 1 \quad and \quad |z_m| \gg 1,$$

then we have

$$\mathcal{J}_{\lambda,\mu}(\underline{z}^m) < \mathcal{J}_{\lambda,\mu}(\underline{z}^{m-1}) + \mathcal{E}_{\lambda,0}(\phi, B)$$

for fixed $\varepsilon, \mu > 0$ sufficiently small which are independent of m and $\underline{z}^m$.

Proof. We recall that $\widetilde{v}_{\underline{z}^m} + \widetilde{\eta}_{\underline{z}^m} = (\widetilde{\psi}_{\underline{z}^m, 0} + \widetilde{\xi}_{\underline{z}^m}, \widetilde{A}_{\underline{z}^m, 0} + D_{\underline{z}^m})$, by direct calculations,

$$\mathcal{J}_{\lambda,\mu}(\underline{z}^m) = \mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m, 0}) + \mathcal{O}_\lambda^1(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\lambda^2(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\mu^3(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) \quad (6.1)$$

where

$$
\begin{aligned}
&\mathcal{O}_\lambda^1(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) \\
&= \int_{\mathbb{R}^2} (Re(\overline{\nabla_{\widetilde{A}_{\underline{z}^m,0}} \widetilde{\psi}_{\underline{z}^m,0}} \cdot (\nabla_{\widetilde{A}_{\underline{z}^m,0}} \widetilde{\xi}_{\underline{z}^m} - i\widetilde{\psi}_{\underline{z}^m,0} D_{\underline{z}^m} - i\widetilde{\xi}_{\underline{z}^m} D_{\underline{z}^m})) \\
&\quad + \int_{\mathbb{R}^2} (\nabla \times \widetilde{A}_{\underline{z}^m,0})(\nabla \times D_{\underline{z}^m}) \\
&\quad + \int_{\mathbb{R}^2} \frac{\lambda}{4}(|\widetilde{\psi}_{\underline{z}^m,0}|^2 - 1)(2Re(\overline{\widetilde{\psi}_{\underline{z}^m,0}} \widetilde{\xi}_{\underline{z}^m}) + |\widetilde{\xi}_{\underline{z}^m}|^2)),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_\lambda^2(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla_{\widetilde{A}_{\underline{z}^m,0}} \widetilde{\xi}_{\underline{z}^m} - i\widetilde{\psi}_{\underline{z}^m,0} D_{\underline{z}^m} - i\widetilde{\xi}_{\underline{z}^m} D_{\underline{z}^m}|^2 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \times D_{\underline{z}^m}|^2 + \frac{\lambda}{4}(2Re(\overline{\widetilde{\psi}_{\underline{z}^m,0}} \widetilde{\xi}_{\underline{z}^m}) + |\widetilde{\xi}_{\underline{z}^m}|^2)^2,
\end{aligned}
$$

and

$$
\mathcal{O}_\mu^3(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) = \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(2Re(\overline{\widetilde{\psi}_{\underline{z}^m,0}} \widetilde{\xi}_{\underline{z}^m}) + |\widetilde{\xi}_{\underline{z}^m}|^2).
$$

For the term $\mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m,0})$, by (1.7) and [22, Lemma 12],

$$
\begin{aligned}
\mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m,0}) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \prod_{j=1}^m f_j|^2 + \int_{\mathbb{R}^2} \prod_{j=1}^m f_j^2 |\sum_{j=1}^m (B_j - \nabla\theta_j)|^2 \\
&\quad + \int_{\mathbb{R}^2} |\nabla \times (\sum_{j=1}^m B_j)|^2 + \frac{\lambda}{4}(\prod_{j=1}^m f_j^2 - 1)^2 \\
&\quad + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(\prod_{j=1}^m f_j^2 - 1) \\
&= \mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1},0}) + \mathcal{E}_{\lambda,0}(\phi, B) + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(f_m^2 - 1) \\
&\quad + \mathcal{U}_{\lambda,m} + O(\sum_{j=1}^{m-1} e^{-2(1-\sigma)m_\lambda|z_j - z_m|}), \quad\quad (6.2)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{U}_{\lambda,m} &= \int_{\mathbb{R}^2} (\sum_{j=1}^{m-1} \nabla \times B_j)\nabla \times B_m \\
&\quad + \int_{\mathbb{R}^2} \prod_{j=1}^m f_j^2 (\sum_{j=1}^{m-1}(B_j - \nabla\theta_j))(B_m - \nabla\theta_m) \\
&\quad + \int_{\mathbb{R}^2} f_m^2 (\prod_{j=1}^{m-1} f_j^2 - 1)|B_m - \nabla\theta_m|^2 \\
&\quad + \int_{\mathbb{R}^2} (f_m^2 - 1)\prod_{j=1}^{m-1} f_j^2 (\sum_{j=1}^{m-1} |B_j - \nabla\theta_j|^2). \quad\quad (6.3)
\end{aligned}
$$

By (1.7) and [22, Lemma 12],

$$|\mathcal{U}_{\lambda,m}| \lesssim \sum_{j=1}^{m-1} e^{-(1-\sigma)|z_j-z_m|}. \tag{6.4}$$

For the terms $\mathcal{O}_\lambda^1(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\lambda^2(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\mu^3(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})$, by the fact that

$$div(D_{\underline{z}^m}) = Im(\overline{\widetilde{\psi}_{\underline{z}^m,0}}\widetilde{\xi}_{\underline{z}^m})$$

and integrating by parts, we observed that

$$\mathcal{O}_\lambda^1(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\lambda^2(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\mu^3(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})$$
$$= \langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}), \widetilde{\eta}_{\underline{z}^m}\rangle + \frac{1}{2}\langle\widetilde{\mathcal{L}}_m(\widetilde{\eta}_{\underline{z}^m}), \widetilde{\eta}_{\underline{z}^m}\rangle + \int_{\mathbb{R}^2}(\int_0^{\widetilde{\eta}_{\underline{z}^m}} \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, t)dt)$$

where, for the sake of simplicity, we re-denote $\widetilde{\mathcal{L}}_{\underline{z}^m,-\chi^*} = \widetilde{\mathcal{L}}_m$. Thus, by denoting

$$\varphi_{\underline{z}^m} = (\varsigma_{\underline{z}^m}, C_{\underline{z}^m}) = \widetilde{\eta}_{\underline{z}^m} - \widetilde{\eta}_{\underline{z}^{m-1}},$$

we have

$$\mathcal{O}_\lambda^1(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\lambda^2(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m}) + \mathcal{O}_\mu^3(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})$$
$$= \langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}), \widetilde{\eta}_{\underline{z}^{m-1}}\rangle + \langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle$$
$$+ \langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle + \frac{1}{2}\langle\widetilde{\mathcal{L}}_{m-1}(\varphi_{\underline{z}^m}), \varphi_{\underline{z}^m}\rangle + \langle\widetilde{\mathcal{L}}_{m-1}(\widetilde{\eta}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle$$
$$+ \langle \mathcal{G}_m(\widetilde{\eta}_{\underline{z}^m}), \widetilde{\eta}_{\underline{z}^m}\rangle + \int_{\mathbb{R}^2}(\int_0^{\widetilde{\eta}_{\underline{z}^{m-1}}} (\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, t) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, t))dt)$$
$$+ \int_{\mathbb{R}^2}(\int_{\widetilde{\eta}_{\underline{z}^{m-1}}}^{\widetilde{\eta}_{\underline{z}^{m-1}}+\varphi_{\underline{z}^m}} \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, t)dt) + \mathcal{J}_{\lambda,\mu}(\underline{z}^{m-1}) - \mathcal{E}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1},0}),$$

where $\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^m}) = (\widetilde{\varpi}_m(\widetilde{\eta}_{\underline{z}^m}), \widetilde{G}_m(\widetilde{\eta}_{\underline{z}^m}))$ with

$$\widetilde{\varpi}_m(\widetilde{\eta}_{\underline{z}^m})$$
$$= \Big(2i(b_m\nabla\theta_m - \nabla\widetilde{\omega}_m) \cdot \nabla\widetilde{\xi}_{\underline{z}^m} + |b_m\nabla\theta_m - \nabla\widetilde{\omega}_m|^2\widetilde{\xi}_{\underline{z}^m}$$
$$+ (2\sum_{j=1}^{m-1}(b_j\nabla\theta_j - \nabla\widetilde{\omega}_j) \cdot (b_m\nabla\theta_m - \nabla\widetilde{\omega}_m))\widetilde{\xi}_{\underline{z}^m}$$
$$+ \prod_{j=1}^{m-1} f_j^2(f_m^2 - 1)\frac{1+2\lambda}{2}\widetilde{\xi}_{\underline{z}^m}\Big)$$
$$+ \Big(\prod_{j=1}^{m-1} e^{2i(\theta_j-\widetilde{\omega}_j)} f_j^2(f_m^2 e^{2i(\theta_m-\widetilde{\omega}_m)} - 1)\frac{\lambda-1}{2}\overline{\widetilde{\xi}_{\underline{z}^m}}$$
$$+ 2i((f_m e^{i(\theta_m-\widetilde{\omega}_m)} - 1)\sum_{j=1}^{m-1}\prod_{j=1}^{m-1} e^{i(\theta_j-\widetilde{\omega}_j)} \prod_{j\neq l} f_j(f_l'x_l^0 - if_l(b_l\nabla\theta_l - \nabla\theta_l) \cdot D_{\underline{z}^m})$$
$$+ 2ie^{i(\theta_m-\widetilde{\omega}_m)}(\prod_{j=1}^{m-1} e^{i(\theta_j-\widetilde{\omega}_j)} f_j(f_m'x_m^0 - if_m(b_m\nabla\theta_m - \nabla\theta_m)) \cdot D_{\underline{z}^m})\Big), \tag{6.5}$$

$$\widetilde{G}_m(\widetilde{\eta}_{\underline{z}^m})$$

$$= 2Im\bigg(\prod_{j=1}^{m}e^{i(\theta_j-\widetilde{\omega}_j)}\prod_{l=1}^{m-1}f_l(f'_m x^0_m + if_m(b_m\nabla\theta_m - \nabla\theta_m)\widetilde{\xi}_{\underline{z}^m}\bigg)$$

$$+2Im\bigg((f_m e^{i(\theta_m-\widetilde{\omega}_m)}-1)\prod_{j=1}^{m-1}e^{i(\theta_j-\widetilde{\omega}_j)}\prod_{j\neq l}f_j(f'_l x^0_l + if_l(b_l\nabla\theta_l - \nabla\theta_l)\widetilde{\xi}_{\underline{z}^m}\bigg)$$

$$+\prod_{j=1}^{m-1}f_j^2(f_m^2-1)D_{\underline{z}^m}. \qquad\qquad (6.6)$$

By (1.7), Lemma 2.1, Proposition 4.1 and [22, Lemma 12],

$$|\langle\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}),\widetilde{\eta}_{\underline{z}^{m-1}}\rangle|$$

$$+|\int_{\mathbb{R}^2}(\int_0^{\widetilde{\eta}_{\underline{z}^{m-1}}}(\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m},t) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}},t))dt)|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}}+\mu)(\sum_{j=1}^{m-1}e^{-(1-\sigma)|z_j-z_m|}+\mu e^{-(1-\sigma_0)|z_m|})$$

and

$$|\langle\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}),\varphi_{\underline{z}^m}\rangle|$$

$$\lesssim \sum_{j=1}^{m-1}e^{-2(1-\sigma)|z_j-z_m|}+\mu e^{-2(1-\sigma_0)|z_m|}+\|\varphi_{\underline{z}^m}\|_{H^1}^2.$$

Since $\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^m})$ is linear for $\widetilde{\eta}_{\underline{z}^m}$, we have

$$\begin{aligned}\langle\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^m}),\widetilde{\eta}_{\underline{z}^m}\rangle &= \langle\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}}),\widetilde{\eta}_{\underline{z}^{m-1}}\rangle + \langle\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}}),\varphi_{\underline{z}^m}\rangle\\ &\quad+\langle\mathcal{G}_m(\varphi_{\underline{z}^m}),\widetilde{\eta}_{\underline{z}^{m-1}}\rangle + \langle\mathcal{G}_m(\varphi_{\underline{z}^m}),\varphi_{\underline{z}^m}\rangle,\end{aligned}$$

which together with Lemma 2.1, (1.7), Proposition 4.1 and [22, Lemma 12], implies that

$$|\frac{1}{2}\langle\widetilde{\mathcal{L}}_{m-1}(\varphi_{\underline{z}^m}),\varphi_{\underline{z}^m}\rangle| + |\langle\mathcal{G}_m(\widetilde{\eta}_{\underline{z}^m}),\widetilde{\eta}_{\underline{z}^m}\rangle|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}}+\mu)^2(\sum_{j=1}^{m-1}e^{-(1-\sigma)|z_j-z_m|}+\mu e^{-(1-\sigma_0)|z_m|}) + \|\varphi_{\underline{z}^m}\|_{H^1}^2.$$

By (5.1), Lemma 2.1, Proposition 4.1 and the Taylor expansion,

$$\langle \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle + \langle \widetilde{\mathcal{L}}_{m-1}(\widetilde{\eta}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle$$

$$+ \int_{\mathbb{R}^2}\Big(\int_{\widetilde{\eta}_{\underline{z}^{m-1}}}^{\widetilde{\eta}_{\underline{z}^{m-1}}+\varphi_{\underline{z}^m}} \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, t)dt\Big)$$

$$= \langle \sum_{j,k} \widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})\widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}, \varphi_{\underline{z}^m}\rangle$$

$$-\langle \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle + \int_{\mathbb{R}^2}\Big(\int_{\widetilde{\eta}_{\underline{z}^{m-1}}}^{\widetilde{\eta}_{\underline{z}^{m-1}}+\varphi_{\underline{z}^m}} \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, t)dt\Big)$$

$$= \langle \sum_{j,k} \widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})\widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}, \varphi_{\underline{z}^m}\rangle + O(\|\varphi_{\underline{z}^m}\|_{H^1}^2)$$

$$+\langle \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^{m-1}}) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle.$$

By Proposition 4.1, (1.7), Lemma 2.1 and [22, Lemma 12],

$$|\langle \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^{m-1}}) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}}), \varphi_{\underline{z}^m}\rangle|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2\Big(\sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_j-z_m|} + \mu e^{-2(1-\sigma_0)|z_m|}\Big) + \|\varphi_{\underline{z}^m}\|_{H^1}^2.$$

We recall that $\varphi_{\underline{z}^m} = \widetilde{\eta}_{\underline{z}^m} - \widetilde{\eta}_{\underline{z}^{m-1}}$. Then, by the orthogonal conditions satisfied by $\widetilde{\eta}_{\underline{z}^m}$ and $\widetilde{\eta}_{\underline{z}^{m-1}}$, and (1.7), (4.6), Proposition 4.1 and [22, Lemma 12] once more, we have

$$|\widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})\langle \widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}, \varphi_{\underline{z}^m}\rangle|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)|\langle \widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}, \widetilde{\eta}_{\underline{z}^m}\rangle|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\widetilde{\xi}_m\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2} |1 - f_m||(\nabla_{B_j}\phi_j)_k|$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 e^{-(1-\sigma)|z_j-z_m|},$$

which implies that

$$|\sum_{j,k} \widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})\langle \widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}, \varphi_{\underline{z}^m}\rangle| \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 \sum_{j=1}^{m-1} e^{-(1-\sigma)|z_j-z_m|}.$$

Therefore, inserting the above estimates into (6.1) and by (6.4), we will arrive at

$$\mathcal{J}_{\lambda,\mu}(\underline{z}^m) = \mathcal{J}_{\lambda,\mu}(\underline{z}^{m-1}) + \mathcal{E}_{\lambda,0}(\phi, B) + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(f_m^2 - 1)$$

$$+O(\|\varphi_{\underline{z}^m}\|_{H^1(\mathbb{R}^2)}^2) + O\Big(\sum_{j=1}^{m-1} e^{-(1-\sigma)|z_j-z_m|}\Big)$$

$$+O((e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 e^{-(1-\sigma_0)|z_m|}). \tag{6.7}$$

Here, we remark that this estimate is obtained by taking

$$\min_{j=1,2,\cdots,m-1} |z_m - z_j| >> 1 \quad \text{and} \quad |z_m| >> 1$$

for fixed ε, μ. In what follows, we shall use the secondary reduction argument to estimate the term $\|\varphi_{\underline{z}^m}\|^2_{H^1(\mathbb{R}^2)}$, as that in [8]. We recall that $\widetilde{\eta}_m = \widetilde{\eta}_{\underline{z},0}$ and

$$\widetilde{\eta}_{\underline{z},0} = (e^{-i\chi_m^*}\xi_{\underline{z},0}, D_{\underline{z},0}) = (\widetilde{\xi}_{\underline{z},0}, D_{\underline{z},0}),$$

where $\chi_m^* = -F_{\underline{z}} - \sum_{j=1}^m \widetilde{\omega}_j$. We divide $\varphi_{\underline{z}^m}$ by

$$\varphi_{\underline{z}^m} = \sum_{j,k} \widetilde{\beta}^m_{j,k} \widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k} + \varphi^\perp_{\underline{z}^m},$$

where $\langle \varphi^\perp_{\underline{z}^m}, \widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k} \rangle = 0$ for all j, k. By the orthogonal conditions satisfied by $\widetilde{\eta}_{\underline{z}^m}$ and $\widetilde{\eta}_{\underline{z}^{m-1}}$, Proposition 4.1, (1.7) and [22, Lemma 12],

$$
\begin{aligned}
|\widetilde{\beta}^m_{j,k}| &\lesssim \|\widetilde{\xi}_{m-1}\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |1 - f_m||(\nabla_{B_j}\phi_j)_k| \\
&\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)e^{-(1-\sigma)|z_m - z_j|}
\end{aligned}
\tag{6.8}
$$

for $j = 1, 2, \cdots, m-1$ and $k = 1, 2$ and

$$
\begin{aligned}
|\widetilde{\beta}^m_{m,k}| &\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu) \int_{\mathbb{R}^2} W^*_{\underline{z}^{m-1}, \sigma}|(\nabla_{B_m}\phi_m)_k| \\
&\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)(\sum_{j=1}^{m-1} e^{-(1-\sigma)|z_m - z_j|} + e^{-(1-\sigma_0)|z_m|}).
\end{aligned}
\tag{6.9}
$$

It follows that

$$
\begin{aligned}
&\|\sum_{j,k} \widetilde{\beta}^m_{j,k} \widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k}\|^2_{H^1(\mathbb{R}^2)} \\
&\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 (\sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m - z_j|} + e^{-2(1-\sigma_0)|z_m|}).
\end{aligned}
\tag{6.10}
$$

To estimate $\varphi^\perp_{\underline{z}^m}$, we recall that by gauge invariance, $\widetilde{\eta}_m$ satisfies the following equation:

$$\widetilde{\mathcal{L}}_m(\widetilde{\eta}_{\underline{z}^m}) = -(\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) + \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})) + \sum_{j,k} \widetilde{\alpha}^{m, -\chi_m^*}_{j,k}(\eta_{\underline{z}^m}) \widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k}.$$

Thus, $\varphi^\perp_{\underline{z}^m}$ satisfies

$$
\begin{aligned}
\widetilde{\mathcal{L}}_m(\varphi^\perp_{\underline{z}^m}) &= (\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}) + \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}})) \\
&\quad -(\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}) + \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})) + \sum_{j,k} \widetilde{\alpha}^{\underline{z}^m, -\chi^*}_{j,k}(\widetilde{\eta}_{\underline{z}^m}) \widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k} \\
&\quad -\sum_{j,k} \widetilde{\alpha}^{\underline{z}^{m-1}, -\chi_{m-1}^*}_{j,k}(\widetilde{\eta}_{\underline{z}^{m-1}}) \widetilde{T}^{\underline{z}^{m-1}, -\chi_{m-1}^*}_{j,k} \\
&\quad -\sum_{j,k} \widetilde{\beta}^m_{j,k} \widetilde{\mathcal{L}}_m(\widetilde{T}^{\underline{z}^m, -\chi_m^*}_{j,k}) - (\widetilde{\omega}_m(\widetilde{\eta}_{\underline{z}^{m-1}}), \widetilde{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}})),
\end{aligned}
\tag{6.11}
$$

where

$$[\widetilde{\mathcal{L}}_m(\varphi_{\underline{z}^m})]_\psi$$

$$= -\Delta_{\widetilde{A}_{\underline{z}^m,-\chi_m^*}} \varsigma_{\underline{z}^m} + (\frac{\lambda}{2} + \frac{1}{2}) \prod_{j=1}^m f_j^2 \varsigma_{\underline{z}^m} + \frac{\lambda}{2}(\prod_{j=1}^m f_j^2 - 1)\varsigma_{\underline{z}^m}$$

$$+ \frac{\lambda-1}{2} \prod_{j=1}^m e^{2i(\theta_j - \widetilde{\omega}_j)} f_j^2 \overline{\varsigma_{\underline{z}^m}}$$

$$+ 2i \sum_{j=1}^m \prod_{j=1}^m e^{i(\theta_j - \widetilde{\omega}_j)} \prod_{l \neq j} f_l (f_j' x_j^0 - i f_j(b_j \nabla \theta_j - \nabla \theta_j)) \cdot C_{\underline{z}^m},$$

$$[\widetilde{\mathcal{L}}_m(\varphi_{\underline{z}^m})]_A$$

$$= -\Delta C_{\underline{z}^m} + \prod_{j=1}^m f_j^2 C_{\underline{z}^m}$$

$$+ 2Im(\sum_{j=1}^m \prod_{j=1}^m e^{i(\theta_j - \widetilde{\omega}_j)} \prod_{l \neq j} f_l(f_j' x_j^0 + i f_j(b_j \nabla \theta_j - \nabla \theta_j)\varsigma_{\underline{z}^m})$$

and $(\widetilde{\omega}_m(\widetilde{\eta}_{\underline{z}^{m-1}}), \widetilde{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}}))$ is given by (6.5) and (6.6). We claim that

$$\|\varphi_{\underline{z}^m}^\perp\|_{H^1(\mathbb{R}^2)}^2 \lesssim \|g_{\underline{z}^m}\|_{L^2(\mathbb{R}^2)}^2 \tag{6.12}$$

for $\varepsilon, \mu > 0$ sufficiently small, where $g_{\underline{z}^m}$ is the right hand side of (6.11). Indeed, we assume the contrary that there exist $\varepsilon_n \to 0$ as $n \to \infty$, $m_n \in \mathbb{N}$ and $\underline{z}^{m_n} \in \mathcal{M}_{\varepsilon_n}$ such that $\|g_{\underline{z}^{m_n}}\|_{L^2(\mathbb{R}^2)}^2 \to 0$ as $n \to \infty$ and $\|\varphi_{\underline{z}^{m_n}}^\perp\|_{H^1(\mathbb{R}^2)}^2 = 1$. By similar arguments as that for (3.10) and using the fact that $\mathbb{R}^2$ is paracompact, we can show that

$$\|\varphi_{\underline{z}^{m_n}}^\perp\|_{H^1(\mathbb{R}^2)}^2 \lesssim \|\varphi_{\underline{z}^{m_n}}^\perp\|_{L^4(\mathbb{R}^2)}^2 + o_n(1). \tag{6.13}$$

Indeed, by similar arguments as that for (3.10), we have

$$\int_{B_1(y)} |\nabla(\varsigma_{\underline{z}^{m_n}})^\perp|^2 + |(\varsigma_{\underline{z}^{m_n}})^\perp|^2$$

$$\lesssim \int_{B_2(y)} |(\varsigma_{\underline{z}^{m_n}})^\perp|^2 + |(C_{\underline{z}^{m_n}})^\perp|^2 + \int_{B_2(y)} |g_{\underline{z}^{m_n}}|^2$$

$$\lesssim \left(\int_{B_2(y)} |(\varsigma_{\underline{z}^{m_n}})^\perp|^4\right)^{\frac{1}{2}} + \left(\int_{B_2(y)} |(C_{\underline{z}^{m_n}})^\perp|^4\right)^{\frac{1}{2}} + \int_{B_2(y)} |g_{\underline{z}^{m_n}}|^2$$

and by the second equation of (6.11) and (1.7), we also have

$$\int_{B_1(y)} |\nabla(C_{\underline{z}^{m_n}})^\perp|^2 + |(C_{\underline{z}^{m_n}})^\perp|^2$$

$$\lesssim \int_{B_2(y)} |(\varsigma_{\underline{z}^{m_n}})^\perp|^2 + |(C_{\underline{z}^{m_n}})^\perp|^2 + \int_{B_2(y)} |g_{\underline{z}^{m_n}}|^2$$

$$\lesssim \left(\int_{B_2(y)} |(\varsigma_{\underline{z}^{m_n}})^\perp|^4\right)^{\frac{1}{2}} + \left(\int_{B_2(y)} |(C_{\underline{z}^{m_n}})^\perp|^4\right)^{\frac{1}{2}} + \int_{B_2(y)} |g_{\underline{z}^{m_n}}|^2$$

where $\varphi_{\underline{z}^{m_n}}^{\perp} = ((\varsigma_{\underline{z}^{m_n}})^{\perp}, (C_{\underline{z}^{m_n}})^{\perp})$. Then, by the fact that $\mathbb{R}^2$ is paracompact, $\mathbb{R}^2 \subset \bigcup_{j=1}^{\infty} B_2(y_j)$ such that for every $x \in \mathbb{R}^2$, it is only covered by uniformly finitely many times. It follows that

$$
\begin{aligned}
\|\varphi_{\underline{z}^{m_n}}^{\perp}\|_{H^1(\mathbb{R}^2)}^2 &\lesssim \sum_{j=1}^{\infty} \|\varphi_{\underline{z}^{m_n}}^{\perp}\|_{H^1(B_1(y_j))}^2 \\
&\lesssim \sum_{j=1}^{\infty} \left(\int_{B_2(y_j)} |(\varsigma_{\underline{z}^{m_n}})^{\perp}|^4 \right)^{\frac{1}{2}} + \sum_{j=1}^{\infty} \left(\int_{B_2(y_j)} |(C_{\underline{z}^{m_n}})^{\perp}|^4 \right)^{\frac{1}{2}} \\
&\quad + \sum_{j=1}^{\infty} \int_{B_2(y_j)} |g_{\underline{z}^{m_n}}|^2 \\
&\lesssim \|\varphi_{\underline{z}^{m_n}}^{\perp}\|_{L^4(\mathbb{R}^2)}^2 + \|g_{\underline{z}^{m_n}}\|_{L^2(\mathbb{R}^2)}^2,
\end{aligned}
$$

which is the desired estimate (6.13). Thus, by combining with similar arguments as that for (3.12) and the Lions lemma ([26, Lemma I.1], see also [49, Lemma 1.21]), we can show that there exist $\{y_n\}$ such that $\varphi_{\underline{z}^{m_n}}^{\perp}(\cdot + y_n) \to \varphi_0 \neq 0$ strongly in $C_{loc}^{1,\alpha}(\mathbb{R}^2)$ as $n \to \infty$. If $|y_n - z_{j_n}^{m_n}| \lesssim 1$ for some $\{z_{j_n}^{m_n}\}$ up to a subsequence, then $\varphi_{\underline{z}^{m_n}}^{\perp}(\cdot + z_{j_n}^{m_n}) \to \varphi_* \neq 0$ strongly in $C_{loc}^{1,\alpha}(\mathbb{R}^2)$ as $n \to \infty$. Since $\langle \varphi_{\underline{z}^{m_n}}^{\perp}, \widetilde{T}_{j,k}^{z^{m_n}, -\chi_{m_n}^*} \rangle = 0$ for all j, k, we can obtain a contradiction by applying similar arguments as that used in the proof of Proposition 3.1. Thus, $|y_n - z_j^{m_n}| \to +\infty$ for all j. In this case, by (1.7), (3.4) and (3.7), φ_0 satisfies the following equation:

$$
\begin{cases}
-\Delta \xi_0 + (\frac{\lambda}{2} + \frac{1}{2})\xi_0 + \frac{\lambda-1}{2}\bar{\xi}_0 = 0, \\
-\Delta D_0 + D_0 = 0.
\end{cases}
$$

It is still impossible since $\varphi_0 \in H^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$. Therefore, it remains to estimate $\|g_{\underline{z}^m}\|_{L^2(\mathbb{R}^2)}^2$. By (1.7)-(2.2) and [22, Lemma 12], we have

$$
\|\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m})\|_{L^2(\mathbb{R}^2)}^2 \lesssim \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m - z_j|} + \mu^2 e^{-2(1-\sigma_0)|z_m|}.
$$

By (6.10), Proposition 4.1 and similar estimates in (4.7), we have

$$
\begin{aligned}
&\|\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}})) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})\|_{L^2(\mathbb{R}^2)}^2 \\
&\lesssim \|\mathcal{N}_{\lambda}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}})) - \mathcal{N}_{\lambda}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^{m-1}})\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \|\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^{m-1}})) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})\|_{L^2(\mathbb{R}^2)}^2 \\
&\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m - z_j|} + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\varphi_{\underline{z}^m}\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + (e^{-\frac{1}{2\varepsilon}} + \mu)^2 e^{-2(1-\sigma_0)|z_m|} \\
&\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m - z_j|} + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\varphi_{\underline{z}^m}^{\perp}\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 e^{-2(1-\sigma_0)|z_m|},
\end{aligned}
$$

where

$$\mathcal{N}_\lambda(v_{\underline{z},\chi}, \eta) = \left(\frac{\lambda}{2}(2\psi_{\underline{z},\chi}\overline{\xi} + \overline{\psi_{\underline{z},\chi}}\xi + |\xi|^2)\xi + |D|^2(\psi_{\underline{z},\chi} + \xi) \right.$$
$$+ i[Im(\overline{\psi}_{\underline{z},\chi}\xi)\xi + 2D\nabla_{A_{\underline{z},\chi}}\xi],$$
$$\left. - Im(\overline{\xi}\nabla_{A_{\underline{z},\chi}}\xi) + D(2\text{Re}(\overline{\psi_{\underline{z},\chi}}\xi) + |\xi|^2) \right).$$

Thus, by (4.2), the choice of cutoff functions and similar estimates as used before,

$$\| \sum_{j,k;j=1}^{m-1} (\widetilde{\alpha}_{j,k}^{\underline{z}^m,-\chi^*}(\widetilde{\eta}_{\underline{z}^m})\widetilde{T}_{j,k}^{\underline{z}^m,-\chi_m^*} - \widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})\widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*})\|_{L^2(\mathbb{R}^2)}^2$$

$$\lesssim \sum_{j,k;j=1}^{m-1} |\widetilde{\alpha}_{j,k}^{\underline{z}^m,-\chi_m^*}(\widetilde{\eta}_{\underline{z}^m}) - \widetilde{\alpha}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}(\widetilde{\eta}_{\underline{z}^{m-1}})|^2$$

$$\lesssim \|\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}})) - \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m}, \widetilde{\eta}_{\underline{z}^m})\|_{L^2(\mathbb{R}^2)}^2$$
$$+ \|\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}) - \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m})\|_{L^2(\mathbb{R}^2)}^2$$
$$+ \left| \langle \mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}, \widetilde{\eta}_{\underline{z}^{m-1}})) + \mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^{m-1}}), \sum_{j,k;j=1}^{m-1} (\widetilde{T}_{j,k}^{\underline{z}^m,-\chi_m^*} - \widetilde{T}_{j,k}^{\underline{z}^{m-1},-\chi_{m-1}^*}) \rangle \right|^2$$

$$\lesssim \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m-z_j|} + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)\|\varphi_{\underline{z}^m}^{\perp}\|_{L^2(\mathbb{R}^2)}^2$$
$$+ (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 e^{-2(1-\sigma_0)|z_m|}.$$

Since it has been established in [22] that

$$\|\widetilde{\mathcal{L}}_m(T_{j,k}^{\underline{z}^m,-\chi_m^*})\|_{L^2(\mathbb{R}^2)}^2 \lesssim \sum_{l\neq j} e^{-2(1-\sigma)|z_j-z_l|}.$$

By the choice of cutoff functions and Lemma 2.1, we know that

$$\|\widetilde{\mathcal{L}}_m(\widetilde{T}_{j,k}^{\underline{z}^m,-\chi_m^*})\|_{L^2(\mathbb{R}^2)}^2 \lesssim 1$$

for all fixed j. Thus, by (6.8) and (6.9), we have

$$\| \sum_{j,k} \widetilde{\beta}_{j,k}^m \widetilde{\mathcal{L}}_m(\widetilde{T}_{j,k}^{\underline{z}^m,-\chi_m^*})\|_{L^2(\mathbb{R}^2)}^2$$

$$\lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 (\sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m-z_j|} + e^{-2(1-\sigma_0)|z_m|}).$$

For the term $(\widetilde{\varpi}_m(\widetilde{\eta}_{\underline{z}^{m-1}}), \widetilde{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}}))$, by Lemma 2.1, (1.7) and Proposition 4.1, we can obtain the following estimates by the similar arguments as above:

$$\|(\widetilde{\varpi}_m(\widetilde{\eta}_{\underline{z}^{m-1}}), \widetilde{G}_m(\widetilde{\eta}_{\underline{z}^{m-1}}))\|_{L^2(\mathbb{R}^2)}^2 \lesssim (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m-z_j|}$$
$$+ (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2 e^{-2(1-\sigma_0)|z_m|}.$$

It remains to estimate the term $\|\widetilde{\alpha}_{j,k}^{z^m,-\chi^*}(\widetilde{\eta}_{\underline{z}^m})\widetilde{T}_{j,k}^{z^m,-\chi_m^*}\|_{L^2(\mathbb{R}^2)}^2$. By Lemma 2.1 and (1.7)-(2.2),

$$|\mathcal{F}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m})| \lesssim \sum_{j=1}^{m-1} e^{-(1-\sigma)|x-z_j|} + \mu e^{-(1-\sigma_0)|x|}.$$

On the other hand, we recall that

$$\widetilde{\eta}_{\underline{z}^m} = \widetilde{\eta}_{\underline{z}^{m-1}} + \varphi_{\underline{z}^m} = \widetilde{\eta}_{\underline{z}^{m-1}} + \sum_{j,k}\widetilde{\beta}_{j,k}^m\widetilde{T}_{j,k}^{z^m,-\chi_m^*} + \varphi_{\underline{z}^m}^{\perp}.$$

Thus, by Proposition 4.1 and (6.8),

$$
\begin{aligned}
|\mathcal{N}_{\lambda,\mu}(\widetilde{v}_{\underline{z}^m},\widetilde{\eta}_{\underline{z}^m})| \quad &\lesssim \quad (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)(\sum_{j=1}^{m-1} e^{-(1-\sigma)|x-z_j|} + e^{-(1-\sigma_0)|x|} + |\varphi_{\underline{z}^m}^{\perp}|) \\
&\quad + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2(\sum_{j=1}^{m-1} e^{-(1-\sigma)|z_m-z_j|} + e^{-(1-\sigma_0)|z_m|}).
\end{aligned}
$$

Thus, by (1.7), (4.2) and [22, Lemma 12],

$$
\begin{aligned}
|\widetilde{\alpha}_{j,k}^{z^m,-\chi_m^*}(\widetilde{\eta}_{\underline{z}^m})|^2 \quad &\lesssim \quad \sum_{j=1}^{m-1} e^{-2(1-\sigma)|z_m-z_j|} + (e^{-\frac{1}{2\varepsilon}} + \mu)^2 e^{-2(1-\sigma_0)|z_m|} \\
&\quad + (e^{-\frac{1-\sigma}{2\varepsilon}} + \mu)^2\|\varphi_{\underline{z}^m}^{\perp}\|_{H^1}^2.
\end{aligned}
$$

Therefore, by (6.12),

$$\|\varphi_{\underline{z}^m}^{\perp}\|_{H^1(\mathbb{R}^2)}^2 = O\left((e^{-\frac{1}{2\varepsilon}} + \mu)^2(\sum_{j=1}^{m-1} e^{-(1-\sigma)|z_m-z_j|} + e^{-(1-\sigma_0)|z_m|})\right)$$

for $\varepsilon,\mu > 0$ sufficiently small which are independent of m and $\underline{z}^m$. Thus, by (6.7), we have

$$
\begin{aligned}
\mathcal{J}_{\lambda,\mu}(\underline{z}^m) \quad &= \quad \mathcal{J}_{\lambda,\mu}(\underline{z}^{m-1}) + \mathcal{E}_{\lambda,0}(\phi,B) \\
&\quad + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(f_m^2 - 1) + O(\sum_{j=1}^{m-1} e^{-(1-\sigma)|z_m-z_j|}) \\
&\quad + O((e^{-\frac{1}{2\varepsilon}} + \mu)^2 e^{-(1-\sigma_0)|z_m|}). \quad\quad\quad\quad (6.14)
\end{aligned}
$$

Since by the slow decaying condition (V_1) and (1.7), it is easy to estimate

$$\int_{\mathbb{R}^2} V(x)(f_m^2 - 1) \lesssim -e^{-(1-\sigma_0)|z_m|},$$

we obtain the conclusion by (6.14) and $\sigma < \sigma_0$. $\square$

7. CRITICAL POINTS OF THE REDUCED FUNCTIONAL

So far, by Proposition 5.2, we have reduced the problem (1.3) to find critical points of $\mathcal{J}_{\lambda,\mu}(\underline{z})$, and by Proposition 6.1, we have established a good estimate for e_m, where

$$e_m = \min_{\underline{z}^m \in \mathcal{M}_\varepsilon} \mathcal{J}_{\lambda,\mu}(\underline{z}^m). \quad\quad\quad\quad (7.1)$$

Since our above arguments do not include the case $m = 1$, which is the start point of our iteration arguments for the general case $m \geq 2$, we shall first consider the case $m = 1$. Since in this situation, we only have one vortex, we will re-denote $\underline{z}^1$ by z. Moreover, we point out that $\mathcal{M}_\varepsilon = \mathbb{C}$ in this case. Thus, for $m = 1$, we actually consider the following minimizing problem:

$$e_1 = \min_{z \in \mathbb{C}} \mathcal{J}_{\lambda,\mu}(z). \tag{7.2}$$

Proposition 7.1. *The minimizing problem* (7.2) *has a solution* $\widehat{\underline{z}}^1$ *for* $\mu > 0$ *sufficiently small.*

Proof. The ingredient of this proof is essentially contained in our above arguments, so we only sketch it here. The approximate solution in the case $m = 1$ is given by $v_{z,\chi} = (\psi_{z,\chi}, A_{z,\chi})$, where

$$\psi_{z,\chi} = e^{i(F_z(x)+\chi(x))}\phi(x - z) \quad \text{and} \quad A_{z,\chi} = B(x - z) + \nabla(F_z(x) + \chi(x))$$

with $F_z(x) = z \cdot B(x - z)$. Since $u = (\phi, B)$ is the fundamental vortex solution of (1.4), the error of this approximate solution is given by

$$\mathcal{F}_{\lambda,\mu}(v_{z,\chi}) = ([\mathcal{F}_{\lambda,\mu}(v_{z,\chi})]_\psi, [\mathcal{F}_{\lambda,\mu}(v_{z,\chi})]_A) = (\mu V(x)\psi_{z,\chi}, 0).$$

By (1.7) and the assumption (V_1),

$$\|\mathcal{F}_{\lambda,\mu}(v_{z,\chi})\|^2_{L^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)} \lesssim \mu^2.$$

We now consider the linear problem $\widetilde{\mathcal{L}}_{z,\chi}(\eta) = g$ in $H^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$, where $g \in L^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$. Then by similar arguments as that used for (6.12), we can prove the following a-priori estimate:

$$\|\eta\|_{H^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)} \lesssim \|g\|_{L^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)}, \tag{7.3}$$

where $\eta \in \mathbb{X}_{z,\chi}^\perp$ and $g \in L^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$ such that

$$\langle(\xi, D), \widetilde{T}_k^{z,\chi}\rangle = 0$$

for $k = 1, 2$ with $g = (\xi, D)$. Here, with a bit of abuse of notations, we use $\widetilde{T}_k^{z,\chi}$ to denote translational zero modes of the equation (1.4) at $v_{z,\chi}$, as that of $\widetilde{T}_{j,k}^{z,\chi}$. As in Proposition 4.1, the nonlinear problem $\mathcal{F}_{\lambda,\mu}(v_{z,\chi} + \eta) = 0$ can be solved now by the contraction mapping theorem with the help of (7.3), that is, there exists a unique $\eta_{z,\chi} \in \mathbb{X}_{z,\chi}^\perp$ such that

$$\mathcal{F}_{\lambda,\mu}(v_{z,\chi} + \eta_{z,\chi}) = \sum_{k=1}^2 \alpha_k^{z,\chi}(\eta_{z,\chi})\widetilde{T}_k^{z,\chi}$$

where

$$\alpha_k^{z,\chi}(\eta) = \langle\mathcal{F}_{\lambda,\mu}(v_{z,\chi}) + \mathcal{N}_{\lambda,\mu}(v_{z,\chi}, \eta), \widetilde{T}_k^{z,\chi}\rangle.$$

Moreover, $\eta_{z,\chi}$ is smooth for z and $\|\eta_{z,\chi}\|^2_{H^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)} \lesssim \mu^2$. Now, let $\{z_n\}$ be a minimizing sequence of e_1. We claim that $\{z_n\}$ is bounded for $\mu > 0$ sufficiently small. Suppose the contrary that $|z_n| \to +\infty$ as $n \to \infty$ up to a subsequence. Without loss of generality, we assume that $|z_n| \to +\infty$ as $n \to \infty$. By the Taylor expansion, we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(z) &= \mathcal{E}_{\lambda,0}(\phi, B) + \frac{\mu}{2}\int_{\mathbb{R}^2} V(x)(f(x - z)^2 - 1) \\
&\quad + \langle\mathcal{F}_{\lambda,\mu}(v_{z,0}), \eta_{z,0}\rangle + O(\|\eta_{z,\chi}\|^2_{H^2(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)}).
\end{aligned}$$

Therefore, by (1.7), the assumption (V_1) and [22, Lemma 12], we have

$$\mathcal{J}_{\lambda,\mu}(z_n) = \mathcal{E}_{\lambda,0}(\phi, B) - \frac{\mu}{2} e^{-(1-\sigma_0)|z_n|} + O(\mu^2),$$

which implies $e_1 = \mathcal{E}_{\lambda,0}(\phi, B) + O(\mu^2)$. On the other hand, we choose $|z_\varepsilon| = \frac{1}{\varepsilon}$. Then by the Taylor expansion once more and taking $\varepsilon > 0$ sufficiently small, we can use (1.7) and [22, Lemma 12] to show that

$$\mathcal{J}_{\lambda,\mu}(z_\varepsilon) \le \mathcal{E}_{\lambda,0}(\phi, B) - \frac{\mu}{2} e^{-\frac{1-\sigma_0}{\varepsilon}} + O(\mu^2).$$

Since $e_1 \le \mathcal{J}_{\lambda,\mu}(z_\varepsilon)$, we must have $e^{-\frac{1-\sigma_0}{\varepsilon}} \lesssim \mu$, which is impossible for $\mu > 0$ sufficiently small. Thus, $\{z_n\}$ is bounded and $z_n \to \widehat{\underline{z}}^1$ as $n \to \infty$ up to a subsequence. It follows from Proposition 5.2 that e_1 is attained by $\widehat{\underline{z}}^1$ for $\mu > 0$ sufficiently small. $\qquad\square$

Let us now solve the minimizing problem (7.1) in the general case $m \ge 2$.

Proposition 7.2. *The minimizing problem (7.1) has a solution $\widehat{\underline{z}}^m$ in the interior of $\mathcal{M}_\varepsilon$ for $\varepsilon, \mu > 0$ sufficiently small which are all independent of m and $\underline{z}^m$.*

Proof. The conclusion for any $m \ge 2$ will be proved by iterations. By Proposition 7.1, the conclusion holds for $m = 1$. Suppose that the conclusion has been already true for $m = 1, 2, \cdots, k-1$ and then let us consider the case $m = k$. We consider a configuration

$$\underline{z}^k = \{\widehat{z}_1^{k-1}, \widehat{z}_2^{k-1}, \cdots, \widehat{z}_{k-1}^{k-1}, z_k\} \in \mathcal{M}_\varepsilon,$$

where $\widehat{\underline{z}}^{k-1}$ is the minimizer of e_{k-1} and

$$\min_{1 \le j \le m-1} |\widehat{z}_j^{k-1} - z_k| \gg 1 \quad \text{and} \quad |z_k| \gg 1.$$

Then by Proposition 6.1, we have

$$e_k < e_{k-1} + \mathcal{E}_{\lambda,0}(\phi, B) \tag{7.4}$$

for $\varepsilon, \mu > 0$ sufficiently small. As for the case $m = 1$, we can prove that $\{\underline{z}_n^k\}$ is bounded. Otherwise, without loss of generality, we may assume that $|z_{k,n}^k| \to \infty$ as $n \to \infty$. Let us go back to (6.3). By [22, Lemma 12] and (1.7), we have

$$\int_{\mathbb{R}^2} f_m^2 (\prod_{j=1}^{m-1} f_j^2 - 1)|B_m - \nabla\theta_m|^2 + (f_m^2 - 1)\prod_{j=1}^{m-1} f_j^2 (\sum_{j=1}^{m-1} |B_j - \nabla\theta_j|^2)$$

$$= O\Big(\sum_{j=1}^{m-1} d_j \Big(\frac{1}{|z_j - z_m|}\Big)^{\frac{1}{2}} e^{-|z_j - z_m|}\Big)$$

for $\varepsilon > 0$ sufficiently small. On the other hand, by (1.7) and Lemma 8.1, we have

$$\int_{\mathbb{R}^2} (\sum_{j=1}^{m-1} \nabla \times B_j)\nabla \times B_m + \prod_{j=1}^{m} f_j^2 (\sum_{j=1}^{m-1} (B_j - \nabla\theta_j))(B_m - \nabla\theta_m)$$

$$\sim \sum_{j=1}^{m-1} d_j' |z_j - z_m|^{\frac{1}{2}} e^{-|z_j - z_m|}$$

for $\varepsilon > 0$ sufficiently small. Thus, using these two new estimates in (6.2) and using the assumption $|z_{k,n}^k| \to \infty$ as $n \to \infty$, we can rewrite the estimate (6.14) for $\underline{z}_n^k$ as follows:

$$
\begin{aligned}
e_k \;\geq\; & \mathcal{J}_{\lambda,\mu}(\underline{z}_n^{k-1}) + \mathcal{E}_{\lambda,0}(\phi, B) + O\left(\sum_{j=1}^{m-1} d_j \left(\frac{1}{|z_j^n - z_m^n|} \right)^{\frac{1}{2}} e^{-|z_j^n - z_m^n|} \right) \\
& + \sum_{j=1}^{m-1} d_j' |z_j^n - z_m^n|^{\frac{1}{2}} e^{-|z_j^n - z_m^n|} + o_n(1) \qquad\qquad (7.5) \\
\geq\; & e_{k-1} + \mathcal{E}_{\lambda,0}(\phi, B) + o_n(1).
\end{aligned}
$$

It contradicts (7.4). Thus, $\{\underline{z}_n^k\}$ must be bounded. Without loss of generality, we may assume that $\underline{z}_n^k \to \widehat{\underline{z}}^k$ as $n \to \infty$. It remains to prove that $\widehat{\underline{z}}^k$ belongs to the interior of $\mathcal{M}_\varepsilon$ for $k \geq 2$. Suppose the contrary that $\widehat{\underline{z}}^k \in \partial \mathcal{M}_\varepsilon$. Then without loss of generality, we may assume that $|\widehat{z}_1^k - \widehat{z}_k^k| = \varepsilon^{-1}$. Similar to (7.5), by (6.14), the energy estimate for $\widehat{\underline{z}}^k$ now reads as:

$$
\begin{aligned}
e_k \;\geq\; & \mathcal{J}_{\lambda,\mu}(\widehat{\underline{z}}^{k-1}) + \mathcal{E}_{\lambda,0}(\phi, B) + O\left(\sum_{j=1}^{m} d_j |\widehat{z}_j^{k-1} - \widehat{z}_m^{k-1}|^{-\frac{1}{2}} e^{-|\widehat{z}_j^{k-1} - \widehat{z}_m^{k-1}|} \right) \\
& + \sum_{j=1}^{m} d_j' |\widehat{z}_j^{k-1} - \widehat{z}_m^{k-1}|^{\frac{1}{2}} e^{-|\widehat{z}_j^{k-1} - \widehat{z}_m^{k-1}|} + O(\mu) \\
\geq\; & e_{k-1} + \mathcal{E}_{\lambda,0}(\phi, B) + d_1'' e^{-\frac{1}{\varepsilon}} + O(\mu)
\end{aligned}
$$

for $\varepsilon > 0$ sufficiently small. Hence, by (7.4) and taking $\mu << e^{-\frac{1}{\varepsilon}}$, we will obtain a contradiction. Therefore, we must have $\widehat{\underline{z}}^k$ belong to the interior of $\mathcal{M}_\varepsilon$ and thus, $\widehat{\underline{z}}^k$ is a critical point of $\mathcal{J}_{\lambda,\mu}(\underline{z}^m)$ for $\varepsilon, \mu > 0$ all sufficiently small. It completes the proof. $\qquad\qquad\square$

We close this section by the proof of Theorem 1.1.

Proof of Theorem 1.1: It follows immediately from Propositions 4.1, 5.2 and 7.2. $\qquad\qquad\square$

8. Appendix: A useful estimate

Lemma 8.1. *Let $w \in H^1(\mathbb{R}^2)$ such that $w(|x|) \sim |x|^{-\frac{1}{2}} e^{-|x|}$ as $|x| \to +\infty$. Suppose $e_1 \in \mathbb{R}^2$ such that $|e_1| = 1$. Then as $R \to +\infty$,*

$$
\int_{\mathbb{R}^2} w(x) w(x - R e_1) dx \sim R^{\frac{1}{2}} e^{-R}.
$$

Proof. The proof is almost same as that of [50, Lemma 4.1], we give it here for reader's convenience. Without loss of generality, we assume that $e_1 = (0, 1)$. Thus,

$$\int_{\mathbb{R}^2} w(x)w(x - Re_1)dx$$

$$= \int_{\{|x|\leq M\}} w(x)w(x - Re_1)dx + \int_{\{|x-Re_1|\leq M\}} w(x)w(x - Re_1)dx$$

$$+ \int_{\{M<|x|\leq \frac{R}{2}\}} w(x)w(x - Re_1)dx + \int_{\{M<|x-Re_1|\leq \frac{R}{2}\}} w(x)w(x - Re_1)dx$$

$$+ \int_{\{|x|>\frac{R}{2}\}\cap\{|x-Re_1|>\frac{R}{2}\}} w(x)w(x - Re_1)dx$$

for $R > 0$ sufficiently large, where $M > 0$ is a sufficiently large constant such that $w(|x|) \sim |x|^{-\frac{1}{2}}e^{-|x|}$ for $|x| \geq M$. For $\int_{\{|x|\leq M\}} w(x)w(x - Re_1)dx$, we estimate it as follows:

$$\int_{\{|x|\leq M\}} w(x)w(x - Re_1)dx \quad \sim \quad \int_{\{|x|\leq M\}} w(x)|x - Re_1|^{-\frac{1}{2}}e^{-|x-Re_1|}dx$$

$$\lesssim \quad R^{-\frac{1}{2}}e^{-R}\int_{\{|x|\leq M\}} w(x)e^{|x|}dx$$

as $R \to +\infty$. For $\int_{\{|x-Re_1|\leq M\}} w(x)w(x - Re_1)dx$, the estimate is similar to that of $\int_{\{|x|\leq M\}} w(x)w(x - Re_1)dx$. For $\int_{\{|x|>\frac{R}{2}\}\cap\{|x-Re_1|>\frac{R}{2}\}} w(x)w(x - Re_1)dx$, we estimate it as follows:

$$\int_{\{|x|>\frac{R}{2}\}\cap\{|x-Re_1|>\frac{R}{2}\}} w(x)w(x - Re_1)dx$$

$$\sim \quad \int_{\{|x|>\frac{R}{2}\}\cap\{|x-Re_1|>\frac{R}{2}\}} (|x||x - Re_1|)^{-\frac{1}{2}}e^{-|x|}e^{-|x-Re_1|}dx$$

$$\lesssim \quad R^{-\frac{1}{2}}e^{-\frac{R}{2}}\int_{\frac{R}{2}}^{+\infty} e^{-r}r^{\frac{1}{2}}dr$$

$$\sim \quad e^{-R}$$

as $R \to +\infty$. For $\int_{\{M<|x|\leq \frac{R}{2}\}} w(x)w(x - Re_1)dx$, we denote $x = (x', x_1)$. Then $x' = |x|\cos\rho$ and $x_1 = |x|\sin\rho$. It follows that

$$|x - Re_1| - R \geq -x_1 \quad \text{for } x \in \{M < |x| \leq \frac{R}{2}\}.$$

On the other hand, we observe that

$$|Re_1 - x| - R = \frac{-2Rx_1 + |x|^2}{|Re_1 - x| + R}.$$

Thus, if $|Re_1 - x| - R \leq 0$, then $|Re_1 - x| + R \leq 2R$ while if $|Re_1 - x| - R \geq 0$, then $|Re_1 - x| + R \geq 2R$, which implies that

$$|Re_1 - x| - R \leq -x_1 + \frac{|x|^2}{2R} \quad \text{for } x \in \{M < |x| \leq \frac{R}{2}\}.$$

Then by symmetry, we can estimate the upper bound of $\int_{\{M<|x|\leq\frac{R}{2}\}} w(x)w(x - Re_1)dx$ as follows:

$$\int_{\{M<|x|\leq\frac{R}{2}\}} w(x)w(x - Re_1)dx$$

$$\sim \int_{\{M<|x|\leq\frac{R}{2}\}\cap\{0\leq\rho\leq\pi\}} w(x)w(x - Re_1)dx$$

$$\lesssim R^{-\frac{1}{2}}e^{-R}\left(\int_0^{\frac{\pi}{2}}\int_M^{\frac{R}{2}} e^{-(r-r\sin\rho)}r^{\frac{1}{2}}drd\rho + \int_0^{\frac{\pi}{2}}\int_M^{\frac{R}{2}} e^{-(r-r\cos\rho)}r^{\frac{1}{2}}drd\rho\right)$$

$$\sim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{2}}\int_M^{\frac{R}{2}} e^{-(r-rsin\rho)}r^{\frac{1}{2}}drd\rho$$

$$\lesssim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{2}}\int_M^{\frac{R}{2}} e^{-\frac{1}{2}r(\cos\rho)^2}r^{\frac{1}{2}}drd\rho$$

$$= R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{2}}\int_M^{\frac{R}{2}} e^{-\frac{1}{2}r\rho^2(\frac{\sin\rho}{\rho})^2}r^{\frac{1}{2}}drd\rho$$

$$\lesssim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{2}}\int_{\frac{2M}{\pi^2}}^{\frac{R}{4}} e^{-r\rho^2}r^{\frac{1}{2}}drd\rho$$

$$\sim R^{-\frac{1}{2}}e^{-R}\int_{\frac{2M}{\pi^2}}^{\frac{R}{4}} dr\int_0^{+\infty} e^{-y^2}dy$$

$$\sim R^{\frac{1}{2}}e^{-R}.$$

For the lower bound of $\int_{\{M<|x|\leq\frac{R}{2}\}} w(x)w(x - Re_1)dx$, the estimates is similar to that of the upper bound:

$$\int_{\{M<|x|\leq\frac{R}{2}\}} w(x)w(x - Re_1)dx$$

$$\sim R^{-\frac{1}{2}}e^{-R}\int_{\{M<|x|\leq\frac{R}{2}\}} |x|^{-\frac{1}{2}}e^{-(|x|+\frac{|x|^2}{2R}-x_1)}dx$$

$$\gtrsim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{4}}\int_M^{\frac{R}{2}} e^{-(r+\frac{r^2}{2R}-r\sin\rho)}r^{\frac{1}{2}}drd\rho$$

$$\gtrsim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{4}}\int_M^{\frac{R}{2}} e^{-2r\cos^2\rho}r^{\frac{1}{2}}drd\rho$$

$$\gtrsim R^{-\frac{1}{2}}e^{-R}\int_0^{\frac{\pi}{4}}\int_{\frac{M}{2}}^{\frac{R}{2}} e^{-2r\cos^2\rho}r^{\frac{1}{2}}drd\rho$$

$$\sim R^{-\frac{1}{2}}e^{-R}\int_{\frac{M}{2}}^{\frac{R}{2}} dr\int_0^{+\infty} e^{-y^2}dy$$

$$\sim R^{\frac{1}{2}}e^{-R}.$$

The estimate of $\int_{\{|x-Re_1|\leq M\}} w(x)w(x - Re_1)dx$ is also similar. Thus, the proof is completed. $\qquad\square$

9. Acknowledgements

References

[1] A. Abrikosov, On the magnetic properties of superconductors of the second group, *Soviet Physics JETP,* **5** (1957), 1174–1182.

[2] A. Aftalion, S. Alama, L. Bronsard, Giant vortex and the breakdown of strong pinning in a rotating Bose-Einstein condensate, *Arch. Rational Mech. Anal.,* **178** (2005), 247–286.

[3] S. Alama, L. Bronsard, Pinning effects and their breakdown for a Ginzburg-Landau model with normal inclusions, *J. Math. Phys.,* **46** (2005), 095102.

[4] S. Alama, L. Bronsard, Vortices and pinning effects for the Ginzburg-Landau model in multiply connected domains, *Comm. Pure Appl. Math.,* **59** (2006), 36–70.

[5] S. Alama, L. Bronsard, T. Giorgi, Uniqueness of symmetric vortex solutions in the Ginzburg-Landau model of superconductivity, *J. Funct. Anal.,* **167** (1999), 399–424.

[6] N. Andre, P. Bauman and D. Phillips, Vortex pinning with bounded fields for the Ginzburg-Landau equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **20** (2003), 705–729.

[7] N. Andre, I. Shafrir, Asymptotic behavior of minimizers for the Ginzburg-Landau functional with weight. I, II, *Arch. Rational Mech. Anal.,* **142** (1998), 45–73, 75–98.

[8] W. Ao, J. Wei, Infinitely many positive solutions for nonlinear equations with non-symmetric potentials, *Calc. Var. PDEs,* **51** (2014), 761–798.

[9] W. Ao, J. Wei, J. Zeng, An optimal bound on the number of interior spike solutions for the Lin-Ni-Takagi problem, *J. Funct. Anal.,* **265**(2013), 1324–1356.

[10] W. Ao, J. Wei, J. Yang, Vortex helices for inhomogeneous Gross-Pitaevskii equation in three dimensional spaces, 2016, preprint.

[11] F. Bethuel, H. Brezis, F. Helein, Ginzburg-Landau Vortices. Basel: Birkhäuser, 1994.

[12] I. Bialynicki-Birula, Z. Bialynicka-Birula, C. Śliwa, Motion of vortex lines in quantum mechanics, *Phys. Rev. A,* **61** (2000), 032110.

[13] M. Berger, Y. Chen, Symmetric vortices for the nonlinear Ginzburg-Landau equations of superconductivity, and the nonlinear desingularization phenomena, *J. Funct. Anal.,* **82** (1989), 259–295.

[14] G. Cerami, D. Passaseo, S. Solimini, Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients, *Comm. Pure Appl. Math.,* **66** (2013), 372–413.

[15] C. Clemons, An existence and uniqueness result for symmetric vortices for the Ginzberg-Landau equations of superconductivity, *J. Differential Equations,* **157** (1999), 150–162.

[16] S. Chapman, Q. Du, M. Gunzburger, A Ginzburg-Landau type model of superconducting/normal junctions including Josephson junctions, *Eur. J. Appl. Math.,* **6** (1995), 97–114.

[17] Q. Du, M. Gunzburger, A model for superconducting thin films having variable thickness, *Phys. D,* **69** (1993), 215–231.

[18] Q. Du, M. Gunzburger, J. Peterson, Computational simulations of type II superconductivity including pinning phenomena, *Phys. Rev. B,* **51** (1995), 16194–16203.

[19] F. Lin, Q. Du, Ginzburg-Landau vortices: dynamics, pinning, and hysteresis, *SIAM J. Math. Anal.,* **28** (1997), 1265–1293.

[20] V. Ginzburg, L. Landau, Collected papers of L.D.Landau, Edited by D. Ter. Haar, Pergamon Press, Oxford 1965.

[21] S. Gustafson, I.M. Sigal, Stability of Magnetic Vortices, *Commun. Math. Phys.,* **212** (2000), 257–275.

[22] S. Gustafson, I.M. Sigal, Effective dynamics of magnetic vortices, *Adv. Math.,* **199** (2006), 448–498.

[23] S. Gustafson, I.M. Sigal, T. Tzaneteas, Statics and dynamics of magnetic vortices and of NielsenCOlesen (Nambu) strings, *J. Math. Phys.,* **51** (2010), 015217.

[24] S. Gustafson, F. Ting, Dynamic stability and instability of pinned fundamental vortices, *J. Nonlinear Sci.,* **19** (2009), 341–374.

[25] A. Jaffe, C. Taubes, Vortices and Monopoles. Basel-Boston: Birkhäuser, 1980.

[26] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **1** (1984), 223-283.

[27] M. Musso, F. Pacard, J. Wei, Finite energy sigh-changing solutions with dihedral symmetry for the stationary nonlinear schrödinger equation, *J. Eur. Math. Soc.,* **14** (2012), 1923–1953.

[28] M. Mink, C. Morais Smith, R. Duine, Vortex-lattice pinning in two-component Bose-Einstein condensates, *Phys. Rev. A,* **79** (2009), 013605.

[29] Y. Ovchinnikov, I.M. Sigal, Symmetry breaking solutions to the Ginzburg-Landau equations, *Sov. Phys. JETP,* **99** (2004), 1090.

[30] B. Plohr, Princeton Ph.D Thesis, 1978.

[31] F. Pacard, T. Riviere, Linear and nonlinear aspects of vortices. The GinzburgCLandau model, Progr. Nonlinear Differential Equations Appl., vol. 39, Birkhäuser Boston, Boston, MA, 2000.

[32] L. Pismen, J. Rubinstein, Motion of vortex lines in the ginzburg-Landau model, *Physica D: Nonlinear Phenomena,* **47** (1991), 353–360.

[33] A. Pakylak, F. Ting, J. Wei, Multi-vortex solutions to Ginzburg-Landau equations with external potential, *Arch. Ritional Mech. Anal.,* **204** (2012), 313–354.

[34] J. Reijnders, R. Duine, Pinning of vortices in a Bose-Einstein condensate by an optical lattice, *Phys. Rev. Lett.,* **93** (2004), 060401.

[35] D. Stuart, Dynamics of Abelian Higgs vortices in the near Bogomolny regime, *Commun. Math. Phys.,* **159** (1994), 51–91.

[36] A. Svidzinsky, A. Fetter, Dynamics of a vortex in trapped Bose-Einstein condensate, *Phys. Rev. A,* 62 (2000), no. 6, 063617.

[37] E. Sandier, S. Serfaty, Vortices in the Magnetic Ginzburg-Landau Model, Progress in Nonlinear Differential Equations and their Applications. Vol. 70, Basel-Boston: Birkhäuser, 2007.

[38] S. Serfaty, Coulomb gases and Ginzburg-Landau vortices. Zürich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2015.

[39] I.M. Sigal, F. Ting, Pining of magnetic vortices by an external potential, *St. Petersburg Math. J.,* **16** (2005), 211–236.

[40] I.M. Sigal, T. Tzaneteas, On stability of Abrikosov vortex lattices, *Adv. Math.,* **326** (2018), 108–199.

[41] Y. Strauss, I.M. Sigal, Effective Dynamics of a Magnetic Vortex in a Local Potential, *J. Nonlinear Sci.,* **16** (2006), 123–157.

[42] C. Taubes, Arbitrary n-vortex solutions to the first order GinzburgCLandau equations, *Commun. Math. Phys.,* **72** (1980), 277–292.

[43] C. Taubes, On the equivalence of the first and second order equations for gauge theories, *Commun. Math. Phys.,* **75** (1980), 207–227.

[44] F. Ting, Effective dynamics of multi-vortices in an external potential for the GinzburgCLandau gradient flow, *Nonlinearity,* **23** (2010), 179–210.

[45] S. Tung, V. Schweikhard, E. Cornell, Observation of vortex pinning in Bose-Einstein condensates, *Phys. Rev. Lett.,* **97** (2006), 240402.

[46] F. Ting, J. Wei, Multi-vortex non-radial solutions to the magnetic Ginzburg-Landau equations, *Comm. Math. Phys.,* **317** (2013), 69–97.

[47] T. Lin, J. Wei, J. Yang, Vortex rings for the Gross-Pitaevskii equation in $\mathbb{R}^3$, *J. Math. Pures Appl.,* **100** (2013), 69–112.

[48] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York (1975).

[49] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

[50] J. Wei, Y. Wu, Ground states of nonlinear elliptic systems with mixed couplings, *J. Math. Pures Appl.,* **141** (2020), 50–88.

[51] J. Wei, J. Yang, Vortex rings pinning for the Gross-Pitaevskii equation in three dimensional space, *SIAM J. Math. Anal.,* **44** (2012), 3991–4047.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., CANADA, V6T 1Z2

E-mail address: `jcwei@math.ubc.ca`

SCHOOL OF MATHEMATICS, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, XUZHOU, 221116, P.R. CHINA

E-mail address: `wuyz850306@cumt.edu.cn`