CHAPTER 2

# Maximum Principles

## 2.1. Guide

In this chapter we discuss maximum principles and their applications. Two kinds of maximum principles are discussed. One is due to Hopf and the other to Alexandroff. The former gives the estimates of solutions in terms of the $L^\infty$-norm of the nonhomogeneous terms, while the latter gives the estimates in terms of the $L^n$-norm. Applications include various a priori estimates and the moving plane method.

Most of the statements in Section 2.2 are rather simple. One probably needs to go over Theorem 2.11 and Proposition 2.13. Section 2.3 is often the starting point of the a priori estimates. Section 2.4 can be omitted in the first reading, as we will look at it again in Section 5.2. The moving plane method explained in Section 2.6 has many recent applications. We choose a very simple example to illustrate such a method. The result goes back to Gidas-Ni-Nirenberg, but the proof contains some recent observations in the paper [**1**]. The classical paper of Gilbarg-Serrin [7] may be a very good supplement to this chapter. It may also be a good idea to assume the Harnack inequality of Krylov-Safonov in Section 5.3, and to ask students to improve some of the results in the paper [7].

## 2.2. Strong Maximum Principle

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator $L$ in $\Omega$

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We always assume that $a_{ij}$, $b_i$, and $c$ are continuous and hence bounded in $\bar{\Omega}$ and that $L$ is uniformly elliptic in $\Omega$ in the following sense

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda$.

LEMMA 2.1. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu > 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. If $u$ has a nonnegative maximum in $\bar{\Omega}$, then $u$ cannot attain this maximum in $\Omega$.*

PROOF. Suppose $u$ attains its nonnegative maximum of $\bar{\Omega}$ in $x_0 \in \Omega$. Then $D_iu(x_0) = 0$ and the matrix $B = \big(D_{ij}(x_0)\big)$ is semi-negative definite. By ellipticity condition the matrix $A = (a_{ij}(x_0))$ is positive definite. Hence the matrix $AB$ is semi-negative definite with a nonpositive trace, that is, $a_{ij}(x_0)D_{ij}u(x_0) \leq 0$. This implies $Lu(x_0) \leq 0$, which is a contradiction. $\square$

REMARK 2.2. If $c(x) \equiv 0$, then the requirement for nonnegativeness can be removed. This remark also holds for some results in the rest of this section.

THEOREM 2.3 (Weak Maximum Principle). *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. Then $u$ attains on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$.*

PROOF. For any $\varepsilon > 0$, consider $w(x) = u(x)+\varepsilon e^{\alpha x_1}$ with $\alpha$ to be determined. Then we have

$$Lw = Lu + \varepsilon e^{\alpha x_1}(a_{11}\alpha^2 + b_1\alpha + c)\,.$$

Since $b_1$ and $c$ are bounded and $a_{11}(x) \geq \lambda > 0$ for any $x \in \Omega$, by choosing $\alpha > 0$ large enough we get

$$a_{11}(x)\alpha^2 + b_1(x)\alpha + c(x) > 0 \quad \text{for any } x \in \Omega\,.$$

This implies $Lw > 0$ in $\Omega$. By Lemma 2.1, $w$ attains its nonnegative maximum only on $\partial\Omega$, that is,

$$\sup_\Omega w \leq \sup_{\partial\Omega} w^+\,.$$

Then we obtain

$$\sup_\Omega u \leq \sup_\Omega w \leq \sup_{\partial\Omega} w^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon \sup_{x\in\partial\Omega} e^{\alpha x_1}\,.$$

We finish the proof by letting $\varepsilon \to 0$. □

As an application we have the uniqueness of solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the following Dirichlet boundary value problem for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

if $c(x) \leq 0$ in $\Omega$.

REMARK 2.4. The boundedness of domain $\Omega$ is essential, since it guarantees the existence of maximum and minimum of $u$ in $\bar{\Omega}$. The uniqueness does not hold if the domain is unbounded. Some examples are given in Remark 1.9 of Section 1.2 in Chapter 1. Equally important is the nonpositiveness of the coefficient $c$.

EXAMPLE. Set $\Omega = \{(x, y) \in \mathbb{R}^2; 0 < x < \pi, 0 < y < \pi\}$. Then $u = \sin x \sin y$ is a nontrivial solution for the problem

$$\begin{aligned} \Delta u + 2u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

THEOREM 2.5 (Hopf Lemma). *Let $B$ be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup \{x_0\})$ satisfies $Lu \geq 0$ in $B$ with $c(x) \leq 0$ in $B$. Assume in addition that*

$$u(x) < u(x_0) \quad \textit{for any } x \in B \textit{ and } u(x_0) \geq 0\,.$$

*Then for each outward direction $\nu$ at $x_0$ with $\nu \cdot \mathbf{n}(x_0) > 0$ there holds*

$$\liminf_{t\to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] > 0\,.$$

REMARK 2.6. If in addition $u \in C^1(B \cup \{x_0\})$, then we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0 .$$

PROOF. We may assume that $B$ has the center at the origin with radius $r$. We assume further that $u \in C(\bar{B})$ and $u(x) < u(x_0)$ for any $x \in \bar{B} \setminus \{x_0\}$ (since we can construct a tangent ball $B_1$ to $B$ at $x_0$ and $B_1 \subset B$).

Consider $v(x) = u(x) + \varepsilon h(x)$ for some nonnegative function $h$. We will choose $\varepsilon > 0$ appropriately such that $v$ attains its nonnegative maximum only at $x_0$. Denote $\Sigma = B \cap B_{\frac{1}{2}r}(x_0)$. Define $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$ with $\alpha$ to be determined. We check in the following that

$$Lh > 0 \quad \text{in } \Sigma .$$

Direct calculation yields

$$\begin{aligned} Lh &= e^{-\alpha|x|^2}\left\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n} a_{ii}(x) - 2\alpha \sum_{n=1}^{n} b_i(x)x_i + c\right\} - ce^{-\alpha r^2} \\ &\geq e^{-\alpha|x|^2}\left\{4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x)x_i x_j - 2\alpha \sum_{i=1}^{n}[a_{ii}(x) + b_i(x)x_i] + c\right\} . \end{aligned}$$

By ellipticity assumption, we have

$$\sum_{i,j=1}^{n} a_{ij}(x)x_i x_j \geq \lambda|x|^2 \geq \lambda\left(\frac{r}{2}\right)^2 > 0 \quad \text{in } \Sigma .$$

So for $\alpha$ large enough, we conclude $Lh > 0$ in $\Sigma$. With such $h$, we have $Lv = Lu + \varepsilon Lh > 0$ in $\Sigma$ for any $\varepsilon > 0$. By Lemma 2.1, $v$ cannot attain its nonnegative maximum inside $\Sigma$.

Next we prove for some small $\varepsilon > 0$ $v$ attains at $x_0$ its nonnegative maximum. Consider $v$ on the boundary $\partial\Sigma$.

(i) For $x \in \partial\Sigma \cap B$, since $u(x) < u(x_0)$, so $u(x) < u(x_0) - \delta$ for some $\delta > 0$. Take $\varepsilon$ small such that $\varepsilon h < \delta$ on $\partial\Sigma \cap B$. Hence for such $\varepsilon$ we have $v(x) < u(x_0)$ for $x \in \partial\Sigma \cap B$.

(ii) On $\Sigma \cap \partial B$, $h(x) = 0$ and $u(x) < u(x_0)$ for $x \neq x_0$. Hence $v(x) < u(x_0)$ on $\Sigma \cap \partial B \setminus \{x_0\}$ and $v(x_0) = u(x_0)$.

Therefore we conclude

$$\frac{v(x_0) - v(x_0 - t\nu)}{t} \geq 0 \quad \text{for any small } t > 0 .$$

Hence we obtain by letting $t \to 0$

$$\liminf_{t \to 0} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] \geq -\varepsilon \frac{\partial h}{\partial \nu}(x_0) .$$

By definition of $h$, we have

$$\frac{\partial h}{\partial \nu}(x_0) < 0 .$$

This finishes the proof. □

THEOREM 2.7 (Strong Maximum Principle). *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in $\Omega$. Then the nonnegative maximum of $u$ in $\bar{\Omega}$ can be assumed only on $\partial\Omega$ unless $u$ is a constant.*

PROOF. Let $M$ be the nonnegative maximum of $u$ in $\bar{\Omega}$. Set $\Sigma = \{x \in \Omega; u(x) = M\}$. It is relatively closed in $\Omega$. We need to show $\Sigma = \Omega$.

We prove by contradiction. If $\Sigma$ is a proper subset of $\Omega$, then we may find an open ball $B \subset \Omega \setminus \Sigma$ with a point on its boundary belonging to $\Sigma$. (In fact, we may choose a point $p \in \Omega \setminus \Sigma$ such that $d(p, \Sigma) < d(p, \partial\Omega)$ first and then extend the ball centered at $p$. It hits $\Sigma$ before hitting $\partial\Omega$.) Suppose $x_0 \in \partial B \cap \Sigma$. Obviously we have $Lu \geq 0$ in $B$ and

$$u(x) < u(x_0) \quad \text{for any } x \in B \text{ and } u(x_0) = M \geq 0\,.$$

Theorem 2.5 implies $\frac{\partial u}{\partial n}(x_0) > 0$ where $\mathbf{n}$ is the outward normal direction at $x_0$ to the ball $B$. While $x_0$ is the interior maximal point of $\Omega$, hence $Du(x_0) = 0$. This leads to a contradiction. □

COROLLARY 2.8 (Comparison Principle). *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$ in $\Omega$. If $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in $\Omega$. In fact, either $u < 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.*

In order to discuss the boundary value problem with general boundary condition, we need the following result, which is just a corollary of Theorem 2.5 and Theorem 2.7.

COROLLARY 2.9. *Suppose $\Omega$ has the interior sphere property and that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$ with $c(x) \leq 0$. Assume $u$ attains its nonnegative maximum at $x_0 \in \bar{\Omega}$. Then $x_0 \in \partial\Omega$ and for any outward direction $\boldsymbol{\nu}$ at $x_0$ to $\partial\Omega$*

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

*unless $u$ is constant in $\bar{\Omega}$.*

APPLICATION. Suppose $\Omega$ is bounded in $\mathbb{R}^n$ and satisfies the interior sphere property. Consider the the following boundary value problem

$$(*) \qquad \begin{aligned} Lu &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. Assume in addition that $c(x) \leq 0$ in $\Omega$ and $\alpha(x) \geq 0$ on $\partial\Omega$. Then the problem $(*)$ has a unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ if $c \not\equiv 0$ or $\alpha \not\equiv 0$. If $c \equiv 0$ and $\alpha \equiv 0$, the problem $(*)$ has a unique solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ up to additive constants.

PROOF. Suppose $u$ is a solution to the following homogeneous equation

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}$$

CASE 1. $c \not\equiv 0$ or $\alpha \not\equiv 0$. We want to show $u \equiv 0$.

Suppose that $u$ has a positive maximum at $x_0 \in \bar{\Omega}$. If $u \equiv$ const. $> 0$, this contradicts the condition $c \not\equiv 0$ in $\Omega$ or $\alpha \not\equiv 0$ on $\partial\Omega$. Otherwise $x_0 \in \partial\Omega$ and $\frac{\partial u}{\partial n}(x_0) > 0$ by Corollary 2.9, which contradicts the boundary value. Therefore $u \equiv 0$.

CASE 2. $c \equiv 0$ and $\alpha \equiv 0$. We want to show $u \equiv$ const.

Suppose $u$ is a nonconstant solution. Then its maximum in $\bar{\Omega}$ is assumed only on $\partial\Omega$ by Theorem 2.7, say at $x_0 \in \partial\Omega$. Again Corollary 2.9 implies $\frac{\partial u}{\partial n}(x_0) > 0$. This is a contradiction. □

The following theorem, due to Serrin, generalizes the comparison principle under no restriction on $c(x)$.

THEOREM 2.10. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$. If $u \leq 0$ in $\Omega$, then either $u < 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.*

PROOF. *Method* 1. Suppose $u(x_0) = 0$ for some $x_0 \in \Omega$. We will prove that $u \equiv 0$ in $\Omega$.

Write $c(x) = c^+(x) - c^-(x)$ where $c^+(x)$ and $c^-(x)$ are the positive part and negative part of $c(x)$ respectively. Hence $u$ satisfies

$$a_{ij}D_{ij}u + b_iD_iu - c^-u \geq -c^+u \geq 0\,.$$

So we have $u \equiv 0$ by Theorem 2.7.

*Method* 2. Set $v = ue^{-\alpha x_1}$ for some $\alpha > 0$ to be determined. By $Lu \geq 0$, we have

$$a_{ij}D_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]D_iv + \left(a_{11}\alpha^2 + b_1\alpha + c\right)v \geq 0\,.$$

Choose $\alpha$ large enough such that $a_{11}\alpha^2 + b_1\alpha + c > 0$. Therefore $v$ satisfies

$$a_{ij}D_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]D_iv \geq 0\,.$$

Hence we apply Theorem 2.7 to $v$ to conclude that either $v < 0$ in $\Omega$ or $v \equiv 0$ in $\Omega$. □

The next result is the general maximum principle for the operator $L$ with no restriction on $c(x)$.

THEOREM 2.11. *Suppose there exists a $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $w > 0$ in $\bar{\Omega}$ and $Lw \leq 0$ in $\Omega$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in $\Omega$, then $\frac{u}{w}$ cannot assume in $\Omega$ its nonnegative maximum unless $\frac{u}{w} \equiv$ const. If, in addition, $\frac{u}{w}$ assumes its nonnegative maximum at $x_0 \in \partial\Omega$ and $\frac{u}{w} \not\equiv$ const, then for any outward direction $\nu$ at $x_0$ to $\partial\Omega$ there holds*

$$\frac{\partial}{\partial \nu}\left(\frac{u}{w}\right)(x_0) > 0$$

*if $\partial\Omega$ has the interior sphere property at $x_0$.*

PROOF. Set $v = \frac{u}{w}$. Then $v$ satisfies

$$a_{ij}D_{ij}v + B_iD_iv + \left(\frac{Lw}{w}\right)v \geq 0$$

where $B_i = b_i + \frac{2}{w} a_{ij} D_{ij} w$. We may apply Theorem 2.7 and Corollary 2.9 to $v$. □

REMARK 2.12. If the operator $L$ in $\Omega$ satisfies the condition of Theorem 2.11, then $L$ has the comparison principle. In particular, the Dirichlet boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

has at most one solution.

The next result is the so-called maximum principle for *narrow domain.*

PROPOSITION 2.13. *Let $d$ be a positive number and $\mathbf{e}$ be a unit vector such that $|(y - x) \cdot \mathbf{e}| < d$ for any $x, y \in \Omega$. Then there exists a $d_0 > 0$, depending only on $\lambda$ and the sup-norm of $b_i$ and $c^+$, such that the assumptions of Theorem 2.11 are satisfied if $d \leq d_0$.*

PROOF. By choosing $\mathbf{e} = (1, 0, \ldots, 0)$ we suppose $\bar{\Omega}$ lies in $\{0 < x_1 < d\}$. Assume in addition $|b_i|, c^+ \leq N$ for some positive constant $N$. We construct $w$ as follows: Set $w = e^{\alpha d} - e^{\alpha x_1} > 0$ in $\bar{\Omega}$. By direct calculation we have

$$Lw = -\big(a_{11}\alpha^2 + b_1\alpha\big)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -\big(a_{11}\alpha^2 + b_1\alpha\big) + Ne^{\alpha d}.$$

Choose $\alpha$ so large that

$$a_{11}\alpha^2 + b_1\alpha \geq \lambda\alpha^2 - N\alpha \geq 2N.$$

Hence $Lw \leq -2N + Ne^{\alpha d} = N(e^{\alpha d} - 2) \leq 0$ if $d$ is small such that $e^{\alpha d} \leq 2$. □

REMARK 2.14. Some results in this section, including Proposition 2.13, hold for unbounded domain. Compare Proposition 2.13 with Theorem 2.32.

## 2.3. A Priori Estimates

In this section we derive a priori estimates for solutions to the Dirichlet problem and the Neumann problem.

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator $L$ in $\Omega$

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We assume that $a_{ij}$, $b_i$, and $c$ are continuous and hence bounded in $\bar{\Omega}$ and that $L$ is uniformly elliptic in $\Omega$, that is,

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

where $\lambda$ is a positive number. We denote by $\Lambda$ the sup-norm of $a_{ij}$ and $b_i$, that is,

$$\max_{\Omega} |a_{ij}| + \max_{\Omega} |b_i| \leq \Lambda.$$

PROPOSITION 2.15. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

*for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \le 0$, then there holds*

$$|u(x)| \le \max_{\partial\Omega} |\varphi| + C \max_{\Omega} |f| \quad \text{for any } x \in \Omega$$

*where $C$ is a positive constant depending only on $\lambda$, $\Lambda$, and* $\operatorname{diam}(\Omega)$.

PROOF. We will construct a function $w$ in $\Omega$ such that

(i) $L(w \pm u) = Lw \pm f \le 0$, or $Lw \le \mp f$ in $\Omega$;
(ii) $w \pm u = w \pm \varphi \ge 0$, or $w \ge \mp\varphi$ on $\partial\Omega$.

Denote $F = \max_\Omega |f|$ and $\Phi = \max_{\partial\Omega} |\varphi|$. We need

$$\begin{aligned} Lw &\le -F \quad \text{in } \Omega \\ w &\ge \Phi \quad \text{on } \partial\Omega\,. \end{aligned}$$

Suppose the domain $\Omega$ lies in the set $\{0 < x_1 < d\}$ for some $d > 0$. Set $w = \Phi + (e^{\alpha d} - e^{\alpha x_1})F$ with $\alpha > 0$ to be chosen later. Then we have by direct calculation

$$\begin{aligned} -Lw &= (a_{11}\alpha^2 + b_1\alpha)Fe^{\alpha x_1} - c\Phi - c(e^{\alpha d} - e^{\alpha x_1})F \\ &\ge (a_{11}\alpha^2 + b_1\alpha)F \ge (\alpha^2\lambda + b_1\alpha)F \ge F \end{aligned}$$

by choosing $\alpha$ large such that $\alpha^2\lambda + b_1(x)\alpha \ge 1$ for any $x \in \Omega$. Hence $w$ satisfies (i) and (ii). By Corollary 2.8 (the comparison principle) we conclude $-w \le u \le w$ in $\Omega$, in particular

$$\sup_\Omega |u| \le \Phi + (e^{\alpha d} - 1)F$$

where $\alpha$ is a positive constant depending only on $\lambda$ and $\Lambda$. □

PROPOSITION 2.16. *Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$\begin{cases} Lu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u = \varphi & \text{on } \partial\Omega \end{cases}$$

*where $\mathbf{n}$ is the outward normal direction to $\partial\Omega$. If $c(x) \le 0$ in $\Omega$ and $\alpha(x) \ge \alpha_0 > 0$ on $\partial\Omega$, then there holds*

$$|u(x)| \le C\Big\{ \max_{\partial\Omega} |\varphi| + \max_\Omega |f| \Big\} \quad \text{for any } x \in \Omega$$

*where $C$ is a positive constant depending only on $\lambda$, $\Lambda$, $\alpha_0$, and* $\operatorname{diam}(\Omega)$.

PROOF. SPECIAL CASE. $c(x) \le -c_0 < 0$.
We will show

$$|u(x)| \le \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \quad \text{for any } x \in \Omega\,.$$

Define $v = \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \pm u$. Then we have

$$Lv = c(x)\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm f \le -F \pm f \le 0 \quad \text{in } \Omega$$

$$\frac{\partial v}{\partial n} + \alpha v = \alpha\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm \varphi \ge \Phi \pm \varphi \ge 0 \qquad \text{on } \partial\Omega\,.$$

If $v$ has a negative minimum in $\bar{\Omega}$, then $v$ attains it on $\partial\Omega$ by Theorem 2.5, say, at $x_0 \in \partial\Omega$. This implies $\frac{\partial v}{\partial n}(x_0) \le 0$ for $\mathbf{n} = \mathbf{n}(x_0)$, the outward normal direction at $x_0$. Therefore we get

$$\left(\frac{\partial v}{\partial n} + \alpha v\right)(x_0) \le \alpha v(x_0) < 0$$

which is a contradiction. Hence we have $v \ge 0$ in $\bar{\Omega}$, in particular,

$$|u(x)| \le \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \quad \text{for any } x \in \Omega\,.$$

Note that for this special case $c_0$ and $\alpha_0$ are independent of $\lambda$ and $\Lambda$.

GENERAL CASE. $c(x) \le 0$ for any $x \in \Omega$.

Consider the auxiliary function $u(x) = z(x)w(x)$ where $z$ is a positive function in $\bar{\Omega}$ to be determined. Direct calculation shows that $w$ satisfies

$$a_{ij}D_{ij}w + B_iD_iw + \left(c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z}\right)w = \frac{f}{z} \quad \text{in } \Omega$$

$$\frac{\partial w}{\partial n} + \left(\alpha + \frac{1}{z}\frac{\partial z}{\partial n}\right)w = \frac{\varphi}{z} \quad \text{on } \partial\Omega$$

where $B_i = \frac{1}{z}(a_{ij} + a_{ji})D_jz + b_i$. We need to choose the function $z > 0$ in $\bar{\Omega}$ such that there hold in

$$c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z} \le -c_0(\lambda, \Lambda, d, \alpha_0) < 0 \quad \text{in } \Omega$$

$$\alpha + \frac{1}{z}\frac{\partial z}{\partial n} \ge \frac{1}{2}\alpha_0 \qquad \text{on } \partial\Omega\,,$$

or

$$\frac{a_{ij}D_{ij}z + b_iD_iz}{z} \le -c_0 < 0 \qquad \text{in } \Omega$$

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \le \frac{1}{2}\alpha_0 \qquad \text{on } \partial\Omega\,.$$

Suppose the domain $\Omega$ lies in $\{0 < x_1 < d\}$. Choose $z(x) = A + e^{\beta d} - e^{\beta x_1}$ for $x \in \Omega$ for some positive $A$ and $\beta$ to be determined. Direct calculation shows

$$-\frac{1}{z}\left(a_{ij}D_{ij}z + b_iD_iz\right) = \frac{(\beta^2 a_{11} + \beta b_1)e^{\beta x_1}}{A + e^{\beta d} - e^{\beta x_1}} \ge \frac{\beta^2 a_{11} + \beta b_1}{A + e^{\beta d}} \ge \frac{1}{A + e^{\beta d}} > 0\,,$$

if $\beta$ is chosen such that $\beta^2 a_{11} + \beta b_1 \ge 1$. Then we have

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \le \frac{\beta}{A}e^{\beta d} \le \frac{1}{2}\alpha_0$$

if $A$ is chosen large. This reduces to the special case we just discussed. The new extra first-order term does not change the result. We may apply the special case to $w$. □

REMARK 2.17. The result fails if we just assume $\alpha(x) \geq 0$ on $\partial\Omega$. In fact, we cannot even get the uniqueness.

## 2.4. Gradient Estimates

The basic idea in the treatment of gradient estimates, due to Bernstein, involves differentiation of the equation with respect to $x_k$, $k = 1, \ldots, n$, followed by multiplication by $D_k u$ and summation over $k$. The maximum principle is then applied to the resulting equation in the function $v = |Du|^2$, possibly with some modification. There are two kinds of gradient estimates, global gradient estimates and interior gradient estimates. We will use semi-linear equations to illustrate the idea.

Suppose $\Omega$ is a bounded and connected domain in $\mathbb{R}^n$. Consider the equation

$$a_{ij}(x)D_{ij}u + b_i(x)D_i u = f(x, u) \quad \text{in } \Omega$$

for $u \in C^2(\Omega)$ and $f \in C(\Omega \times \mathbb{R})$. We always assume that $a_{ij}$ and $b_i$ are continuous and hence bounded in $\bar{\Omega}$ and that the equation is uniformly elliptic in $\Omega$ in the following sense

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n$$

for some positive constant $\lambda$.

PROPOSITION 2.18. *Suppose $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_i u = f(x, u) \quad \text{in } \Omega \tag{2.1}$$

*for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$\sup_{\Omega} |Du| \leq \sup_{\partial\Omega} |Du| + C$$

*where $C$ is a positive constant depending only on $\lambda$,* $\mathrm{diam}(\Omega)$, *$|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$, and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.*

PROOF. Set $L \equiv a_{ij}D_{ij} + b_i D_i$. We calculate $L(|Du|^2)$ first. Note

$$D_i(|Du|^2) = 2D_k u D_{ki} u$$

and

$$D_{ij}(|Du|^2) = 2D_{ki}D_{kj}u + 2D_k u D_{kij} u\,. \tag{2.2}$$

Differentiating (2.1) with respect to $x_k$, multiplying by $D_k u$, and summing over $k$, we have by (2.2)

$$\begin{aligned} a_{ij}D_{ij}(|Du|^2) + b_i D_i(|Du|^2) = {} & 2a_{ij}D_{ki}u D_{kj}u - 2D_k a_{ij} D_k u D_{ij} u \\ & - 2D_k b_i D_k u D_i u + 2D_z f |Du|^2 + 2D_k f D_k u\,. \end{aligned}$$

Ellipticity assumption implies

$$\sum_{i,j,k} a_{ij} D_{ki} u D_{kj} u \geq \lambda |D^2 u|^2 .$$

By Cauchy inequality, we have

$$L(|Du|^2) \geq \lambda |D^2 u|^2 - C|Du|^2 - C$$

where $C$ is a positive constant depending only on

$$\lambda , \quad |a_{ij}, b_i|_{C^1(\bar{\Omega})} , \quad \text{and} \quad |f|_{C^1(\bar{\Omega}\times[-M,M])} .$$

We need to add another term $u^2$. We have by ellipticity assumption

$$L(u^2) = 2a_{ij} D_i u D_j u + 2u\{a_{ij} D_{ij} u + b_i D_i u\} \geq 2\lambda |Du|^2 + 2uf .$$

Therefore we obtain

$$L(|Du|^2 + \alpha u^2) \geq \lambda |D^2 u|^2 + (2\lambda\alpha - C)|Du|^2 - C \geq \lambda |D^2 u|^2 + |Du|^2 - C$$

if we choose $\alpha > 0$ large, with $C$ depending in addition on $M$. In order to control the constant term we may consider another function $e^{\beta x_1}$ for $\beta > 0$. Hence we get

$$L\big(|Du|^2 + \alpha u^2 + e^{\beta x_1}\big) \geq \lambda |D^2 u|^2 + |Du|^2 + \big\{\beta^2 a_{11} e^{\beta x_1} + \beta b_1 e^{\beta x_1} - C\big\} .$$

If we put the domain $\Omega \subset \{x_1 > 0\}$, then $e^{\beta x_1} \geq 1$ for any $x \in \Omega$. By choosing $\beta$ large, we may make the last term positive. Therefore, if we set $w = |Du|^2 + \alpha |u|^2 + e^{\beta x_1}$ for large $\alpha$, $\beta$ depending only on $\lambda$, $\operatorname{diam}(\Omega)$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$, then we obtain

$$Lw \geq 0 \quad \text{in } \Omega .$$

By the maximum principle we have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w .$$

This finishes the proof. □

Similarly, we can discuss the interior gradient bound. In this case, we just require the bound of $\sup_\Omega |u|$.

PROPOSITION 2.19. *Suppose $u \in C^3(\Omega)$ satisfies*

$$a_{ij}(x) D_{ij} u + b_i(x) D_i u = f(x, u) \quad \text{in } \Omega$$

*for $a_{ij}, b_i \in C^1(\bar{\Omega})$, and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then there holds for any compact subset $\Omega' \Subset \Omega$*

$$\sup_{\Omega'} |Du| \leq C$$

*where $C$ is a positive constant depending only on* $\lambda$, $\operatorname{diam}(\Omega)$, $\operatorname{dist}(\Omega', \partial\Omega)$, $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$, *and* $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.

PROOF. We need to take a cut-off function $\gamma \in C_0^\infty(\Omega)$ with $\gamma \geq 0$ and consider the auxiliary function with the following form:

$$w = \gamma|Du|^2 + \alpha|u|^2 + e^{\beta x_1}.$$

Set $v = \gamma|Du|^2$. Then we have for operator $L$ defined as before

$$Lv = (L\gamma)|Du|^2 + \gamma L(|Du|^2) + 2a_{ij} D_i\gamma D_j|Du|^2.$$

Recall an inequality in the proof of Proposition 2.18,

$$L(|Du|^2) \geq \lambda|D^2u|^2 - C|Du|^2 - C.$$

Hence we have

$$Lv \geq \lambda\gamma|D^2u|^2 + 2a_{ij} D_k u D_i\gamma D_{kj}u - C|Du|^2 + (L\gamma)|Du|^2 - C.$$

Cauchy inequality implies for any $\varepsilon > 0$

$$|2a_{ij} D_k u D_i\gamma D_{kj}u| \leq \varepsilon|D\gamma|^2|D^2u|^2 + c(\varepsilon)|Du|^2.$$

For the cut-off function $\gamma$, we require that

$$|D\gamma|^2 \leq C\gamma \quad \text{in } \Omega.$$

Therefore we have by taking $\varepsilon > 0$ small

$$Lv \geq \lambda\gamma|D^2u|^2\left(1 - \varepsilon\frac{|D\gamma|^2}{\gamma}\right) - C|Du|^2 - C \geq \frac{1}{2}\lambda\gamma|D^2u|^2 - C|Du|^2 - C.$$

Now we may proceed as before. □

In the rest of this section we use barrier functions to derive the boundary gradient estimates. We need to assume that the domain $\Omega$ satisfies the uniform exterior sphere property.

PROPOSITION 2.20. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_i u = f(x, u) \quad \textit{in } \Omega$$

*for $a_{ij}, b_i \in C(\bar{\Omega})$ and $f \in C(\bar{\Omega} \times \mathbb{R})$. Then there holds*

$$|u(x) - u(x_0)| \leq C|x - x_0| \quad \textit{for any } x \in \Omega \textit{ and } x_0 \in \partial\Omega$$

*where $C$ is a positive constant depending only on $\lambda$, $\Omega$, $|a_{ij}, b_i|_{L^\infty(\Omega)}$, $M = |u|_{L^\infty(\Omega)}$, $|f|_{L^\infty(\Omega\times[-M,M])}$ and $|\varphi|_{C^2(\bar{\Omega})}$ for some $\varphi \in C^2(\bar{\Omega})$ with $\varphi = u$ on $\partial\Omega$.*

PROOF. For simplicity we assume $u = 0$ on $\partial\Omega$. As before set $L = a_{ij}D_{ij} + b_i D_i$. Then we have

$$L(\pm u) = \pm f \geq -F \quad \text{in } \Omega$$

where we denote $F = \sup_\Omega|f(\cdot, u)|$. Now fix $x_0 \in \partial\Omega$. We will construct a function $w$ such that

$$Lw \leq -F \text{ in } \Omega, \quad w(x_0) = 0, \quad w|_{\partial\Omega} \geq 0.$$

Then by the maximum principle we have

$$-w \leq u \leq w \quad \text{in } \Omega.$$

Taking normal derivative at $x_0$, we have

$$\left|\frac{\partial u}{\partial n}(x_0)\right| \le \frac{\partial w}{\partial n}(x_0)\,.$$

So we need to bound $\frac{\partial w}{\partial n}(x_0)$ independently of $x_0$.

Consider the exterior ball $B_R(y)$ with $\bar{B}_R(y) \cap \bar{\Omega} = \{x_0\}$. Define $d(x)$ as the distance from $x$ to $\partial B_R(y)$. Then we have

$$0 < d(x) < D \equiv \operatorname{diam}(\Omega) \quad \text{for any } x \in \Omega\,.$$

In fact, $d(x) = |x-y| - R$ for any $x \in \Omega$. Consider $w = \psi(d)$ for some function $\psi$ defined in $[0,\infty)$. Then we need

$$\begin{aligned}
&\psi(0) = 0 && (\Longrightarrow w(x_0) = 0)\\
&\psi(d) > 0 \quad \text{for } d > 0 && (\Longrightarrow w|_{\partial\Omega} \ge 0)\\
&\psi'(0) \quad \text{is controlled}\,.
\end{aligned}$$

From the first two inequalities, it is natural to require that $\psi'(d) > 0$. Note

$$Lw = \psi'' a_{ij} D_i d D_j d + \psi' a_{ij} D_{ij}\, d + \psi' b_i D_i d\,.$$

Direct calculation yields

$$\begin{aligned}
D_i d(x) &= \frac{x_i - y_i}{|x-y|}\\
D_{ij} d(x) &= \frac{\delta_{ij}}{|x-y|} - \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3}
\end{aligned}$$

which imply $|Dd| = 1$ and with $\Lambda = \sup |a_{ij}|$

$$\begin{aligned}
a_{ij} D_{ij} d = \frac{a_{ii}}{|x-y|} - \frac{a_{ij}}{|x-y|} D_i d D_j d &\le \frac{n\Lambda}{|x-y|} - \frac{\lambda}{|x-y|}\\
&= \frac{n\Lambda - \lambda}{|x-y|} \le \frac{n\Lambda - \lambda}{R}\,.
\end{aligned}$$

Therefore we obtain by ellipticity

$$Lw \le \psi'' a_{ij} D_i d D_j d + \psi' \left(\frac{n\Lambda - \lambda}{R} + |b|\right) \le \lambda\psi'' + \psi'\left(\frac{n\Lambda - \lambda}{R} + |b|\right)$$

if we require $\psi'' < 0$. Hence in order to have $Lw \le -F$ we need

$$\lambda\psi'' + \psi'\left(\frac{n\Lambda - \lambda}{R} + |b|\right) + F \le 0\,.$$

To this end, we study the equation for some positive constants $a$ and $b$

$$\psi'' + a\psi' + b = 0$$

whose solution is given by

$$\psi(d) = -\frac{b}{a} d + \frac{C_1}{a} - \frac{C_2}{a} e^{-ad}$$

for some constants $C_1$ and $C_2$. For $\psi(0) = 0$, we need $C_1 = C_2$. Hence we have for some constant $C$

$$\psi(d) = -\frac{b}{a}d + \frac{C}{a}(1 - e^{-ad})$$

which implies

$$\psi'(d) = Ce^{-ad} - \frac{b}{a} = e^{-ad}\left(C - \frac{b}{a}e^{ad}\right)$$

$$\psi''(d) = -Cae^{-ad}.$$

In order to have $\psi'(d) > 0$, we need $C \geq \frac{b}{a}e^{aD}$. Since $\psi'(d) > 0$ for $d > 0$, so $\psi(d) > \psi(0) = 0$ for any $d > 0$. Therefore we take

$$\psi(d) = -\frac{b}{a}d + \frac{b}{a^2}e^{aD}\left(1 - e^{-ad}\right) = \frac{b}{a}\left\{\frac{1}{a}e^{aD}\left(1 - e^{-ad}\right) - d\right\}.$$

Such $\psi$ satisfies all the requirements we imposed. This finishes the proof. □

## 2.5. Alexandroff Maximum Principle

Suppose $\Omega$ is a bounded domain in $\mathbb{R}^n$ and consider a second-order elliptic operator $L$ in $\Omega$

$$L \equiv a_{ij}(x)D_{ij} + b_i(x)D_i + c(x)$$

where coefficients $a_{ij}, b_i, c$ are at least continuous in $\Omega$. Ellipticity means that the coefficient matrix $A = (a_{ij})$ is positive definite everywhere in $\Omega$. We set $D = \det(A)$ and $D^* = D^{\frac{1}{n}}$ so that $D^*$ is the geometric mean of the eigenvalues of $A$. Throughout this section we assume

$$0 < \lambda \leq D^* \leq \Lambda$$

where $\lambda$ and $\Lambda$ are two positive constants, which denote, respectively, the minimal and maximal eigenvalues of $A$.

Before stating the main theorem we first introduce the concept of contact sets. For $u \in C^2(\Omega)$ we define

$$\Gamma^+ = \{y \in \Omega;\ u(x) \leq u(y) + Du(y)\cdot(x - y) \text{ for any } x \in \Omega\}.$$

The set $\Gamma^+$ is called the upper contact set of $u$ and Hessian matrix $D^2u = (D_{ij}u)$ is nonpositive on $\Gamma^+$. In fact, the upper contact set can also be defined for continuous function $u$ by the following:

$$\Gamma^+ =$$
$$\{y \in \Omega;\ u(x) \leq u(y) + p\cdot(x - y) \text{ for any } x \in \Omega \text{ and some } p = p(y) \in \mathbb{R}^n\}.$$

Clearly, $u$ is concave if and only if $\Gamma^+ = \Omega$. If $u \in C^1(\Omega)$, then $p(y) = Du(y)$ and any support hyperplane must then be a tangent plane to the graph.

Now we consider the equation of the following form

$$Lu = f \quad \text{in } \Omega$$

for some $f \in C(\Omega)$.