5. Properties of Entire Solutions

5.1. Gradient bound. The following gradient bound was proven by Modica.

Proposition 5.1. *Suppose u is a solution to (1.1) in $\mathbb{R}^n$. Then*

$$|\nabla u(x)|^2 \le 2F\big(u(x)\big), \quad x \in \mathbb{R}^n. \tag{5.1}$$

Proof. Define

$$G(x) := F\big(u(x)\big) - \frac{1}{2}|\nabla u(x)|^2, \quad x \in \mathbb{R}^n.$$

The standard gradient estimate for elliptic equation implies $G(x) > -C > -\infty, x \in \mathbb{R}^n$ for some constant C.

Assume that there exists a sequence $\{x_n\}$ such that $\inf_{x\in\mathbb{R}^n} G(x) = \lim_{n\to\infty} G(x_n)$. We define $u_n(x) := u(x_n + x), x \in \mathbb{R}^n$ and $G_n(x) := G(x_n + x), x \in \mathbb{R}^n$. It is easy to see that $||u_n||_{C^{2,\alpha}(\mathbb{R}^n)} \le C < \infty$ and a subsequence of $\{u_n\}$ has a limit u_∞ in $C^2_{loc}(\mathbb{R}^n)$. It is easy to see that u_∞ is also a solution to (1.1) and $G_\infty(x) := F\big(u_\infty(x)\big) - \frac{1}{2}|\nabla u_\infty(x)|^2, \quad x \in \mathbb{R}^n$ satisfies

$$G_\infty(0) = \inf_{x\in\mathbb{R}^n} G_\infty(x) = \inf_{x\in\mathbb{R}^n} G(x).$$

Hence we may assume that $G(0) = \inf_{x\in\mathbb{R}^n} G(x)$ since we otherwise we can replace u by u_∞. We shall prove that $G(0) \ge 0$.

Straightforward computations lead to

$$\begin{aligned}|\nabla u(x)|^2 \Delta G(x) &= \Sigma_{i=1}^n b_i(x) G_{x_i}(x) \\ &+ \Sigma_{i=1}^n \big(\Sigma_{j=1}^n u_{x_j}(x) u_{x_j x_i}(x)\big)^2 - \Sigma_{i,j=1}^n (u_{x_j x_i}(x) |\nabla u(x)|^2\end{aligned}$$

where

$$b_i(x) = G_{x_i}(x) + 2\Sigma_{j=1}^n u_{x_j}(x) u_{x_j x_i}(x).$$

Since

$$\begin{aligned}&\Sigma_{i=1}^n \big(\Sigma_{j=1}^n u_{x_j}(x) u_{x_j x_i}(x)\big)^2 \\ &\le \Sigma_{i=1}^n \big(\Sigma_{j=1}^n (u_{x_j x_i})^2\big)|\nabla u|^2 \le \big(\Sigma_{i,j=1} (u_{x_j x_i})^2\big)|\nabla u|^2,\end{aligned}$$

we obtain

$$|\nabla u(x)|^2 \Delta G(x) = \Sigma_{i=1}^n b_i(x) G_{x_i}(x) \le 0, \quad x \in \mathbb{R}^n.$$

If $|\nabla u(0)|^2 = 0$, we conclude immediately $G(x) \ge 0$ and (5.1) holds.

If $|\nabla u(0)|^2 \ne 0$, by the strong maximum principle we obtain $G(x) \equiv C$ since G attains the maximum at interior point $x = 0$. Assume that $\lim_{n\to\infty} u(\xi_n) = M := \max_{x\in\mathbb{R}^n} u(x)$ for some sequence $\{\xi_n\}$. We obtain that $C = \lim_{n\to\infty} G(\xi_n) = F(M) \ge 0$.

Therefore we have proven $G(x) \ge 0$ for $x \in \mathbb{R}^n$.

□

5.2. A Symmetry Result.

Proposition 5.2. *(Caffareli-Gafafelo-Segale) Suppose u is a solution of* (1.1) *and*

$$\frac{1}{2}|\nabla u|^2 = F(u(x)),\ x \in \mathbb{R}^n, \tag{5.2}$$

then $u(x) = g(ax+b)$ for some $a \in S^{n-1}, b \in \mathbb{R}^n$.

Proof. Let $v(x) = g^{-1}(u(x)), x \in \mathbb{R}^n$, i.e. $g(v(x)) = u(x)$.
Then

$$u_{x_i} = g'(v(x))v_{x_i},$$

hence

$$\begin{aligned} |\nabla u|^2 &= |g'(v(x))|^2|\nabla v|^2 \\ &= 2F(g(v(x)))|\nabla v|^2 \\ &= 2F(u(x))|\nabla v|^2. \end{aligned}$$

By (5.2), we get

$$|\nabla v|^2 = 1,\ x \in \mathbb{R}^n. \tag{5.3}$$

Furthermore,

$$\begin{aligned} u_{x_ix_i} &= g''(v(x))v_{x_i}^2 + g'(v(x))v_{x_ix_i}, \\ \triangle u &= g''(v(x))|\nabla v|^2 + g'(v(x))\triangle v \\ &= F'(g(v(x))) + g'(v(x))\triangle v(x) \\ &= F'(u(x)) + g'(v(x))\triangle v(x). \end{aligned}$$

By (1.1), we obtain

$$\triangle v = 0,\ x \in \mathbb{R}^n. \tag{5.4}$$

(5.3)-(5.4) implies

$$v(x) = a \cdot x + b,\ x \in \mathbb{R}^n \text{ for } a \in \mathbb{S}^n, b \in \mathbb{R}^n.$$

□

□

A generalization is the following.

Proposition 5.3. *Suppose u is a solution to* (0.1) *with*

$$F(u(x)) - \frac{1}{2}|\nabla u|^2 = C > 0,\ x \in \mathbb{R}^n. \tag{5.5}$$

Then $C = F(M) = F(m)$, where $M = \sup_{x\in\mathbb{R}^n} u(x)$, $m = \inf_{x\in\mathbb{R}^n} u(x)$. Furthermore,

$$u(x) = g_M(ax+b),\ x \in \mathbb{R}^n. \tag{5.6}$$

Proof. From last part of proof of Proposition 5.3, we know $C = F(M) = F(m)$.

Define

$$v_\alpha(x) = g_\alpha^{-1}(u(x)),\ x \in \mathbb{R}^n, \alpha > M.$$

Since $g'_\alpha(s) < 0,\ s \in [m, M]$, we know $v_\alpha(x) \in C^{2,\alpha}(\mathbb{R}^n)$ and $0 < v_\alpha(x) < T_\alpha,\ x \in \mathbb{R}^n$. We compute

$$\begin{aligned} u_{x_i} &= g'_\alpha(v_\alpha(x))(v_\alpha)_{x_i}, \\ |\nabla u|^2(x) &= |g'_\alpha(v_\alpha(x))|^2|\nabla v_\alpha|^2 = 2(F(g_\alpha(v_\alpha(x))) - F(\alpha))|\nabla v_\alpha|^2 \\ &= 2(F(u(x)) - F(\alpha))|\nabla v_\alpha|^2,\ x \in \mathbb{R}^n. \end{aligned}$$

Hence for $1 > \alpha > M$,

$$|\nabla v_\alpha|^2 = \frac{F(u(x)) - F(M)}{F(u(x)) - F(\alpha)} < 1,\ x \in \mathbb{R}^n. \tag{5.7}$$

Furthermore

$$\begin{aligned} u_{x_i x_i} &= g''_\alpha(v_\alpha(x))(v_\alpha)^2_{x_i} + g'_\alpha(v_\alpha(x))(v_\alpha)_{x_i x_i}, \\ \triangle u &= g''_\alpha(v_\alpha(x))|v_\alpha|^2 + \sqrt{2(F(g_\alpha(v_\alpha(x))) - F(\alpha))}\triangle v_\alpha \\ &= F'(g_\alpha(v_\alpha(x)))|v_\alpha|^2 + \sqrt{2(F(u(x)) - F(\alpha))}\triangle v_\alpha. \end{aligned}$$

Hence by (0.1), for any $\alpha > M$,

$$\begin{aligned} \triangle v_\alpha &= \frac{F'(u(x))(1 - |\nabla v_\alpha|^2)}{[2(F(u(x)) - F(\alpha)]^{1/2}} \\ &= \frac{F'(u(x))(F(\alpha) - F(M))}{[2(F(u(x)) - F(\alpha)]^{3/2}},\ x \in \mathbb{R}^n. \end{aligned} \tag{5.8}$$

Therefore for $x \in \Omega = \{x | m < u < M\}$, $v \in C^{2,\alpha}(\Omega) \cap C(\mathbb{R}^n)$ and letting $\alpha \longrightarrow M$, we have

$$\triangle v = 0, |\nabla v|^2 = 1, x \in \Omega, \tag{5.9}$$

$$0 = \triangle|\nabla v|^2 = \sum_{i,j}^{n} (v_{x_i x_j})^2,$$

which gives

$$\begin{aligned} v_{x_i x_j} &\equiv 0, x \in \Omega, \forall i, j, \\ v &\equiv a \cdot x + b, a \in \mathbb{S}^n, \end{aligned}$$

in any connected set. We also have

$$|\nabla v|^2 \leq 1, |v| \leq T_M, x \in \mathbb{R}^n. \tag{5.10}$$

We claim $\mathbb{R}^n \backslash \Omega$ non intersecting points, lines, planes and hyperplanes.

Suppose $u(x_0) = M_0$. If $p \in \mathbb{R}^n \backslash \Omega$, $u(p) = M$.

$$u(x)\ M + \sum_{i=1}^{n} a_i(x_i - p_i) + \sum_{i=1}^{n} b_i(x_i - p_i)^2 + o(|x-p|^2) \text{ for } x \text{ near } p.$$

Since $\nabla u(p) = 0$, $\triangle u(p) = F'(M) < 0$, $u(x) \leq M$, x near p, then $a_i = 0$, $i = 1, 2, \cdots, n$, $b_i \leq 0$, $i = 1, 2, \cdots, n$, and $b_{i_0} < 0$ for some i_0. Then if $u(x) = M$, $x \in B_{r(p)}(p)$, then $x_i = p_i$ for i with $b_i < 0$.

Since $\mathbb{R}^n \backslash \Omega$ is closed, then for any bounded region $(\mathbb{R}^n \backslash \Omega) \cap \bar{B}_R$, there exists finite many points, lines, planes, $\cdots$, hyperplanes such that their union contains $(\mathbb{R}^n \backslash \Omega) \cap \bar{B}_R$. (by compactness)

By (5.9) and (5.10), we conclude that any connected component Ω_1 has diameter in direction of $\vec{a}$ at most T_M. On the other hand, $\partial\Omega_1 \subset \{ax+b = T_M or 0\}$ must have distant in directioin $\vec{a}$ equal to T_M. Then solving (*), we have $\Omega_1 = \{x | 0 \leq ax+b \leq T_M\}$, $u(x) = g_M(ax+b)$ in Ω_1.

Similarly, we can show $u(x) = g_M(\tilde{a}x + b), x \in \Omega_2$. Further, $\tilde{a} = a, \tilde{b} = -2T_M + b$. So $u(x) = g_M(ax+b)$ in $x \in \Omega_2$. Keep going, we can conclude $u(x) = g_M(ax+b)$, $x \in \mathbb{R}^n$. □

5.3. **Monotonicity Formula.** We shall show a monotonicity formula regarding the energy of entire solutions in balls. We first show the Pohazaev identitiy.

Proposition 5.4. *Let $B_r = \{x \in \mathbb{R}^n : \frac{x}{r} \in A\}$, then*

$$\int_{B_r} (n-2)|\nabla u|^2 + 2nF(u)dx = r\int_{\partial A_r} (|\nabla u|^2 + 2F(u))ds - 2\int_{\partial B_r} (\nabla u \cdot v_r)(\nabla u \cdot x)ds. \quad (5.11)$$

Proof. Let

$$\psi(r) = r^{-n} \int_{B_r} (|\nabla u|^2 + 2F(u))dx,$$

then

$$\psi'(r) = -nr^{-n-1} \int_{B_r} (|\nabla u|^2 + 2F(u))dx + r^{-n} \int_{\partial B_r} (|\nabla u|^2 + 2F(u))dS.$$