In the last lecture we used the reflection method to solve the boundary value problem for the wave equation on the half-line. We would like to apply the same method to the boundary value problems on the finite interval, which correspond to the physically realistic case of a finite string. Consider the Dirichlet wave problem on the finite line

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & 0 < x < l, 0 < t < \infty, \\ v(x,0) = \phi(x), & v_t(x,0) = \psi(x), \quad x > 0, \\ v(0,t) = v(l,t) = 0, & t > 0. \end{cases} \tag{25}$$

The homogeneous Dirichlet conditions correspond to the vibrating string having fixed ends, as is the case for musical instruments. Using our intuition from the half-line problems, where the wave reflects from the fixed end, we can imagine that in the case of the finite interval the wave bounces back and forth infinitely many times between the endpoints. In spite of this, we can still use the reflection method to find the value of the solution to problem (25) at any point (x, t).

Recall that the idea of the reflection method is to extend the initial data to the whole line in such a way, that the boundary conditions are automatically satisfied. For the homogeneous Dirichlet data the appropriate choice is the odd extension. In this case, we need to extend the initial data ϕ, ψ, which are defined only on the interval $0 < x < l$, in such a way that the resulting extensions are odd with respect to both $x = 0$, and $x = l$. That is, the extensions must satisfy

$$f(-x) = -f(x) \qquad \text{and} \qquad f(l-x) = -f(l+x). \tag{26}$$

Notice that for such a function $f(0) = -f(0)$ from the first condition, and $f(l) = -f(l)$ from the second condition, hence, $f(0) = f(l) = 0$. Subsequently, the solution to the IVP with such data will be odd with respect to both $x = 0$ and $x = l$, and the boundary conditions will be automatically satisfied. Notice that the conditions (26) imply that functions that are odd with respect to both $x = 0$ and $x = l$ satisfy $f(2l+x) = -f(-x) = f(x)$, which means that such functions must be $2l-$periodic. Using this we can define the extensions of the initial data ϕ, ψ as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{for } 0 < x < l, \\ -\phi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l - \text{periodic}, \end{cases} \qquad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{for } 0 < x < l, \\ -\psi(-x) & \text{for } -l < x < 0, \\ \text{extended to be } 2l - \text{periodic}. \end{cases} \tag{27}$$

Now, consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{ext}}(x), u_t(x,0) = \psi_{\text{ext}}(x). \end{cases}$$

For the solution of this IVP we automatically have $u(0,t) = u(l,t) = 0$, and the restriction

$$v(x,t) = u(x,t)\big|_{0 \le x \le l},$$

will solve the boundary value problem (25). By d'Alambert's formula, the solution will be given as

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds \tag{28}$$

for $0 < x < l$. Although formula (28) contains all the information about our solution, we would like to have an expression in terms of the original initial data, so that the values of the solution can be directly computed using the given functions $\phi(x)$ and $\psi(x)$. For this, we need to "bring" the points $x - ct$ and $x + ct$ into the interval $(0, l)$ using the periodicity and oddity of the extended data. To illustrate how this is done, let us fix a point (x, t) and try to find the value of the solution at this point by tracing it back in time along the characteristics to the initial data. The sketch of the backwards characteristics from this point appears in Figure 5 above.

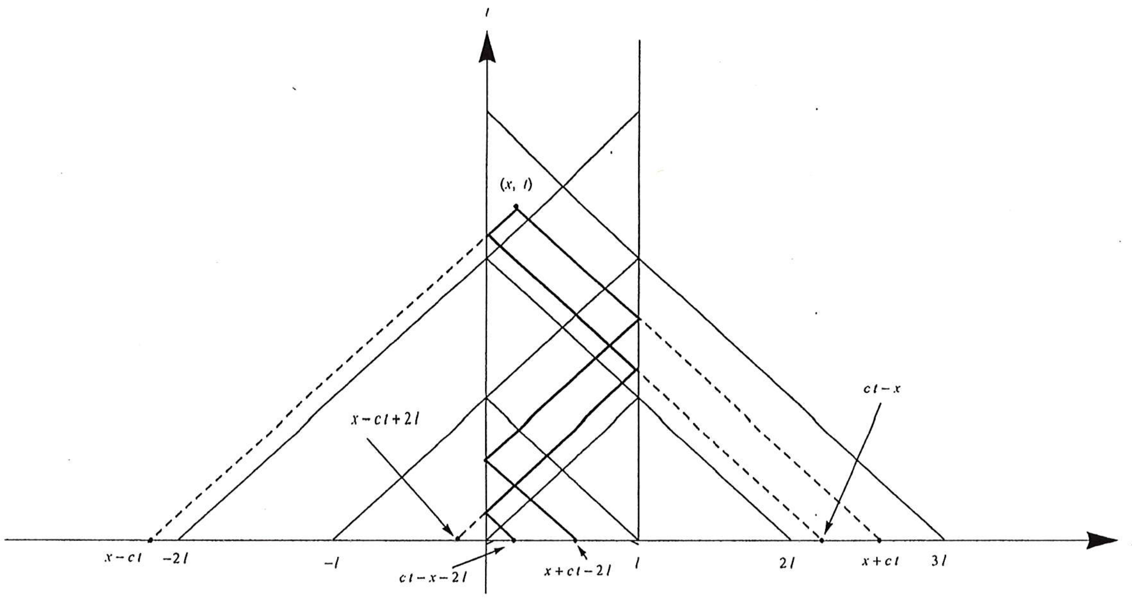


Figure 5: The backwards characteristics from the point (x, t).

In general, the points $x + ct$ and $x - ct$ will end up either in the interval $(0, l)$ or $(-l, 0)$ after finitely many translations by the period $2l$. If the point ends up in $(0, l)$ (even number of reflections), then the value of the initial data picked up by the reflected characteristic will be taken with a positive sign. If, however, the point ends up in the interval $(-l, 0)$ (odd number of reflections), then we need to reflect this point with respect to $x = 0$, and the corresponding value of the initial data will be taken with a negative sign.

From Figure 5 we see that $x + ct$ goes into the interval $(0, l)$ (2 reflections) after translating it to the left by one period $2l$, but the point $x - ct$ goes into the interval $(-l, 0)$ (3 reflections) after a right translation by $2l$, so we need to reflect the resulting point $x - ct + 2l$ to arrive at the point $ct - x - 2l$ in the interval $(0, l)$. The solution will then be

$$u(x,t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

For the integral term, we can break it into two integrals as follows

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds = \frac{1}{2c}\int_{x-ct}^{ct-x} \psi_{\text{ext}}(s)\, ds + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi_{\text{ext}}(s)\, ds.$$

Notice that from the oddity of ψ_{ext}, the integral over the interval $[x - ct, ct - x]$ will be zero, while by periodicity, we can bring the interval $[ct - x, x + ct]$ into the interval $(0, l)$ by subtracting one period $2l$. Thus, the solution can be written as

$$u(x,t) = \frac{1}{2}[\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c}\int_{ct-x-2l}^{x+ct-2l} \psi(s)\, ds. \tag{29}$$

Clearly, the derivation of the above expression for the solution depends on the chosen point, which in turn determines how many reflections the backward characteristics undergo before arriving at the x axis. Hence, the solution will be given by different expressions, depending on the region from which the point is taken. These different regions are depicted in Figure 6, where the labels (m, n) show how many times each of the two backward characteristics gets reflected before reaching the x axis. Expression (29) will be valid for all the points in the region $(3, 2)$.

The method used to arrive at the expression (29) can be used to find the value of the solution at any point (x, t), although it is quite impractical to derive the expression for each of the regions depicted in Figure 6. Furthermore, it does not generalize to higher dimensions, nor does it apply to the heat equation (no characteristics to trace back). Later on we will study another method, called *separation of variables*, which allows for a more general way of approaching boundary value problems on finite intervals.

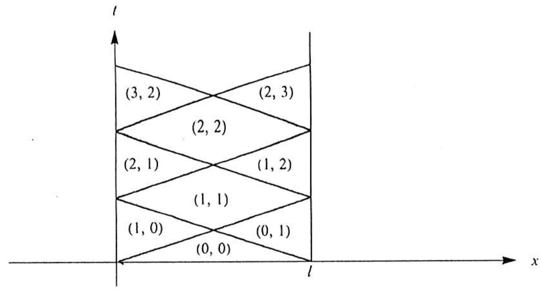


Figure 6: Regions of $(x, t) \in (0, l) \times (0, \infty)$ with the different number of reflections.

Example 14.1. Consider the Dirichlet wave problem on the finite interval

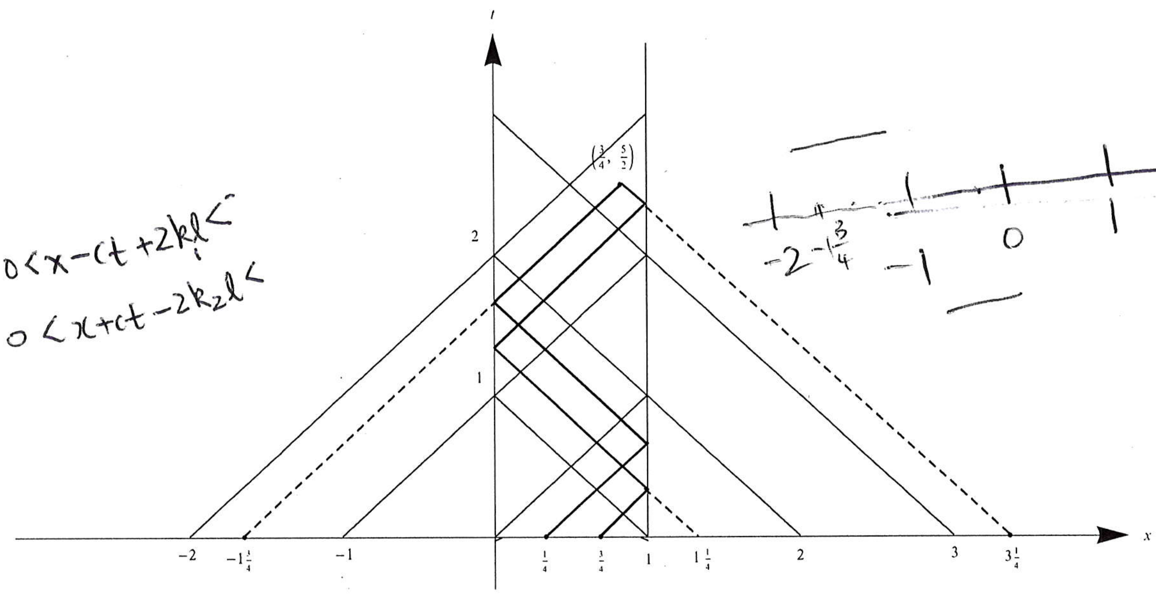
$$\begin{cases} u_{tt} - u_{xx} = 0, \quad \text{for } 0 < x < 1, \\ u(x, 0) = x(1 - x), u_t(x, 0) = x^2, \\ u(0, t) = u(1, t) = 0. \end{cases}$$

Find the value of the solution at the point $(\frac{3}{4}, \frac{5}{2})$.

Notice that in this problem $c = 1$, and $l = 1$, so the period of the extended data will be $2l = 2$. The sketch of the backward characteristics from the point $(x, t) = (\frac{3}{4}, \frac{5}{2})$ appears in the figure below.

$0 < x - ct + 2k_1 l <$

$0 < x + ct - 2k_2 l <$



-2 $-1\frac{3}{4}$ -1 0 1 2 3

Figure 7: The backwards characteristics from the point $(\frac{3}{4}, \frac{5}{2})$.

The characteristics intersect the x axis at the points

$$x - t = \tfrac{3}{4} - \tfrac{5}{2} = -1\tfrac{3}{4} \qquad \text{and} \qquad x + t = \tfrac{3}{4} + \tfrac{5}{2} = 3\tfrac{1}{4}.$$

The point $-1\frac{3}{4}$ goes to the point $\frac{1}{4}$ after a right translation by one period, while the point $3\frac{1}{4}$ goes to the point $1\frac{1}{4}$ after a left translation by one period. After a reflection with respect to $x = 1$, this point will end up at $\frac{3}{4}$, thus, the value of the initial data must be taken with a negative sign at this point. Also, the integral over the

$$\int_{-1\frac{3}{4}}^{3\frac{1}{4}} \psi(s)\,ds = \int_{-1\frac{3}{4}}^{\frac{1}{4}} \psi(s)\,ds + \int_{\frac{1}{4}}^{1\frac{3}{4}} \psi(s)\,ds \quad \int_{\frac{1}{4}}^{\frac{3}{4}} \psi(s)\,ds \quad \int_{\frac{3}{4}}^{1\frac{1}{4}} \psi = 0.$$

interval $[\frac{3}{4}, 1\frac{1}{4}]$ of ψ_{ext} will be zero due to its oddity with respect to $x = 1$. The value of the solution is then

$$u(\tfrac{3}{4}, \tfrac{5}{2}) = \frac{1}{2}[-\phi(\tfrac{3}{4}) + \phi(\tfrac{1}{4})] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} \psi(s)\, ds = \frac{1}{2}\left[-\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\right] + \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} x^2\, dx$$

$$= \frac{x^3}{6}\Bigg|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{6}\left(\frac{27}{64} - \frac{1}{64}\right) = \frac{13}{192}.$$

□

14.1 The parallelogram rule

Recall from a homework problem, that for the vertices of a characteristic parallelogram A, B, C and D as for example in Figure 8, the values of the solution of the wave equation are related as follows

$$u(A) + u(C) = u(B) + u(D).$$

Hence, we can find the value at the vertex A from the values at the three other vertices.

$$u(A) = u(B) + u(D) - u(C).$$

Notice that the values at the vertices B and C in Figure 8 can be found from the expression of the solution for the region $(0, 0)$, while the value at D comes from the boundary data. Thus we reduced finding the value at a point in the region $(1, 0)$ to finding values in the region $(0, 0)$. One can always follow this procedure to evaluate the solution in the regions $(m + 1, n)$ and $(m, n + 1)$ via the values in the region (m, n), provided the boundary condition is in the Dirichlet form.

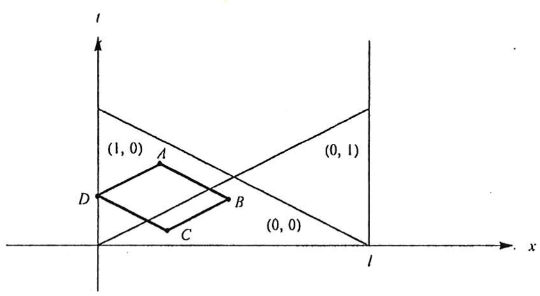


Figure 8: The parallelogram rule.

14.2 Conclusion

We applied the reflection method to derive expressions for the solution to the Dirichlet wave problem on the finite interval. However, the method yields infinitely many expressions for different regions in $(x, t) \in (0, l) \times (0, \infty)$, depending on the number of times the backward characteristics from a point get reflected before reaching the x axis, where the initial data is defined. This makes the method impractical in applications, and is not generalizable to higher dimensions and other PDEs. An alternative method (separation of variables) of solving boundary value problems on the finite interval will be described later in the course.