Multiplicity and concentration of semi-classical solutions to nonlinear Dirac equations

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Dedicated to Professor Paul H. Rabinowitz

Abstract

We study semi-classical solutions to the nonlinear Dirac equation

$$-i\hbar\alpha \cdot \nabla w + a\beta w + M(x)w = f(x, |w|)w$$

for $x \in \mathbb{R}^3$, where $M(x)$ denotes the scaler field $V(x)$ or $V(x)\beta$, and f describes the self-interaction which is either subcritical: $W(x)|w|^{p-2}$, or critical: $W_1(x)|w|^{p-2} + W_2(x)|w|$, with $p \in (2, 3)$.

We prove multiplicity results with the number of solutions obtained depending on the ratio of $\min V$ and $\liminf_{|x|\to\infty} V(x)$, as well as $\max W$ and $\limsup_{|x|\to\infty} W(x)$ for the subcritical case and $\max W_j$ and $\limsup_{|x|\to\infty} W_j(x)$, $j = 1, 2$, for the critical case.

We show also certain concentration phenomenon of the families of semi-classical ground states at saddle points of $M(x) = V(x)\beta$.

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1 Introduction and main results

In quantum theory in order to describe the translation from quantum to classical mechanics, existence of semi-classical solutions of stationary quantum systems possesses an important physical interest. There have been large amounts of works on existence, multiplicity and concentration phenomenon of semi-classical solutions of nonlinear Schrödinger equations arising from non-relativistic quantum mechanics. In comparison, only a few similar results are known for nonlinear Dirac equations arising from relativistic mechanics. In particular, as far as the authors know, there is no result on *multiplicity of semi-classical solutions* to the Dirac equation. There is also no result on *concentration at saddle points*, typically, if the potentials are of the form $V(x)\beta$ with $V:\mathbb{R}^3\to\mathbb{R}$ and $\beta=\mathrm{diag}(I_2,-I_2)$, a 4×4 diagonal matrix (see below). Mathematically, the nonlinear Dirac equation is more difficult because, unlike the spectrum of the Laplacian which is bounded below, the spectrum of the Dirac operator is neither bounded below nor above. Additionally, the concentration phenomena is quite complicated depending on the potentials, and looking for conditions that ensure multiplicity is novel.

In this paper, we are mainly interested in utilizing variational methods to obtain multiplicity results for the Dirac equation, by introducing some new conditions depending on the behaviors of the potentials near the infinity, which can be directly verified. There are two new ingredients. One is to give a representation of ground state of the associated linear autonomous problem (the so-called limit equation) which yields the comparison conditions and etc. The other is to construct subspaces on which the relative energy functional is bounded above, say by b, and satisfies the Palais-Smale condition below the level b, and thus we are able to apply an abstract critical point theorem. We also consider the concentration phenomena for the Dirac equation with scale potential which is incomparable at different space points. Moreover, we will deal with the case of critical nonlinearity.

We now recall the problems and state our results. Consider the nonlinear Dirac equation, which occurs in the attempt to model extended relativistic particles with external fields (see e.g. [22, 33]), given by

$$-i\hbar\partial_t\psi = ic\hbar\sum_{k=1}^{3}\alpha_k\partial_k\psi - mc^2\beta\psi - P(x)\psi + Q_\psi(x,\psi)$$

for the (wave) function $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ which represents the state of a relativistic electron. Here $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_k = \partial/\partial x_k$, c is the speed of light, $m > 0$ is the mass of the electron, $\hbar$ denotes Planck's constant, and $\alpha_1, \alpha_2, \alpha_3, \beta$ are 4×4 complex matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \qquad k = 1, 2, 3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix potential $P(x)$ stands for external fields, and the nonlinearity $Q : \mathbb{R}^3 \times \mathbb{C}^4 \to \mathbb{R}$ represents a nonlinear self-coupling. Assuming that $Q(x, e^{i\theta}\psi) = Q(x, \psi)$ for all $\theta \in [0, 2\pi]$, the standing wave solutions are of the form $\psi(t, x) = e^{\frac{i\theta t}{\hbar}} w(x)$, and searching for such solutions is reduced to finding solutions of the equation

$$-i\hbar\alpha \cdot \nabla w + a\beta w + M(x)w = F_w(x, w) \tag{1.1}$$

for $w : \mathbb{R}^3 \to \mathbb{C}^4$, with $a > 0$, $M(x)$ a 4×4 matrix-valued potential function, $F(x, u)$ a nonlinearity, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$.

In the literature, there are many results concerning existence of solutions of (1.1) under various hypotheses on the potential and the nonlinearity (see [22] for a survey). In [5, 10, 28] the authors studied the problem with $M(x) = V(x)I_4$ and $V(x) \equiv \omega \in (-a, a)$ and the nonlinearity of the so-called Soler model $F(w) = \frac{1}{2}H(\tilde{w}w)$ with $H \in C^2(\mathbb{R}, \mathbb{R})$, $H(0) = 0$, $\tilde{w}w := \langle \beta w, w \rangle$; and in [23] the authors considered the nonlinearity $F(w) = \frac{1}{2}|\tilde{w}w|^2 + b|\tilde{w}\alpha w|^2$ with $\tilde{w}\alpha w := \langle \beta w, \alpha w \rangle$, $\alpha := \alpha_1\alpha_2\alpha_3$, by using shooting methods. Here and in the sequel $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{C}^4$. Such nonlinearities were later studied by using for the first time a variational method in [21], where more general super-linear subcritical $F(w)$ independent of x were considered. Existence and multiplicity results for (1.1) with $M(x)$ and $F(x, w)$ depending periodically on x were obtained in [6] by using a critical point theory. For non-periodic potentials (the Coulomb-type potential is a typical example), existence and multiplicity of solutions were studied in [16] for asymptotically quadratic nonlinearities and in [18] for super-quadratic subcritical nonlinearities, where $M(x)$ and $F(x, w)$ were assumed to have limits as $|x| \to \infty$.

For small $\hbar$, the standing waves are referred to as semi-classical states. To describe the translation from quantum to classical mechanics, existence of solutions $w_\hbar, \hbar$ small, is of great physical importance. Only very recently, existence and concentration phenomena of semi-classical ground states of the Dirac equation (1.1) with $M(x) = V(x)I_4$ and nonlinearity of the form $F_w(x,w) = W(x)h(w)$ have been studied, in [14] for $V(x) \equiv 0$ and $h(w)$ super-linear and subcritical, in [15] for $V : \mathbb{R}^3 \to \mathbb{R}$ and $h(w)$ super-linear and subcritical and in [17] for $V : \mathbb{R}^3 \to \mathbb{R}$ and $h(w) = \big(g(|w|) + |w|\big)w$, $g(|w|)w$ subcritical.

Recall that the matrix-valued potential $M(x)$ is called a *Scalar potential* if

$$M(x) = V(x)\beta \quad \text{or} \quad M(x) = V(x)I_4 \tag{1.2}$$

where $V(x)$ is a real-valued function and I_4 the identity in $\mathbb{C}^4$, see [35]. In this paper we are interested in such potentials. More precisely, for $p \in (2,3)$, writing $\varepsilon \equiv \hbar$ and assuming without loss of generality that $a = 1$, we consider the equation with subcritical nonlinearity

$$-i\varepsilon\alpha \cdot \nabla w + \beta w + M(x)w = W(x)|w|^{p-2}w, \tag{1.3}$$

and the equation with critical nonlinearity

$$-i\varepsilon\alpha \cdot \nabla w + \beta w + M(x)w = W_1(x)|w|^{p-2}w + W_2(x)|w|w. \tag{1.4}$$

We consider the potentials (1.2) with the following assumptions, for (1.3)

(P_0) V, $W \in C^1 \cap L^\infty(\mathbb{R}^3,\mathbb{R})$, $|V|_\infty < 1$, $V(x)$ attains a global minimum, and $W(x)$ attains a global maximum with $\inf_{x\in\mathbb{R}^3} W(x) > 0$;

and for (1.4)

$Q_0)$ V, $W_j \in C^1 \cap L^\infty(\mathbb{R}^3,\mathbb{R})$, $|V|_\infty < 1$, $V(x)$ attains a global minimum, and $W_j(x)$ attains a global maximum with $\inf_{x\in\mathbb{R}^3} W_j(x) > 0$, $j = 1,2$.

Notations: In order to describe our results some notations are in order:

$\tau := \min V$; $\quad \mathscr{V} := \{x : V(x) = \tau\}$; $\quad \tau_\infty := \liminf_{|x|\to\infty} V(x)$;
$\kappa := \max W$; $\quad \mathscr{W} := \{x : W(x) = \kappa\}$; $\quad \kappa_\infty := \limsup_{|x|\to\infty} W(x)$;
$x_v \in \mathscr{V}$ such that $\kappa_v := W(x_v) \equiv \max_{x\in\mathscr{V}} W(x)$;
$x_w \in \mathscr{W}$ such that $\tau_w := V(x_w) \equiv \min_{x\in\mathscr{W}} V(x)$;

and for $j = 1,2$,

$\kappa_j := \max W_j$; $\quad \mathscr{W}_j := \{x : W_j(x) = \kappa_j\}$; $\quad \kappa_{j\infty} := \limsup_{|x|\to\infty} W_j(x)$;

$x_{jv} \in \mathscr{V}$ such that $\kappa_{jv} := W(x_{jv}) \equiv \max_{x\in\mathscr{V}} W_j(x)$.

Case (A): The subcritical case. Firstly, we consider equation (1.3) and we make the assumption

Theorem 1.1. *Let $M(x)$ be of the form* (1.2)*, and assume* (P_0) *holds,* $\tau < \tau_\infty$ *and* $\kappa_\infty \le \kappa_v$*. Let* m *be the largest integer such that*

$$m < \Big(\frac{1+\tau_\infty}{1+\tau}\Big)^{\frac{2(3-p)}{p-2}} \Big(\frac{\kappa_v}{\kappa_\infty}\Big)^{\frac{2}{p-2}}. \tag{1.5}$$

Then there is $\mathcal{E} > 0$ *such that, for* $\varepsilon \le \mathcal{E}$, (1.3) *possesses at least* m *pairs of solutions in* $\bigcap_{s\ge2} W^{1,s}(\mathbb{R}^3)$.

Theorem 1.2. *Let $M(x)$ be of the form* (1.2)*, and assume* (P_0) *holds,* $\tau_w \le \tau_\infty$ *and* $\kappa_\infty < \kappa$*. Let* m *be the largest integer such that*

$$m < \Big(\frac{1+\tau_\infty}{1+\tau_w}\Big)^{\frac{2(3-p)}{p-2}} \Big(\frac{\kappa}{\kappa_\infty}\Big)^{\frac{2}{p-2}}. \tag{1.6}$$

Then there is $\mathcal{E} > 0$ *such that, for* $\varepsilon \le \mathcal{E}$, (1.3) *possesses at least* m *pairs of solutions in* $\bigcap_{s\ge2} W^{1,s}(\mathbb{R}^3)$.

For showing the concentration of ground states we assume additionally

(P_1) One of the following assumptions holds:

(1) $\tau < \tau_\infty$, and there is $R_v > 0$ such that $W(x) \le \kappa_v$ for all $|x| \ge R_v$;

(2) $\kappa > \kappa_\infty$, and there is $R_w > 0$ such that $V(x) \ge \tau_w$ for all $|x| \ge R_w$.

Set

$$\mathscr{A} := \begin{cases} \{x \in \mathscr{V} : W(x) = \kappa_v\} \cup \{x \notin \mathscr{V} : W(x) > \kappa_v\}, & \text{for } (P_1)\text{-}(1); \\ \{x \in \mathscr{W} : V(x) = \tau_w\} \cup \{x \notin \mathscr{W} : V(x) < \tau_w\}, & \text{for } (P_1)\text{-}(2)\,. \end{cases}$$

Obviously, $\mathscr{A}$ is bounded. Moreover, if $\mathscr{V} \cap \mathscr{W} \neq \emptyset$, then $\kappa_v = \kappa$, $\tau_w = \tau$, $\{x \notin \mathscr{V} : W(x) > \kappa_v\} = \emptyset = \{x \notin \mathscr{W} : V(x) < \tau_w\}$, consequently $\mathscr{A} = \mathscr{V} \cap \mathscr{W}$.

Theorem 1.3. *Let $M(x)$ be of the form* (1.2)*, and assume* $(P_0) - (P_1)$ *hold. Then, for sufficiently small* $\varepsilon > 0$, (1.3) *possesses a least energy solution* $w_\varepsilon \in \bigcap_{s\ge2} W^{1,s}(\mathbb{R}^3)$*. Moreover,* w_ε *satisfies:*

(a) *There exists a maximum point* x_ε *of* $|w_\varepsilon|$ *with* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathscr{A}) = 0$ *such that, for any sequence* $x_\varepsilon \to x_0\,(\varepsilon \to 0)$, *the sequence* $u_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges in* $H^1(\mathbb{R}^3)$ *to a least energy solution of*

$$-i\alpha\cdot\nabla u + \beta u + M(x_0)u = W(x_0)|u|^{p-2}u\,. \tag{1.7}$$

If particularly $\mathscr{V}\cap\mathscr{W} \neq \emptyset$ *then* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathscr{V}\cap\mathscr{W}) = 0$ *and* u_ε *converges in* $H^1(\mathbb{R}^3)$ *to a least energy solution of*

$$-i\alpha\cdot\nabla u + \beta u + M_{\min}\,u = \kappa|u|^{p-2}u \tag{1.8}$$

where $M_{\min} = \tau\beta$ *if* $M(x) = V(x)\beta$ *and* $M_{\min} = \tau I_4$ *if* $M(x) = V(x)I_4$.

(b) *For some* $c, C > 0$, $|w_\varepsilon(x)| \le C\exp\big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\big)$ *for all* x.

Case (B): The critical case. Next we consider equation (1.4). In the following let S denote the best constant of the Sobolev inequality

$$S|u|_6^2 \le |\nabla u|_2^2 \qquad \text{for all } u \in H^1(\mathbb{R}^3).$$

Let γ_p denote the least energy of the equation

$$-i\alpha\cdot\nabla u + \beta u = |u|^{p-2}u. \tag{1.9}$$

In addition to (Q_0), we will use the following assumption:

(Q_1) There holds

$$(1+\tau_\infty)^{\frac{2(3-p)}{p-2}}\Big(\frac{\kappa_{2\infty}}{\kappa_{1\infty}^{1/(p-2)}}\Big)^2 \le \frac{S^{3/2}}{6\gamma_p}.$$

For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{R}^2$, we use $\vec{x} \ge \vec{y}$ to denote $x_1 \ge y_1$ and $x_2 \ge y_2$, and $\vec{x} > \vec{y}$ if $\vec{x} \ge \vec{y}$ with $\min\{x_1 - x_2, y_1 - y_2\} > 0$. In what follows, denote, for $\mu \in (-1, \tau_\infty]$ and $\vec{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $\vec{\nu} \ge \vec{0}$,

$$m(\mu, \vec{\nu}) = \min\left\{\Big(\frac{1+\tau_\infty}{1+\mu}\Big)^{\frac{2(3-p)}{p-2}}\Big(\frac{\nu_1}{\kappa_{1\infty}}\Big)^{\frac{2}{p-2}};\quad \Big(\frac{\nu_2}{\kappa_{2\infty}}\Big)^2\right\} \tag{1.10}$$

and let $\vec{\kappa} = (\kappa_1, \kappa_2)$, $\vec{\kappa}_\infty = (\kappa_{1\infty}, \kappa_{2\infty})$, $\vec{\kappa}_v = (\kappa_{1v}, \kappa_{2v})$.

Theorem 1.4. *Let* $M(x)$ *be of the form* (1.2), *and assume that* $(Q_0) - (Q_1)$ *hold,* $\tau < \tau_\infty$, *and* $\vec{\kappa}_v \ge \vec{\kappa}_\infty$. *Let* m *be the smallest integer satisfying* $m \ge m(\tau, \vec{\kappa}_v)$. *Then there exists* $\mathcal{E} > 0$ *such that, for* $\varepsilon \le \mathcal{E}$, (1.4) *possesses at least* m *pairs of solutions in* $\bigcap_{s\ge 2} W^{1,s}(\mathbb{R}^3)$.

For another multiplicity result we assume further the following

(Q_2) $\widetilde{\mathscr{W}} := \mathscr{W}_1 \cap \mathscr{W}_2 \neq \emptyset$.

Let $x_w \in \widetilde{\mathscr{W}}$ be such that $\tau_w \equiv V(x_w) = \min_{x\in\widetilde{\mathscr{W}}} V(x)$.

Theorem 1.5. *Let $M(x)$ be of the form (1.2), and assume that $(Q_0) - (Q_2)$ hold and $\tau_w \leq \tau_\infty$. Let m be the smallest integer satisfying $m \geq m(\tau_w, \vec{\kappa})$. Then there eixsts $\mathcal{E} > 0$ such that, for $\varepsilon \leq \mathcal{E}$, (1.4) possesses at least m pairs of solutions in $\bigcap_{s\geq 2} W^{1,s}(\mathbb{R}^3)$.*

For stating a concentration result we assuming additionally the following:

(Q_3) One of the following assumptions holds:

(1) $\tau < \tau_\infty$, and $\exists\, R_v > 0$ such that $W_j(x) \leq \kappa_{jv}$ if $|x| \geq R_v$ for $j = 1, 2$.

(2) $\vec{\kappa} > \vec{\kappa}_\infty$, and $\exists\, R_w > 0$ such that $V(x) \geq \tau_w$ if $|x| \geq R_w$.

Set, if (Q_3)-(1) holds

$$\mathscr{A}_v := \{x \in \mathscr{V} : W_j(x) = \kappa_{jv}, j = 1, 2\} \cup \{x \notin \mathscr{V} : W_j(x) > \kappa_{jv}, j = 1, 2\};$$

and if (Q_3)-(2) holds

$$\mathscr{A}_w := \{x \in \widetilde{\mathscr{W}} : V(x) = \tau_w\} \cup \{x \notin \widetilde{\mathscr{W}} : V(x) < \tau_w\}.$$

In the following theorem, $\mathscr{A}$ stands for $\mathscr{A}_v$ in the case (Q_3)-(1), and $\mathscr{A}_w$ in the case (Q_3)-(2). Obviously (as before), $\mathscr{A}$ is bounded, and $\mathscr{A} = \mathscr{V} \cap \widetilde{\mathscr{W}}$ if the intersection is not empty.

Theorem 1.6. *Let $M(x)$ be of the form (1.2), and assume $(Q_0) - (Q_3)$ are satisfied. Then, for sufficiently small $\varepsilon > 0$, (1.4) possesses a least energy solution $w_\varepsilon \in \bigcap_{s\geq 2} W^{1,s}(\mathbb{R}^3)$. Moreover, w_ε satisfies:*

(a) There exists a maximum point x_ε of $|w_\varepsilon|$ with $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathscr{A}) = 0$ such that, for any sequence $x_\varepsilon \to x_0\, (\varepsilon \to 0)$, the sequence $u_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon)$ converges in $H^1(\mathbb{R}^3)$ to a least energy solution of

$$-i\alpha \cdot \nabla u + \beta u + M(x_0)u = W_1(x_0)|u|^{p-2}u + W_2(x_0)|u|u\,. \tag{1.11}$$

If particularly $\mathscr{V} \cap \widetilde{\mathscr{W}} \neq \emptyset$ then $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathscr{V} \cap \widetilde{\mathscr{W}}) = 0$ and u_ε converges in $H^1(\mathbb{R}^3)$ to a least energy solution of

$$-i\alpha \cdot \nabla u + \beta u + M_{\min}\, u = \kappa_1|u|^{p-2}u + \kappa_2|u|u\,.$$

(*b*) *For some* $c, C > 0$, $|w_\varepsilon(x)| \le C \exp\big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\big)$ *for all* x.

Remark 1.7. 1) Observe that in (1.5), m can be sufficiently large if τ closes sufficiently to -1, or τ_∞ closes sufficiently to ∞, etc. One may make similar comments on (1.6).

2) In [17], one ground state of (1.1) with either $M(x) \equiv 0$ and $f(x, w) = W(x)(|w|^{p-2} + |w|)w$, or $M(x) = V(x)$ and $W(x) \equiv 1$, was obtained under assumptions different from those in the present paper. Observe that if $W(x) = W_1(x) = W_2(x)$ (hence, $\kappa_\infty = \kappa_{1\infty} = \kappa_{2\infty}$), then (Q_1) reads simply as

$$\Big(\frac{1+\tau_\infty}{\kappa_\infty}\Big)^{\frac{2(3-p)}{p-2}} \le \frac{S^{3/2}}{6\gamma_p},$$

and (1.10) reads as (for $\mu \in (-1, \tau_\infty]$ and $\nu = \nu_1 = \nu_2$)

$$m(\mu, \vec{\nu}) = \Big(\frac{\nu}{\kappa_\infty}\Big)^2.$$

2 Variational setting

Let $|\cdot|_q$ denote the usual L^q-norm, $(\cdot,\cdot)_{L^2}$ the L^2-inner product. Set $A = -i\alpha \cdot \nabla + \beta$, a self-adjoint operator acting on L^2. A Fourier analysis shows that $\sigma(A) = \sigma_c(A) = \mathbb{R} \setminus (-1, 1)$ where $\sigma(\cdot)$ and $\sigma_c(\cdot)$ denote the spectrum and continuous spectrum respectively.

Consider the Hilbert space $E = H^{1/2}$ equipped with the equivalent inner product

$$(u, v) := \Re(|A|^{1/2}u, |A|^{1/2}v)_{L^2}$$

and induced norm $\|u\|^2 := (u, u) = ||A|^{1/2}u|_2^2$. Then there are decompositions

$$L^2 = L^- \oplus L^+, \quad u = u^- + u^+$$

and

$$E = E^- \oplus E^+, \quad E^\pm = E \cap L^\pm,$$

orthogonal with respect to the products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$, such that $A|_{L^-} \le -1$ and $A|_{L^+} \ge 1$. Recall that E embeds into L^q for $q \in [2, 3]$ continuously and L^q_{loc} compactly for $q \in [2, 3)$. In fact we have (see, e.g., [17]):

1) $|u|_2^2 \le \|u\|^2$ for all $u \in E$.

2) For any $q \in [2, 3]$ and for all $u \in E$, $S^{\frac{3(q-2)}{2q}}|u|_q^2 \le \|u\|^2$ where S is the best Sobolev embedding constant.

In the following, let

$M_\varepsilon(x) = M(\varepsilon x)$,

$W_\varepsilon(x) = W(\varepsilon x)$ and $W_{j\varepsilon}(x) = W_j(\varepsilon x)$ for $j = 1, 2$,

$$f(x, |u|) = \begin{cases} W(x)|u|^{p-2}, & \text{in Case (A)}, \\ W_1(x)|u|^{p-2} + W_2(x)|u|, & \text{in Case (B)}, \end{cases}$$

$F(x, u) = \int_0^{|u|} f(x, t)t\, dt$ and $F_\varepsilon(x, u) = F(\varepsilon x, u)$,

$\Psi_\varepsilon(u) = \int_{\mathbb{R}^3} F_\varepsilon(x, u)$.

Define the functional

$$\begin{aligned} \Phi_\varepsilon(u) &= \frac{1}{2} \int_{\mathbb{R}^3} \langle -i\alpha \cdot \nabla u + \beta u + M_\varepsilon(x)u,\, u\rangle - \Psi_\varepsilon(u) \\ &= \frac{1}{2} \left(\|u^+\|^2 - \|u^-\|^2 \right) + \frac{1}{2} \int_{\mathbb{R}^3} \langle M_\varepsilon(x)u, u\rangle - \Psi_\varepsilon(u) \end{aligned} \tag{2.1}$$

where (and in the sequel) $\|u^+\|^2 - \|u^-\|^2$ refers to the splitting $E = E^- \oplus E^+$. Denoting $E_e = E^- \oplus \mathbb{R}^+ e$ and $\hat{E}_e = E^- \oplus \mathbb{R}e$ for $e \in E^+ \setminus \{0\}$, and $\hat{E}_H = E^- \oplus H$ for any finite dimensional linear subspace $H \subset E^+$, it is easy to check the following

Lemma 2.1. *One has:*

1) *Ψ_ε is weakly sequentially lower semicontinuous and Φ'_ε is weakly sequentially continuous.*

2) *Φ_ε possesses the linking structure:*

   1° *There exist $r > 0$ and $\rho > 0$ independent of ε such that $\Phi_\varepsilon|_{B_r^+} \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \rho$, where $B_r^+ = \{u \in E^+ : \|u\| \leq r\}$ and $S_r^+ = \{u \in E^+ : \|u\| = r\}$;*

   2° *For any finite dimensional linear subspace $H \subset E^+$, there exist $R = R_H > 0$ and $C = C_H > 0$ such that $\Phi_\varepsilon(u) < 0$ for all $u \in \hat{E}_H \setminus B_R$ and* $\max \Phi_\varepsilon(\hat{E}_H) \leq C$.

Let c_ε denote the minimax level of Φ_ε deduced by the linking structure

$$c_\varepsilon := \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \Phi_\varepsilon(u) = \inf_{e \in E^+ \setminus \{0\}} \max_{u \in \hat{E}_e} \Phi_\varepsilon(u). \tag{2.2}$$

Let $\mathscr{K}_\varepsilon := \{u \in E : \Phi'_\varepsilon(u) = 0\}$ be the critical set of Φ_ε. Note that if $u \in \mathscr{K}_\varepsilon$ then

$$\Phi_\varepsilon(u) = \Phi_\varepsilon(u) - \frac{1}{2}\Phi'_\varepsilon(u)u = \int_{\mathbb{R}^3} \frac{1}{2} f(\varepsilon x, |u|)|u|^2 - F(\varepsilon x, |u|) \geq 0.$$

Using the same iterative argument of [21, Proposition 3.2] one obtains easily the following

Lemma 2.2. *If $u \in \mathscr{K}_\varepsilon$ with $\Phi_\varepsilon(u) \le C_1$ and $|u|_2 \le C_2$, then, for any $q \in [2,\infty)$, $u \in W^{1,q}(\mathbb{R}^3)$ with $\|u\|_{W^{1,q}} \le \Lambda_q$ where Λ_q depends only on C_1, C_2 and q.*

To describe furthermore c_ε we recall the Mountain-Pass type reduction, see [1] (also [18, 31, 34]). Consider, for a fixed $u \in E^+$, the map $\phi_u : E^- \to \mathbb{R}$ defined by $\phi_u(v) = \Phi_\varepsilon(u+v)$. Observe that, for any $v, w \in E^-$,

$$\phi_u''(v)[w,w] = -\|w\|^2 + \int_{\mathbb{R}^3} \langle M_\varepsilon(x)w, w\rangle - \Psi_\varepsilon''(u+v)[w,w].$$

Since $|V|_\infty < 1$ and Ψ_ε is strictly convex, there is a unique $h_\varepsilon(u) \in E^-$ such that

$$\phi_u(h_\varepsilon(u)) = \max_{v \in E^-} \phi_u(v). \tag{2.3}$$

It is clear that $v \ne h_\varepsilon(u)$ if and only if $\Phi_\varepsilon(u+v) < \Phi_\varepsilon(u + h_\varepsilon(u))$. Define $I_\varepsilon : E^+ \to \mathbb{R}$ by $I_\varepsilon(u) = \Phi_\varepsilon(u + h_\varepsilon(u))$, that is,

$$\begin{aligned} I_\varepsilon(u) = {} & \frac{1}{2}\big(\|u\|^2 - \|h_\varepsilon(u)\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} \langle M_\varepsilon(u + h_\varepsilon(u)),\, u + h_\varepsilon(u)\rangle \\ & - \Psi_\varepsilon(u + h_\varepsilon(u)). \end{aligned}$$

Set

$$\mathscr{N}_\varepsilon := \{u \in E^+ \setminus \{0\} :\ I_\varepsilon'(u)u = 0\}.$$

In the following we will call $\{h_\varepsilon(\cdot), I_\varepsilon(\cdot), \mathscr{N}_\varepsilon\}$ the "mountain-pass" reduction for the equation:

$$-i\alpha \cdot \nabla u + \beta u + M_\varepsilon(x)u = f_\varepsilon(x, |u|)u.$$

Plainly,

$$c_\varepsilon = \inf_{u \in \mathscr{N}_\varepsilon} I_\varepsilon(u).$$

(see [1, 14, 18]). This, jointly with (2.2), implies

Lemma 2.3. *There is a sequence $\{e_n\} \subset E^+ \setminus \{0\}$ such that, denoting $u_n = e_n + h_\varepsilon(e_n)$,*

$$\Phi_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad \Phi_\varepsilon'(u_n) \to 0$$

as $n \to \infty$.

Furthermore, one has the following

Lemma 2.4. *Let $u_n = u_n^+ + u_n^-$ be a $(PS)_c$ sequence for Φ_ε and set $v_n = u_n^+ + h_\varepsilon(u_n^+)$, $z_n = u_n^- - h_\varepsilon(u_n^+)$. Then $\|z_n\| \to 0$ and v_n is also a $(PS)_c$ sequence for Φ_ε, that is, u_n^+ is a $(PS)_c$ sequence for I_ε. Consequently, either $c = 0$ or $c \geq c_\varepsilon$.*

Proof. It suffices to show that $\|z_n\| \to 0$. Observe that

$$0 = \Phi_\varepsilon'(v_n)z_n = -(h_\varepsilon(u_n^+), z_n) + \int_{\mathbb{R}^3} \langle M_\varepsilon v_n, z_n\rangle - \Psi_\varepsilon'(v_n)z_n,$$

and since u_n is a (PS) sequence,

$$o(1) = \Phi_\varepsilon'(u_n)z_n = -(u_n^-, z_n) + \int_{\mathbb{R}^3} \langle M_\varepsilon u_n, z_n\rangle - \Psi_\varepsilon'(u_n)z_n.$$

Thus,

$$o(1) = \|z_n\|^2 - \int_{\mathbb{R}^3} \langle M_\varepsilon z_n, z_n\rangle + \big(\Psi_\varepsilon'(v_n + z_n) - \Psi_\varepsilon'(v_n)\big)z_n.$$

Since $F_\varepsilon(x, u)$ is strictly convex, $\big(\Psi_\varepsilon'(v_n+z_n)-\Psi_\varepsilon'(v_n)\big)z_n \geq 0$, which, together with the fact that $|V|_\infty < 1$, implies

$$o(1) \geq \big(1 - |V|_\infty\big)\|z_n\|^2.$$

Thus, $\|z_n\| \to 0$. Finally, it follows from (2.2) that if $c \neq 0$ then $c \geq c_\varepsilon$. □

Below, for notational convenience, we denote by Φ_0 the energy functional of the equation

$$-i\alpha \cdot \nabla u + \beta u + M(0)u = f(0, |u|)u\,. \tag{2.4}$$

We define correspondingly c_0, the critical set $\mathscr{K}_0$, and the "mountain-pass" reduction $\{h_0, I_0, \mathscr{N}_0\}$ for (2.4).

Lemma 2.5. *We have*

(1) *For any $u \in E^+ \setminus \{0\}$, there is a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathscr{N}_\varepsilon$. Moreover, $\lim_{\varepsilon\to0} t_\varepsilon(u) = t_0(u)$, $\|h_\varepsilon(t_\varepsilon u) - h_0(t_0 u)\| \to 0$, and $\limsup_{\varepsilon\to0} c_\varepsilon \leq I_0(t_0 u)$. In addition, if $u \in \mathscr{N}_0$ then $t_0 = 1$.*

(2) *$\lim_{\varepsilon\to0} c_\varepsilon = c_0$.*

Proof. It follows from [1, 14] that, for any $u \in E^+ \setminus \{0\}$, there is a unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathscr{N}_\varepsilon$, and moreover $\{t_\varepsilon(u)\}_{0\le\varepsilon\le1}$ is bounded. It is easy to check that, if $t_\varepsilon \to t_0$ then $h_\varepsilon(t_\varepsilon u) \to h_0(t_0 u)$. Consequently, $\limsup_{\varepsilon\to0} c_\varepsilon \le I_0(t_0 u)$. It is clear that, since $t_\varepsilon u \in \mathscr{N}_\varepsilon$, one has $t_0 u \in \mathscr{N}_0$. Thus, if $u \in \mathscr{N}_0$, there must be $t_0 = 1$. As a consequence, we see that

$$\limsup_{\varepsilon\to0} c_\varepsilon \le c_0.$$

We now verify that

$$\liminf_{\varepsilon\to0} c_\varepsilon \ge c_0. \tag{2.5}$$

Assume by contradiction that $c_\varepsilon < c_0$ and let $\theta > 0$ be small so that $c_\varepsilon < c_0 - \theta$ along a sequence $\varepsilon \to 0$. For any $e_\varepsilon \in \mathscr{N}_\varepsilon$ with $\Phi_\varepsilon(u_\varepsilon) \le c_0 - \theta$, $u_\varepsilon = e_\varepsilon + h_\varepsilon(e_\varepsilon)$, it is clear that $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in E. A concentration argument shows that there exist $\{y_\varepsilon\} \subset \mathbb{R}^3$ and $R > 0, \sigma > 0$ such that

$$\liminf_{\varepsilon\to0} \int_{B_R(y_\varepsilon)} |u_\varepsilon|^2 \ge \sigma. \tag{2.6}$$

In particular, we choose, by Lemma 2.3, $u_j = e_{\varepsilon_j} + h_{\varepsilon_j}(e_{\varepsilon_j})$, $j \to \infty$, satisfying

$$c_{\varepsilon_j} \le \Phi_{\varepsilon_j}(u_j) \le c_{\varepsilon_j} + \frac{1}{j} \quad \text{and} \quad \|\Phi'_{\varepsilon_j}(u_j)\| \le \frac{1}{j}. \tag{2.7}$$

Note that

$$o(1) + c_{\varepsilon_j} = \int_{\mathbb{R}^3} \mathcal{F}_{\varepsilon_j}(x, u_j)$$

where $\mathcal{F}_{\varepsilon_j}(x,u_j) = \frac{1}{2} f_{\varepsilon_j}(x,u_j)|u_j|^2 - F_{\varepsilon_j}(x,u_j)$. Set $y_j = y_{\varepsilon_j}$. Plainly, if $\{y_j\}$ is bounded then $u_j \rightharpoonup v \ne 0$, a solution of (2.4) with energy

$$c_0 \ge \Phi_0(v) = \int_{\mathbb{R}^3} \mathcal{F}_0(0,v) \le \liminf_{j\to\infty} \int_{\mathbb{R}^3} \mathcal{F}_{\varepsilon_j}(x,u_j) = \liminf_{j\to\infty} c_j,$$

contradicting to $\limsup_{j\to\infty} c_j \le c_0 - \theta$.

Assume that $\{y_j\}$ is unbounded. Set $v_j(x) = u_j(x + y_j)$, $\hat{M}_j(x) = M(\varepsilon_j(x + y_j))$ and $\hat{F}_j(x, v_j) = F((\varepsilon_j(x+y_j)), v_j)$. Then

$$\hat{\Phi}'_j(v_j) = \Phi'_{\varepsilon_j}(u_j) \to 0.$$

One may assume that $v_j \rightharpoonup v$ in E, $v_j \to v$ in L^q_{loc} for $q \in [2,3)$, $v_j(x) \to v(x)$ a.e. for $x \in \mathbb{R}^3$, and $c_{\varepsilon_j} \to \tilde{c}_0$ as $j \to \infty$. By (2.6), $v \ne 0$. Clearly v solves (2.4) with the energy denoted by $\hat{\Phi}_0(v)$. Then

$$o(1) + c_{\varepsilon_j} \ge \liminf_{j\to\infty} \int \hat{\mathcal{F}}_j(x, v_j) \ge \int \hat{\mathcal{F}}(0,v) = \hat{\Phi}_0(v) \ge c_0,$$

again a contradiction. □

In order to establish our multiplicity results, we recall an abstract critical point theorem, see [7, 13]. Let X, Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a countable dense subset. Let $\mathcal{P}$ be the family of semi-norms on E:

$$p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x+y) = |s(x)| + \|y\|, \qquad s \in \mathcal{S}.$$

Denote by $\mathcal{T}_\mathcal{P}$ the topology on E induced by $\mathcal{P}$. Let $\mathcal{T}_{w^*}$ be the weak*-topology of E^*.

For a functional $\Phi : E \to \mathbb{R}$ and numbers $a, b \in \mathbb{R}$ we write $\Phi^a := \{u \in E : \Phi(u) \leq a\}$, $\Phi_a := \{u \in E : \Phi(u) \geq a\}$, and $\Phi_a^b := \Phi_a \cap \Phi^b$. Assume

(Φ_1) $\Phi \in C^1(E, \mathbb{R})$; $\Phi : (E, \mathcal{T}_\mathcal{P}) \to \mathbb{R}$ is upper semicontinuous, and $\Phi' : (\Phi_a, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*})$ is continuous for every $a \in \mathbb{R}$.

(Φ_2) there exists $r > 0$ with $\rho := \inf \Phi(S_r Y) > \Phi(0) = 0$ where $S_r Y := \{y \in Y : \|y\| = r\}$;

(Φ_3) there exist a finite-dimensional subspace $Y_0 \subset Y$ and $R > r$ such that we have for $E_0 := X \times Y_0$ and $B_0 := \{u \in E_0 : \|u\| \leq R\}$: $b := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus B_0) < \inf \Phi(B_r Y)$.

We consider the set $\mathcal{M}(\Phi^c)$ of maps $g : \Phi^c \to E$ with the properties

(i) g is $\mathcal{P}$-continuous and odd;

(ii) $g(\Phi^a) \subset \Phi^a$ for all $a \in [\rho, b]$;

(iii) each $u \in \Phi^c$ has a $\mathcal{P}$-open neigbourhood $O \subset E$ such that the set $(id - g)(O \cap \Phi^c)$ is contained in a finite-dimensional linear subspace.

The pseudo-index of Φ^c is defined by

$$\psi(c) := \min\{\text{gen}(g(\Phi^c) \cap S_r Y) : g \in \mathcal{M}(\Phi^c)\} \in \mathbb{N}_0 \cup \{\infty\}$$

where gen$(\cdot)$ denotes the usual symmetric index. Additionally, set for $d > 0$ fixed

$$\mathcal{M}_0(\Phi^d) := \{g \in \mathcal{M}(\Phi^d) : \ g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d)\}.$$

Then we define for $c \in [0, d]$

$$\psi_d(c) := \min\left\{\text{gen}(g(\Phi^c) \cap S_r Y) : \ g \in \mathcal{M}_0(\Phi^d)\right\}.$$

Note that, by definition, $\psi(c) \leq \psi_d(c)$ for all $c \in [0, d]$.

Theorem 2.6 ([7, 13]). *Let $(\Phi_1) - (\Phi_3)$ be satisfied, and assume that Φ is even and satisfies the $(PS)_c$-condition for $c \in [\rho, b]$. Then Φ has at least $n := \dim Y_0$ pairs of critical points with critical values given by*

$$c_i := \inf\{c \geq 0 : \psi(c) \geq i\} \in [\rho, b], \quad i = 1, ..., n.$$

If Φ has only finitely many critical points in Φ_ρ^b, then $\rho < c_1 < c_2 < \cdots < c_n \leq b$.

Remark 2.7. Setting $X = E^-$ and $Y = E^+$, it flows from the definition and Lema 2.1 that the functional $\Phi = \Phi_\varepsilon$ is even and satisfies (Φ_1) and (Φ_2).

3 A strongly indefinite quadrature form

In order to construct the subspace satisfying the assumption (Φ_3) we make certain preparations in the following two sections.

In general, for any $0 \neq \gamma \in \mathbb{R}$, set $A_\gamma = -i\alpha \cdot \nabla + \gamma\beta$, a self-adjoint operator on L^2 with spectrum $\sigma(A_\gamma) = \mathbb{R} \setminus (-|\gamma|, |\gamma|)$. By $(\cdot, \cdot)_\gamma$, $\|\cdot\|_\gamma$ and $E = E_\gamma^- \oplus E_\gamma^+$ we denote the inner product, norm and orthogonal decomposition associated to the operator $|A_\gamma|^{1/2}$, see, e.g., [17]. Without loss of generality we always assume below $\gamma > 0$. Let S_q denote the Sobolev embedding constant:

$$S_q|u|_q^2 \leq |\nabla u|_2^2 + |u|_2^2$$

for $u \in H^1$. Note that, if $q = 6$, then notation $S_6 = S$ and $S|u|_6^2 \leq |\nabla u|_2^2$. Recall that

$$(3.1) \qquad S^{1/2}|u|_3^2 \leq |\,|A_\gamma|^{1/2}u|_2^2, \qquad \gamma^{(6-2q)/q} S_{2q/(4-q)}^{1/2}|u|_q^2 \leq |\,|A_\gamma|^{1/2}u|_2^2$$

for $q \in [2, 3]$ and all $u \in E$ (see, [17, Remark 3.2]). In particular, if $q = 3$ then $S^{1/2}|u|_3^2 \leq |\,|A_\gamma|^{1/2}u|_2^2$ for all $\gamma \in \mathbb{R}$.

Consider the quadrature form

$$a(w) = \int_{\mathbb{R}^3} \langle A_{\gamma\lambda} w, w\rangle \quad \text{with} \quad A_{\gamma\lambda} = -i\alpha \cdot \nabla + \gamma\beta + \lambda$$

for $u \in E$, where λ stands for a 4×4 symmetric real matrix with norm $|\lambda| < \gamma$. Denote, for $q \in [2, 3]$,

$$\ell_q := \inf_{u \in E_\gamma^+ \setminus \{0\}} \max_{v \in E_\gamma^-} \frac{a(u+v)}{|u+v|_q^2}$$

and let $\sigma(u) \in E_\gamma^-$ be such that

$$\ell(u) := \max_{v \in E_\gamma^-} \frac{a(u+v)}{|u+v|_q^2} = \frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^2}$$

($\sigma(u)$ is unique, see [17]). It is clear that

$$\gamma_q = \frac{q-2}{2q} \ell_q^{\frac{q}{q-2}} \tag{3.2}$$

is the least energy of

$$-i\alpha \cdot \nabla w + \gamma \beta w + \lambda w = |w|^{q-2} w \,. \tag{3.3}$$

Set $a_0(w) = \int_{\mathbb{R}^3} \langle A_\gamma w, w \rangle$ and let $\lambda_{\min}$ denote the minimal eigenvalue of λ. Observe that, by interpolation,

$$\int_{\mathbb{R}^3} |u|^q \leq |u|_2^{2(3-q)} |u|_3^{3(q-2)},$$

which together with (3.1) implies

$$(\gamma + \lambda_{\min})^{2(3-q)/q} \, S^{3(q-2)/2q} \, |u|_q^2 \leq |\, |A_\gamma|^{1/2} u|_2^2.$$

Therefore,

$$\frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^2} \geq \frac{a(u)}{|u|_q^2} \geq (\gamma + \lambda_{\min})^{\frac{2(3-q)}{q}} \, S^{\frac{3(q-2)}{2q}}. \tag{3.4}$$

In particular, by definition, taking $\gamma = 1$ and $\lambda = 0$, (3.2) yields

$$\gamma_p \geq \frac{p-2}{2p} S^{\frac{p}{2(p-2)}} \tag{3.5}$$

(it is also easy to show that $\gamma_p \geq \frac{p-2}{2p} \, S_{2p/(4-p)}^{1/2(p-2)}$).

Remark that, by definition, for any $t \neq 0$, there holds

$$\begin{aligned}\frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^2} &= \frac{a(tu+t\sigma(u))}{|tu+t\sigma(u)|_q^2} \leq \frac{a(tu+\sigma(tu))}{|tu+\sigma(tu)|_q^2} \\ &= \frac{a(u+\frac{1}{t}\sigma(tu))}{|u+\frac{1}{t}\sigma(tu)|_q^2} \leq \frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^2}\end{aligned}$$

hence

$$t\sigma(u) = \sigma(tu) \quad \text{and} \quad \ell(u) = \ell(tu). \tag{3.6}$$

Let, as before, $\{\hat{h}(\cdot), \hat{I}(\cdot), \hat{\mathscr{N}}\}$ be the "mountain-pass" reduction for (3.3). Set

$$\hat{t} = \hat{t}(u) = \Big(\frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^q}\Big)^{\frac{1}{q-2}}, \quad \hat{w} = \hat{t}u + \hat{t}\sigma(u)\,.$$

Plainly, one checks ([17]) that, for any $v \in E_\gamma^- \oplus \mathbb{R}u$,

$$0 = \Re\Big[a(\hat{w}, v) - \int_{\mathbb{R}^3} |\hat{w}|^{q-2}\hat{w}\overline{v}\Big]. \tag{3.7}$$

This implies that $\hat{t}u \in \hat{\mathscr{N}}$, and by the uniqueness of $\hat{t}(u)(>0)$, $\hat{h}(u)$ and $\sigma(u)$ (see [17]), $\hat{h}(\hat{t}u) = \hat{t}\sigma(u)$. Note that, in particular,

$$\hat{h}(u) = \sigma(u) \quad \text{if } u \in \hat{\mathscr{N}}. \tag{3.8}$$

Lemma 3.1. *Assume that $\nu > 0$ and let $\{h(\cdot), I(\cdot), \mathscr{N}\}$ be the "mountain-pass" reduction for the equation ($q \in [2,3]$)*

$$-i\alpha\cdot\nabla w + \gamma\beta w + \lambda w = \nu|w|^{q-2}w\,.$$

Then for all $u \in \mathscr{N}$

$$\frac{a(u+h(u))}{|u+h(u)|_q^2} \ge (\gamma+\lambda_{\min})^{\frac{2(3-q)}{q}}\, S^{\frac{3(q-2)}{2q}}\,..$$

Proof. Let $u \in \mathscr{N}$ and $w = u + h(u)$, and set $\hat{w} = \nu^{1/(q-1)}w = \nu^{1/(q-1)}u + \nu^{1/(q-1)}h(u)$. Then $\hat{w}$ satisfies (3.7) for all $v \in E_\gamma^- \oplus \mathbb{R}u$, that is, $\hat{u} = \nu^{1/(q-2)}u \in \hat{\mathscr{N}}$. Thus, $\sigma(\hat{u}) = \hat{h}(\hat{u}) = \nu^{1/(q-1)}h(u)$ by (3.8) and the uniqueness of $\hat{h}(\cdot)$, and

$$\begin{aligned}\frac{a(u+\sigma(u))}{|u+\sigma(u)|_q^2} &= \frac{a(\nu^{1/(q-1)}u+\nu^{1/(q-1)}\sigma(u))}{|\nu^{1/(q-1)}u+\nu^{1/(q-1)}\sigma(u)|_q^2}\\ &\le \frac{a(\nu^{1/(q-1)}u+\sigma(\nu^{1/(q-1)}u))}{|\nu^{1/(q-1)}u+\sigma(\nu^{1/(q-1)}u)|_q^2}\\ &= \frac{a(\nu^{1/(q-1)}u+\nu^{1/(q-1)}h(u))}{|\nu^{1/(q-1)}u+\nu^{1/(q-1)}h(u)|_q^2}\\ &= \frac{a(u+h(u))}{|u+h(u)|_q^2}.\end{aligned}$$

This, together with (3.4), implies the desired estimates. □

4 Preliminary results

Firstly, we recall a result on the representation of solutions to certain constant coefficient systems .

Lemma 4.1. *Let M be a 4×4 symmetric constant real metric and either $F(u)=\frac{\nu}{p}|u|^p$ or $F(u)=\frac{\nu_1}{p}|u|^p+\frac{\nu_2}{3}|u|^3$, and let u be a solution of*

$$-i\alpha\cdot\nabla u+\beta u+Mu=\nabla F(u),\qquad u\in H^1(\mathbb{R}^3,\mathbb{C}^4).$$

Then the energy

$$\Phi(u)=\frac{1}{6}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle.$$

Proof. By the Pohozev identity ([21])

$$\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle=\frac{3}{2}\int_{\mathbb{R}^3}-\langle(\beta+M)u,u\rangle+2F(u).$$

On the other hand,

$$\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle=\int_{\mathbb{R}^3}-\langle(\beta+M)u,u\rangle+\nabla F(u)\bar{u}.$$

Thus,

$$\frac{1}{2}\int_{\mathbb{R}^3}\langle(\beta+M)u,u\rangle=\int_{\mathbb{R}^3}3F(u)-\nabla F(u)\bar{u}=\frac{\nu_1(3-p)}{p}\int_{\mathbb{R}^3}|u|^p,$$

so the energy functional

$$\begin{aligned}\Phi(u)=&\Phi(u)-\frac{1}{3}\Phi'(u)u\\ =&\frac{1}{6}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle+\langle(\beta+M)u,u\rangle-\frac{\nu_1(3-p)}{3p}\int_{\mathbb{R}^3}|u|^p\\ =&\frac{1}{6}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle.\end{aligned}$$

□

In the following by M_μ we denote the constant matrix $\mu\beta$ or μI_4. Additionally, write $M_\varepsilon^\mu(x)$ for the matrix function $V_\varepsilon^\mu(x)\beta$ or $V_\varepsilon^\mu(x)I_4$ where $V^\mu=\max\{\mu,V(x)\}$, $V_\varepsilon^\mu(x)=V^\mu(\varepsilon x)$ (the identity matrix I_4 will be omitted below. Moreover, set $W^\nu(x)=\min\{\nu,W(x)\}$, $W_j^{\nu_j}(x)=\min\{\nu_j,W_j(x)\}$, and $W_\varepsilon^\nu(x)=W^\nu(\varepsilon x), W_{j\varepsilon}^{\nu_j}(x)=W_j^{\nu_j}(\varepsilon x)$.

Consider, for any $\tau \le \mu \le \tau_\infty$ and $\kappa_\infty \le \nu, \nu_1, \nu_2 \le \kappa$„

$$-i\alpha \cdot \nabla u + \beta u + M_\mu u = \nu |u|^{p-2} u, \tag{4.1}$$

$$-i\alpha \cdot \nabla u + \beta u + M_\mu u = \nu_1 |u|^{p-2} u + \nu_2 |u| u, \tag{4.2}$$

$$-i\alpha \cdot \nabla u + \beta u + M_\varepsilon^\mu(x) u = W_\varepsilon^\nu(x) |u|^{p-2} u, \tag{4.3}$$

$$-i\alpha \cdot \nabla u + \beta u + M_\varepsilon^\mu(x) u = W_{1\varepsilon}^{\nu_1}(x) |u|^{p-2} u + W_{2\varepsilon}^{\nu_2}(x) |u| u\,. \tag{4.4}$$

4.1. The equation (4.1)**.** Its solutions are critical points of the functional

$$\Gamma_{\mu\nu}(u) := \frac{1}{2}\Big(\|u^+\|^2 - \|u^-\|^2\Big) + \frac{1}{2}\int_{\mathbb{R}^3} \langle M_\mu u, u\rangle - \frac{\nu}{p}\int_{\mathbb{R}^3} |u|^p$$

defined for $u = u^+ + u^- \in E$. Denote the critical set, the least energy, and the set of least energy solutions of $\Gamma_{\mu\nu}$ as follows

$$\begin{aligned}
&\mathscr{L}_{\mu\nu} := \{u \in E : \ \Gamma'_{\mu\nu}(u) = 0\},\\
&\gamma_{\mu\nu} := \inf\{\Gamma_{\mu\nu}(u) : \ u \in \mathscr{L}_{\mu\nu} \setminus \{0\}\},\\
&\mathscr{R}_{\mu\nu} := \{u \in \mathscr{L}_{\mu\nu} : \ \Gamma_{\mu\nu}(u) = \gamma_{\mu\nu},\ |u(0)| = |u|_\infty\}.
\end{aligned}$$

The following conclusions are from [18]:

i) $\mathscr{L}_{\mu\nu} \neq \emptyset$, $\gamma_{\mu\nu} > 0$, and $\mathscr{L}_{\mu\nu} \subset \bigcap_{q\ge 2} W^{1,q}$;

ii) $\gamma_{\mu\nu}$ is attained, and $\mathscr{R}_{\mu\nu}$ is compact in $H^1(\mathbb{R}^3, \mathbb{C}^4)$;

iii) there exist $C, c > 0$ such that $|u(x)| \le C \exp(-c|x|)$ for all $x \in \mathbb{R}^3$ and $u \in \mathscr{R}_{\mu\nu}$.

Using γ_p we have the following representation

Lemma 4.2. *Assume $M_\mu = \mu\beta$ and let u be a least energy solution of* (4.1)*. Then*

$$\gamma_{\mu\nu} = \gamma_p \,(1+\mu)^{\frac{2(3-p)}{p-2}} \nu^{\frac{-2}{p-2}}.$$

Proof. Set

$$v(x)=\Big(\frac{\nu}{1+\mu}\Big)^{\frac{1}{p-2}}u\Big(\frac{x}{1+\mu}\Big).$$

Then v is a ground state of (1.9). In virtue of Lemma 4.1, one has

$$\begin{aligned}\gamma_p&=\frac{1}{6}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla v,v\rangle=\Big(\frac{\nu}{1+\mu}\Big)^{\frac{2}{p-2}}(1+\mu)^2\left(\frac{1}{6}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle\right)\\&=\nu^{\frac{2}{p-2}}\big(1+\mu\big)^{-\frac{2(3-p)}{p-2}}\gamma_{\mu\nu}\end{aligned}$$

ending the proof. ◻

Lemma 4.3. *Let $-1<\mu_j<1$ and $\nu_j>0$, $j=1,2$, with $\min\{\mu_2-\mu_1,\ \nu_1-\nu_2\}>0$. Then $\gamma_{\mu_1\nu_1}<\gamma_{\mu_2\nu_2}$. In particular, $\gamma_{\mu_1\nu}<\gamma_{\mu_2\nu}$ if $\mu_1<\mu_2$, and $\gamma_{\mu\nu_1}>\gamma_{\mu\nu_2}$ if $\nu_1<\nu_2$.*

Proof. The conclusion follows from the representation of $\gamma_{\mu\nu}$ in Lemma 4.2 if $M_\mu=\mu\beta$. If $M_\mu=\mu$ then the conclusion follows directly from the representation of $\Gamma_{\mu\nu}$. ◻

Set $M_\infty=M_{\tau_\infty}$ and $\gamma_\infty=\gamma_{\tau_\infty\kappa_\infty}$. Consider

$$-i\alpha\cdot\nabla u+\beta u+M_\infty u=\kappa_\infty|u|^{p-2}u\,. \tag{4.5}$$

Lemma 4.4. *Let $\tau\le\mu\le\tau_\infty$ and $\kappa_\infty\le\nu\le\kappa$.*

$$\gamma_{\mu\nu}\le\Big(\frac{\kappa_\infty}{\nu}\Big)^{\frac{2}{p-2}}\Big(\frac{1+\mu}{1+\tau_\infty}\Big)^{\frac{2(3-p)}{p-2}}\gamma_\infty\,.$$

Proof. Let u be a least energy solution of (4.5) and set

$$v(x)=b\,u(\xi x),\quad b=\Big(\frac{\kappa_\infty(1+\mu)}{\nu(1+\tau_\infty)}\Big)^{\frac{1}{p-2}},\quad \xi=\frac{1+\mu}{1+\tau_\infty}. \tag{4.6}$$

1) $M_\mu=\mu\beta$. It is clear that v is a solution of (4.1) with desired inequality (by using Lemma 4.1).

2) $M_\mu=\mu$. Writing $u=(u_1,u_2)\in\mathbb{C}^2\times\mathbb{C}^2$, observe that (4.1) is equivalent to

$$\begin{cases}-i\sigma\cdot\nabla u_2+(1+\mu)u_1=\nu|u|^{p-2}u_1\\-i\sigma\cdot\nabla u_1-(1-\mu)u_2=\nu|u|^{p-2}u_2\end{cases}$$

with the energy functional

$$\Gamma_{\mu\nu}(u)=\frac{1}{2}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla u,u\rangle+(1+\mu)|u_1|^2-(1-\mu)|u_2|^2\ -\frac{\nu}{p}\int_{\mathbb{R}^3}|u|^p.$$

Now let u be a least energy solution of (4.5) and v be defined by (4.6). Setting

$$\eta = \frac{(1+\mu)(1-\tau_\infty)}{(1+\tau_\infty)(1-\mu)}, \tag{4.7}$$

the function v is a least energy solution of

$$\begin{cases} -i\sigma\cdot\nabla v_2 + (1+\mu)v_1 = \nu|v|^{p-2}v_1 \\ -i\sigma\cdot\nabla v_1 - \eta(1-\mu)v_2 = \nu|v|^{p-2}v_2 \end{cases}$$

with energy

$$I(v) := \frac{1}{2}\int_{\mathbb{R}^3}\langle -i\alpha\cdot\nabla v, v\rangle + (1+\mu)|v_1|^2 - \eta(1-\mu)|v_2|^2 \;-\; \frac{\nu}{p}\int_{\mathbb{R}^3}|v|^p.$$

Since

$$\eta - 1 = \frac{2(\mu-\tau_\infty)}{(1+\tau_\infty)(1-\mu)} \le 0,$$

i.e., $\eta \le 1$, one has $\Gamma_{\mu\nu}(v) \le I(v)$ which, together with Lemma 4.1, implies the desired first inequality. □

Letting $[r]$ denote the integer part of $r\in\mathbb{R}$, as a consequence we have

Lemma 4.5. *There holds* $m\,\gamma_{\mu\nu} \le \gamma_\infty$ *where*

$$m = \left[\Big(\frac{\nu}{\kappa_\infty}\Big)^{\frac{2}{p-2}}\Big(\frac{1+\tau_\infty}{1+\mu}\Big)^{\frac{2(3-p)}{p-2}}\right].$$

4.2. The equation (4.2)**.** Its solutions are critical points of

$$\Gamma_{\mu\vec{\nu}}(u) := \Gamma_{\mu\nu_1}(u) - \frac{\nu_2}{3}\int_{\mathbb{R}^3}|u|^3 = \Gamma_{\mu\nu_2}(u) - \frac{\nu_1}{p}\int_{\mathbb{R}^3}|u|^p$$

on $u\in E$, where

$$\Gamma_{\mu\nu_1}(u) = \frac{1}{2}\Big(\|u^+\|^2 - \|u^-\|^2\Big) + \frac{1}{2}\int_{\mathbb{R}^3}\langle M_\mu u, u\rangle - \frac{\nu_1}{p}\int_{\mathbb{R}^3}|u|^p,$$

$$\Gamma_{\mu\nu_2}(u) = \frac{1}{2}\Big(\|u^+\|^2 - \|u^-\|^2\Big) + \frac{1}{2}\int_{\mathbb{R}^3}\langle M_\mu u, u\rangle - \frac{\nu_2}{3}\int_{\mathbb{R}^3}|u|^3.$$

Let $\gamma_{\mu\vec{\nu}}$, $\gamma_{\mu\nu_1}$, $\gamma_{\mu\nu_2}$ denote the linking levels of $\Gamma_{\mu\vec{\nu}}$, $\Gamma_{\mu\nu_1}$, $\Gamma_{\mu\nu_2}$, respectively. One has

$$\gamma_{\mu\vec{\nu}} < \gamma_{\mu\nu_1}, \quad \gamma_{\mu\vec{\nu}} < \gamma_{\mu\nu_2}. \tag{4.8}$$

Lemma 4.6. *$\gamma_{\mu\vec{\nu}}$ is attained provided that*

$$\gamma_{\mu\vec{\nu}} < \frac{S^{3/2}}{6\nu_2^2}. \tag{4.9}$$

Proof. Let $\{u_n\} \subset \mathscr{M}_{\mu\vec{\nu}}$ be such that $I(u_n) = \Gamma_{\mu\vec{\nu}}(u_n + h_0(u_n)) \to c = \gamma_{\mu\vec{\nu}}$. It is not difficult to check that $\{w_n = u_n + h_0(u_n)\}$ is bounded in E. By a Lions' concentration principle $\{w_n\}$ is either vanishing or non-vanishing.

Assume that $\{w_n\}$ is vanishing. Then $|w_n|_s \to 0$ for $s \in (2,3)$, one gets,

$$\Gamma_{\mu\vec{\nu}}(u_n + h_0(u_n)) = \frac{1}{6}a(w_n) + o(1).$$

Similarly,

$$|w_n|_3^3 = \frac{6c}{\nu_2} + o(1).$$

Let $\hat{\Gamma}_{\mu\vec{\nu}}$ be the energy functional and $(\hat{h}(\cdot), \hat{I}(\cdot), \hat{\mathscr{M}})$ the "mountain-pass" reduction of

$$-i\alpha \cdot \nabla \hat{w} + \beta\hat{w} + M_\mu \hat{w} = \nu_2 |\hat{w}|\hat{w}.$$

Let $t_n > 0$ be such that $\hat{u}_n = t_n u_n \in \hat{\mathscr{M}}$ and set $\hat{w}_n = t_n u_n + \hat{h}(t_n u_n)$. Plainly, $\{t_n\}$ is bounded, hence $|\hat{w}_n|_s \to 0$ for any $s \in (2,3)$. By definition, $\hat{\Gamma}_{\mu\vec{\nu}}(w_n) \le \hat{\Gamma}_{\mu\vec{\nu}}(\hat{w}_n)$ and $\Gamma_{\mu\vec{\nu}}(\hat{w}_n) \le \Gamma_{\mu\vec{\nu}}(w_n)$. Thus

$$\begin{aligned}\Gamma_{\mu\vec{\nu}}(w_n) &= \hat{\Gamma}_{\mu\vec{\nu}}(w_n) + o(1) \le \hat{\Gamma}_{\mu\vec{\nu}}(\hat{w}_n) + o(1) \\ &= \Gamma_{\mu\vec{\nu}}(\hat{w}_n) + o(1) \le \Gamma_{\mu\vec{\nu}}(w_n) + o(1).\end{aligned}$$

This yields that

$$\lim_{n\to\infty} \hat{\Gamma}_{\mu\vec{\nu}}(w_n) = \lim_{n\to\infty} \hat{\Gamma}_{\mu\vec{\nu}}(\hat{w}_n) = c$$

which, together with the fact that $\hat{u}_n \in \hat{\mathscr{M}}$, implies that

$$\hat{\Gamma}_{\mu\vec{\nu}}(\hat{w}_n) = \frac{1}{6}a(\hat{w}_n) = \frac{1}{6}\nu_2 |\hat{w}_n|_3^3 \to c,$$

that is,

$$|\hat{w}_n|_3 = \Big(\frac{6\hat{\Gamma}_{\mu\vec{\nu}}(\hat{w}_n)}{\nu_2}\Big)^{1/3} \quad \text{and} \quad \frac{a(\hat{w}_n)}{|\hat{w}_n|_3^2} = \nu_2|\hat{w}_n|_3 = \nu_2\Big(\frac{6c}{\nu_2}\Big)^{1/3} + o(1).$$

Now, by virtue of Lemma 3.1 with $q = 3$ and $\nu = \nu_2$, we see that

$$c \ge \frac{S^{3/2}}{6\nu_2^2},$$

a contradiction.

Therefore, $\{w_n\}$ is non-vanishing, that is, there exist $r, \delta > 0$ and $x_n \in \mathbb{R}^3$ such that, setting $v_n(x) = w_n(x + x_n)$, along a subsequence,

$$\int_{B_r(0)} |v_n|^2 \geq \delta.$$

Without loss of generality we assume $v_n \rightharpoonup v$. Then $v \neq 0$ and is a solution of (4.2). And so $\gamma_{\mu\vec{\nu}}$ is attained. □

Lemma 4.7. *$\gamma_{\mu\vec{\nu}}$ is attained provided the following:*

$$(4.10) \qquad (1+\mu)^{\frac{2(3-p)}{p-2}} \Big(\frac{\nu_2}{\nu_1^{1/(p-2)}} \Big)^2 \leq \frac{S^{3/2}}{6\gamma_p}.$$

Proof. Assume $M_\mu = \mu\beta$. Consider the norm $\|u\|_\gamma = ||A_\gamma|^{1/2} u|_2$ on E induced by the operator $A_\gamma = -i\alpha \cdot \nabla + (1+\mu)\beta$. Then 1) follows from (4.8) and Lemmas 4.2 and 4.6.

Consider 2). Observe that, since $\mu \leq 0$, for $u \neq 0$,

$$\begin{aligned} \Gamma_{\mu\vec{\nu}}(u) = &\frac{1}{2} \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + (1+\mu)\beta)u,\, u\rangle + \frac{1}{2} \int_{\mathbb{R}^3} \mu \langle (I - \beta)u,\, u\rangle \\ &- \int_{\mathbb{R}^3} \Big(\frac{\nu_1}{p} |u|^p + \frac{\nu_2}{3} |u|^3 \Big) \\ < &\frac{1}{2} \int_{\mathbb{R}^3} \langle (-i\alpha \cdot \nabla + (1+\mu)\beta)u,\, u\rangle - \frac{\nu_1}{p} \int_{\mathbb{R}^3} |u|^p \end{aligned}$$

Now the conclusion follows from (4.8) and Lemmas 4.2 and 4.6. □

In the sequel, by $\vec{\nu_1} \leq \vec{\nu_2}$ (resp. $\vec{\nu_1} < \vec{\nu_2}$) we mean that $\min\{\nu_1^2 - \nu_1^1, \nu_2^2 - \nu_2^1\} \geq 0$ (resp. $\min\{\nu_1^2 - \nu_1^1, \nu_2^2 - \nu_2^1\} > 0$) for any vectors $\vec{\nu_j} = (\nu_1^j, \nu_2^j)$. Additionally, set for $\vec{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2$,

$$M_{\vec{\mu}} = \begin{pmatrix} \mu_1 I_2 & 0 \\ 0 & \mu_2 I_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

For any $\vec{\mu}_k = (\mu_{k1}, \mu_{k2}) \in \mathbb{R}^2$, $k = 1, 2$, $M_{\vec{\mu}_2} \geq M_{\vec{\mu}_1}$ if $\vec{\mu}_1 \leq \vec{\mu}_2$. The following conclusion is clear.

Lemma 4.8. *Let $\mu_j > -1$ and $\vec{\nu}_j > 0$ for $j = 1, 2$. If $\mu_2 \geq \mu_1$ and $\vec{\nu}^1 \geq \vec{\nu}^2$ then $\gamma_{\mu_1\vec{\nu}^1} \leq \gamma_{\mu_2\vec{\nu}^2}$. If $\min\{\mu_2 - \mu_1,\ \vec{\nu}^1 - \vec{\nu}^2\} > 0$ then $\gamma_{\mu_1\vec{\nu}^1} < \gamma_{\mu_2\vec{\nu}^2}$.*

Proof. We only verify the case $M_\mu = \mu\beta$ (the other is clear). In fact, u is a least energy solution of (4.2) if and only if $v(x) = bu(x/(1+\mu))$ is a least energy solution of

$$-i\alpha \cdot \nabla v + \beta v = \frac{1}{1+\mu}\left(\frac{\nu_1}{b^{p-2}}\,|v|^{p-2}v + \frac{\nu_2}{b}\,|v|v\right)$$

with the energy

$$\gamma(v) = (1+\mu)^2 b^2 \gamma(u).$$

The conclusion follows easily by taking $b = 1, \left(\frac{\nu_1}{1+\mu}\right)^{1/(p-2)}$ and $\frac{\nu_2}{1+\mu}$, respectively. □

Remark 4.9. Similarly, if $M_{\vec{\mu}_1} \le M_{\vec{\mu}_2}$ and $\vec{\nu}_1 \ge \vec{\nu}_2$ then $\gamma_{\vec{\mu}_1\vec{\nu}_1} \le \gamma_{\vec{\mu}_2\vec{\nu}_2}$, where $\gamma_{\vec{\mu}\vec{\nu}}$ denotes the energy of (4.2) with M_μ replaced by $M_{\vec{\mu}}$.

Below, let u be a least energy solution of

$$-i\alpha \cdot \nabla u + \beta u + M_\infty u = \kappa_{1\infty}|u|^{p-2}u + \kappa_{2\infty}|u|u \tag{4.11}$$

with the energy denoted by $\gamma_{\vec{\infty}}$ which is attained if $\gamma_{\vec{\infty}} < S^{3/2}/6\kappa_{2\infty}^2$ by Lemma 4.6. For $\tau \le \mu \le \tau_\infty$ and $\kappa_{j\infty} \le \nu_j \le \kappa_j$, set

$$v(x) = bu(\xi x), \quad \xi = \frac{1+\mu}{1+\tau_\infty}, \qquad b = \max\{b_1,\ b_2\},$$

where

$$b_1 = \left(\frac{\xi\kappa_{1\infty}}{\nu_1}\right)^{\frac{1}{p-2}} \quad \text{and} \quad b_2 = \frac{\xi\kappa_{2\infty}}{\nu_2}.$$

Then, if $M_\mu = \mu\beta$, v solves

$$-i\alpha \cdot \nabla v + (1+\mu)\beta v = \frac{\kappa_{1\infty}(1+\mu)}{b^{p-2}\nu_1(1+\tau_\infty)}\,\nu_1|v|^{p-2}v + \frac{\kappa_{2\infty}(1+\mu)}{b\nu_2(1+\tau_\infty)}\,\nu_2|v|v,$$

and, if $M_\mu = \mu$, v solves

$$-i\alpha \cdot \nabla v + \beta v + M_{\vec{u}}v = \frac{\kappa_{1\infty}(1+\mu)}{b^{p-2}\nu_1(1+\tau_\infty)}\,\nu_1|v|^{p-2}v + \frac{\kappa_{2\infty}(1+\mu)}{b\nu_2(1+\tau_\infty)}\,\nu_2|v|v$$

with energy denoted by $I^*(v)$, where $\vec{\mu} = (\mu_1, \mu_2)$ with

$$\mu_1 = \xi(1+\tau_\infty) - 1 = \mu, \quad \mu_2 = 1 - \xi(1-\tau_\infty) > \mu.$$

By dfinition, Remark 4.9 and Lemma 4.1, it is clear that

$$\gamma_{\mu\vec{\nu}} \le I^*(v) = \Big(\frac{b(1+\tau_\infty)}{1+\mu}\Big)^2 \gamma_{\vec{\infty}}.$$

Set

$$m(\mu,\vec{\nu}) = \begin{cases} \Big(\dfrac{1+\tau_\infty}{1+\mu}\Big)^{\frac{2(3-p)}{p-2}} \Big(\dfrac{\nu_1}{\kappa_{1\infty}}\Big)^{\frac{2}{p-2}}, & \text{if } b_1 \ge b_2 \\ \Big(\dfrac{\nu_2}{\kappa_{2\infty}}\Big)^2, & \text{otherwise.} \end{cases} \tag{4.12}$$

Then

$$m(\mu,\vec{\nu})\, I^*(v) = \gamma_{\vec{\infty}},$$

and we have

Lemma 4.10. *For $\tau \le \mu \le \tau_\infty$ and $\kappa_{j\infty} \le \nu_j \le \kappa_j$, there holds*

$$m(\mu,\vec{\nu})\, \gamma_{\mu\vec{\nu}} < \gamma_{\vec{\infty}}.$$

4.3. The equation (4.3). The solutions of (4.3) are critical points of

$$\Phi_\varepsilon^{\mu\nu}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} \langle M_\varepsilon^\mu u, u\rangle - \frac{1}{p}\int_{\mathbb{R}^3} W_\varepsilon^\nu(x)|u|^p$$

on $u \in E = E^+ \oplus E^-$. Let $c_\varepsilon^{\mu\nu}$ denote the Minimax level of $\Phi_\varepsilon^{\mu\nu}$ deduced by the linking structure (see (2.2)). Write $h_\varepsilon^{\mu\nu}$, $I_\varepsilon^{\mu\nu}$, $\mathscr{N}_\varepsilon^{\mu\nu}$, and so on, for the notations associated to the Mountain-Pass induce. Recall that, for any $u \in E^+ \setminus \{0\}$, there is a unique $t = t(u) > 0$ such that $t(u)u \in \mathscr{N}_\varepsilon^{\mu\nu}$ ([18]). It is easy to check that $c_\varepsilon^{\mu\nu} = \inf\{I_\varepsilon^{\mu\nu}(u) : u \in \mathscr{N}_\varepsilon^{\mu\nu}\}$.

In the sequel we denote

$$\Phi_\varepsilon^\infty = \Phi_\varepsilon^{\tau_\infty\kappa_\infty}, \quad c_\varepsilon^\infty = c_\varepsilon^{\tau_\infty\kappa_\infty}, \quad \mathscr{N}_\varepsilon^\infty = \mathscr{N}_\varepsilon^{\tau_\infty\kappa_\infty},$$

and $\Gamma_\infty = \Gamma_{\tau_\infty\kappa_\infty}$, $\gamma_\infty = \gamma_{\tau_\infty\kappa_\infty}$. As a consequence of Lemma 2.5 we have

Lemma 4.11. *$c_\varepsilon^\infty \to \gamma_\infty$ as $\varepsilon \to 0$.*

Remark 4.12. Similar, one obtains easily that $\lim_{\varepsilon\to 0} c_\varepsilon^{\mu\nu} = \gamma_{\mu\nu}$.

As a consequence one has

Lemma 4.13. *$\Phi_\varepsilon^{\mu\nu}$ satisfies the $(PS)_c$ condition for $c < \gamma_\infty$ if ε small.*

Proof. Writing $I(u) = \Phi_\varepsilon^{\mu\nu}(u)$, let $I(u_n) \to c$ and $I'(u_n) \to 0$. Then u_n is bounded and we can assume that $u_n \rightharpoonup u$. Clearly $I'(u) = 0$. Set $z_n = u_n - u$. Note that $z_n \rightharpoonup 0$ in E, $z_n \to 0$ in L^q_{loc} for $q \in [1, 3)$, and $z_n(x) \to 0$ a.e. in x. It is easy to check that $\Phi_\varepsilon^\infty(z_n) \to c - \Phi_\varepsilon^{\mu\nu}(u)$ and $(\Phi_\varepsilon^\infty)'(z_n) \to 0$. If $c = \Phi_\varepsilon^{\mu\nu}(u)$ then $z_n \to 0$ and we are done. If $c - \Phi_\varepsilon^{\mu\nu}(u) \geq c_\varepsilon^\infty$ then $c \geq c_\varepsilon^{\mu\nu} + c_\varepsilon^\infty$, a contradiction. □

4.4. The equation (4.4)**.** Its solutions are critical points of

$$\Phi_\varepsilon^{\mu\vec{\nu}}(u) = \Phi_\varepsilon^{\mu\nu_1}(u) - \frac{1}{3}\int_{\mathbb{R}^3} W_{2\varepsilon}^{\nu_2}(x)|u|^3$$

with $\vec{\nu} = (\nu_1, \nu_2)$. Let $c_\varepsilon^{\mu\vec{\nu}}$ be the linking level (see (2.2)). Write $\Phi_\varepsilon^{\vec{\infty}}$ and $c_\varepsilon^{\vec{\infty}}$ for $\mu = \tau_\infty$ and $\vec{\nu} = (\kappa_{1\infty}, \kappa_{2\infty})$. We have, as Lemma 4.11, the following

Lemma 4.14. $c_\varepsilon^{\vec{\infty}} \to \gamma_{\vec{\infty}}$ *as* $\varepsilon \to 0$.

Also as Lemma 4.13 there holds the following

Lemma 4.15. $\Phi_\varepsilon^{\mu\vec{\nu}}$ *satisfies the* $(PS)_c$ *condition for all* $c < \gamma_{\vec{\infty}}$.

Proof. Denote $I(u) = \Phi_\varepsilon^{\mu\vec{\nu}}$ and let $I(u_n) \to c$, $I'(u_n) \to 0$. One can assume $u_n \rightharpoonup u$ and set $z_n = u_n - u$. Then z_n is a $(PS)_c$ sequence for $\Phi_\varepsilon^{\vec{\infty}}$ where $c = c - I(u)$. By Lemma 4.14, if $I(u) \neq c$ then $c - I(u) \geq \gamma_{\vec{\infty}}$, a contradiction. □

5 Proofs of main results: the subcritical case

Setting $u(x) = w(\varepsilon x)$, the equation (1.3) is equivalent to the following

$$-i\alpha \cdot \nabla u + \beta u + M_\varepsilon(x)u = W_\varepsilon(x)|u|^{p-2}u\,. \tag{5.1}$$

Proof of Theorem 1.1. Without loss of generality, we may assume that $0 \in \mathscr{V}$ and $x_v = 0$. Observe that $\tau = V(0)$ and $\kappa_v = W(0)$. Solutions of (5.1) are critical points of the functional $\Phi_\varepsilon(u) := \Phi_\varepsilon^{\tau\kappa_v}(u)$. For notational convenience we denote $\Phi_0(u) = \Gamma_{\tau\kappa_v}$. We will utilize Theorem 2.6. Obversely, Φ_ε is even, and in virtue of Remark 2.7 the conditions (Φ_1) and (Φ_2) are satisfied. It remains to verify (Φ_3).

Let $u \in \mathscr{R}_{\tau\kappa_v}$ and let $\chi_r \in C_0^\infty(\mathbb{R}^+)$ be such that $\chi_r(s) = 1$ if $s \leq r$ and $\chi_r(s) = 0$ if $s \geq r + 1$. Set $u_r(x) = \chi_r(|x|)u(x)$. Recall that $|u(x)| \leq Ce^{-c|x|}$ for some $C, c > 0$ and all $x \in \mathbb{R}^3$, hence $\|u_r - u\| \to 0$ as $r \to \infty$. Then $\|u_r^\pm - u^\pm\| \leq \|u_r - u\| \to 0$, $\Phi_0(u_r) \to \gamma_{\tau\kappa_v}$ and $\Phi_0'(u_r) \to 0$ as $r \to \infty$. Let $h_0 : E^+ \to E^-$ be defined so that $\Phi_0(u + h_0(u)) = \max_{v\in E^-} \Phi_0(u + v)$ (see

(2.3)). Plainly, $\|u_r^- - h_0(u_r^+)\| \to 0$ and $\|u_r - \hat{u}_r\| \to 0$ where $\hat{u}_r = u_r^+ + h_0(u_r^+)$ (see Lemma 2.4). Therefore,

$$\max_{v \in E^-} \Phi_0(u_r^+ + v) = \Phi_0(\hat{u}_r) = \Phi_0(u_r) + o(1) = \gamma_{\tau\kappa_v} + o(1).$$

Observe that since $V(\varepsilon x) \to \tau$ and $W(\varepsilon x) \to \kappa_v$ as $\varepsilon \to 0$ uniformly in $|x| \le r+1$, we have that, for any $\delta > 0$, there are $r_\delta > 0$ and $\varepsilon_\delta > 0$ such that

$$\max_{w \in E^- \oplus \mathbb{R}u_r} \Phi_\varepsilon(w) < \gamma_{\tau\kappa_v} + \delta \tag{5.2}$$

for all $r \ge r_\delta$ and $\varepsilon \le \varepsilon_\delta$.

Let $y^j = (2j(r+1), 0, 0)$, define $u_j(x) = u(x - y^j) = u(x_1 - 2j(r+1), x_2, x_3)$, $u_{rj}(x) = u_r(x - y^j)$ for $j = 0, 1, ..., m-1$. Setting $r_m = (2m-1)(r+1)$, it is clear that $\operatorname{supp} u_{rj} \subset B_{r_m}(0)$. Obviously $\{u_{rj}^+\}_{j=0}^{m-1}$ are linearly independent. Indeed, if $w^+ = \sum_{j=0}^{m-1} c_j u_{rj}^+ = 0$, denoting $w = \sum_{j=0}^{m-1} c_j u_{rj}$, one has $w = w^- + w^+$ and

$$-\|w^-\|^2 = a_\tau(w) = \sum_j c_j^2 a_\tau(u_{rj}^+) = a_\tau(u_r) \sum_j c_j^2$$

which implies $c_j = 0, j = 0, 1, ..., m-1$. Now set

$$\begin{aligned} E_m &= E^- \oplus \operatorname{span}\{u_{rj} \,:\, j = 0, ..., m-1\} \\ &= E^- \oplus \operatorname{span}\{u_{rj}^+ \,:\, j = 0, ..., m-1\}. \end{aligned}$$

By virtue of Lemma 2.5, let $t_{\varepsilon rj} > 0$ be such that $t_{\varepsilon rj} u_{rj}^+ \in \mathscr{N}_\varepsilon$. Observe that

$$\lim_{\varepsilon \to 0} \lim_{r \to \infty} t_{\varepsilon rj} = \lim_{\varepsilon \to 0} t_{\varepsilon j} = 1, \tag{5.3}$$

$$\lim_{\varepsilon \to 0} \lim_{r \to \infty} h_\varepsilon(t_{\varepsilon rj} u_{rj}^+) = \lim_{\varepsilon \to 0} h_\varepsilon(t_{\varepsilon j} u_{rj}^+) = h_0(u^+) = u^-, \tag{5.4}$$

$$\lim_{\varepsilon \to 0} \lim_{r \to \infty} \|h_\varepsilon(t_{\varepsilon rj} u_{rj}^+) - t_{\varepsilon rj} u_{rj}^-\| = \lim_{\varepsilon \to 0} \|h_\varepsilon(t_{\varepsilon j} u^+) - t_{\varepsilon j} u^-\| = 0. \tag{5.5}$$

It is not difficult to check the following

$$\begin{aligned}
\max_{w\in E_m}\Phi_\varepsilon(w) &= \Phi_\varepsilon\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^{+} + h_\varepsilon(t_{\varepsilon j}u_{rj}^{+})\Big)\\
&= \Phi_\varepsilon\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^{+} + t_{\varepsilon j}u_{rj}^{-}\Big) + o(1_r)\\
&= \Phi_\varepsilon\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}\Big) + o(1_r)\\
&= \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}) + o(1_r)\\
&= \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^{+} + t_{\varepsilon j}u_{rj}^{-}) + o(1_r)\\
&= \sum_{j=0}^{m-1}\Phi_\varepsilon(t_{\varepsilon j}u_{rj}^{+} + h_\varepsilon(t_{\varepsilon j}u_{rj}^{+})) + o(1_r)\\
&= \sum_{j=0}^{m-1}\Phi_0(t_{0j}u_{rj}^{+} + h_0(t_{0j}u_{rj}^{+})) + o(1_{r\varepsilon})\\
&= \sum_{j=0}^{m-1}\Phi_0(u) + o(1_{r\varepsilon})\\
&= m\,\gamma_\kappa + o(1_{r\varepsilon})
\end{aligned}$$

where $o(1_r)$ means arbitrary small as $r\to\infty$, and $o(1_{r\varepsilon})$ means arbitrary small as r sufficiently large and ε sufficiently small.

Now, by assumptions and Lemma 4.5, for any $0<\delta<\gamma_\infty - m\,\gamma_\kappa$, one may choose $r>0$ large and then $\varepsilon_m>0$ small such that, for all $\varepsilon\le\varepsilon_m$, $\max_{w\in E_m}\Phi_\varepsilon(w)\le\gamma_\infty-\delta$. Now by Theorem 2.6 one gets the multiplicity conclusion.

By Lemma 2.2 we see that the solutions are in $\cap_{s\ge 2}W^{1,s}$.

Finally, repeating the proof of [20, Lemma 4.6] gives the exponential decay.

Now, as (5.2), one can choose $r>0$ and $\varepsilon_m>0$ such that, if $\varepsilon\le\varepsilon_m$,

$$\Phi_\varepsilon(w)<\gamma_\infty \quad \text{for all}\, w\in E_m.$$

It follows from Lemmas 4.13 that Φ_ε satisfies the $(PS)_c$-condition for all $c<\gamma_\infty$, that is, the general condition (Φ_3) is satisfied. Now by applying

Theorem 2.6, one sees that either Φ_ε has infinitely many critical points, or has at least m pairs of critical points with different critical values $0 < c_\varepsilon^0 < \cdots < c_\varepsilon^{m-1} \le \sup_{w\in H_m} \Phi_\varepsilon(w) < \gamma_\infty$. $\square$

Proof of Theorem 1.2. We are sketchy. Assume $x_w = 0$ and consider $\mu = \tau_w = V(0)$, $\nu = \kappa = W(0)$ and $\Phi_\varepsilon = \Phi^{\tau_w\kappa}$, $\Phi_0 = \Gamma_{\tau\kappa}$. Let $u \in \mathscr{R}_{\tau_w\kappa}$, $\Phi_0(u) = \gamma_{\tau_w\kappa}$. As before. define u_r and $u_{rj}, j = 0, .., m-1$, and set the m-dimensional subspace E_m. Then one checks that, for $w \in E_m$ with $\ell(w) = \max \ell(E_m)$,

$$\begin{aligned}\Phi_0(w) &= \frac{p-2}{2p\kappa^{2/(p-2)}}\ell(w)^{p/(p-2)} \\ &\le \frac{p-2}{2p\kappa^{2/(p-2)}}(m\ell(u))^{p/(p-2)} + o(1) \\ &= \gamma_p\left(m\Big(\frac{(1+\tau_w)^{3-p}}{\kappa}\Big)^{\frac{2}{p}}\right)^{\frac{p}{p-2}} + o(1) \\ &< \gamma_\infty + o(1)\end{aligned}$$

by (1.6) as $r \to \infty$. Now Theorem 2.6 applies. $\square$

Remark 5.1. Let u be a solution with $\Phi_\varepsilon(u) \le \Lambda$. Plainly one checks the following

$$\Delta u = \Big((1+V_\varepsilon)^2 + i\varepsilon\beta\sum_{k=1}^{3}\alpha_k\partial_k V(\varepsilon x)\Big)u + r_\varepsilon(u)u \tag{5.6}$$

where

$$\begin{aligned}r_\varepsilon(u) = \Big(i\varepsilon\sum_{k=1}^{3}\partial_k W(\varepsilon x) - W_\varepsilon|u|^{p-2}\Big)|u|^{p-2} \\ + i(p-2)W_\varepsilon|u|^{p-3}\sum_{k=1}^{3}\alpha_k\Re\Big[\frac{u\partial_k\overline{u}}{|u|}\Big].\end{aligned}$$

By the Kato's inequality we get (by sgn denoting the sign function)

$$\begin{aligned}\Delta|u| &\ge \Re\big[\Delta u(\operatorname{sgn} u)\big] \\ &\ge \big((1+V_\varepsilon)^2 - \varepsilon|\nabla V(\varepsilon x)|\big)|u| - W_\varepsilon|u|^{2p-3}\end{aligned} \tag{5.7}$$

(cf. [11]) which, together with the regularity estimates, implies that there is $\lambda > 0$ depending on Λ but independent of ε such that

$$\Delta|u| \ge -\lambda|u|.$$

Now the sub-solution estimate shows that there exists $C_0 > 0$ independent of $\varepsilon > 0$ with

$$|u(x)| \le C_0 \int_{B_1(0)} |u(y)|\, dy \tag{5.8}$$

for all $x \in \mathbb{R}^3$. In addition, letting $\sigma = \min\{(1+V(x))^2 : x \in \mathbb{R}^3\}$, the estimate (5.6) implies that there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \le \varepsilon_1$,

$$\Delta|u| \ge \frac{\sigma}{2}|u| - (\kappa|u|^{2(p-2)})|u|. \tag{5.9}$$

Proof of Theorem 1.3. If the potential $M(x) = V(x)I_4$, the concentration had been proved in [15], we need to deal with here the other potential $M(x) = V(x)\beta$. Moreover we give only the proof for the case (2) of (P_1) because the case (1) can be handled similarly.

We may assume $x_w = 0$, $\tau_w = V(0)$ and $\kappa = W(0)$. The existence follows from Theorem 1.2. Note that, by Remark 4.12, the least energy

$$c_\varepsilon \to \gamma_{\tau_w \kappa} \quad \text{as} \quad \varepsilon \to 0. \tag{5.10}$$

The remainder will be argued in several steps.

Step 1) Let $\varepsilon_j \to 0$, $u_j \in \mathscr{K}_j$ where $\mathscr{K}_j = \mathscr{K}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A concentration argument shows that there exist a sequence $\{y_j'\} \subset \mathbb{R}^3$ and constants $R > 0, \delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B_R(y_j')} |u_j|^2 \ge \delta. \tag{5.11}$$

Set $v_j(x) = u_j(x + y_j')$. Then v_j solves, denoting $\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x + y_j'))$ and $\hat{W}_{\varepsilon_j}(x) = W(\varepsilon_j(x + y_j'))$,

$$-i\alpha \cdot \nabla v_j + \big(1 + \hat{V}_{\varepsilon_j}(x)\big)\beta v_j = \hat{W}_{\varepsilon_j}(x)|v_j|^{p-2}v_j \tag{5.12}$$

with energy

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j) = \frac{p-2}{2p}\int_{\mathbb{R}^3} \hat{W}_{\varepsilon_j}(x)|v_j|^p.$$

Plainly,

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}(v_j) = \Phi_{\varepsilon_j}(u_j) = c_{\varepsilon_j}.$$

We may assume additionally $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1,3)$ with $v \ne 0$ by (5.11).

Since V and W are bounded, without loss of generality, we assume $V(\varepsilon_j y_j') \to V_0$ and $W(\varepsilon_j y_j') \to W_0$ as $j \to \infty$. By virtue of the boundedness of ∇V and ∇W one sees that $\hat{V}_{\varepsilon_j}(x) \to V_0$ and $\hat{W}_{\varepsilon_j}(x) \to W_0$ as $j \to \infty$ uniformly on any bounded set of x. Consequently, by (5.12), for any $\varphi \in C_0^\infty$,

$$\begin{aligned}0 &= \lim_{j\to\infty}\int_{\mathbb{R}^3}\big\langle -i\alpha\cdot\nabla v_j + (1+V_{\varepsilon_j}(x))\beta v_j - \hat{W}_{\varepsilon_j}(x)|v_j|^{p-2}v_j,\, \varphi\big\rangle \\ &= \int_{\mathbb{R}^3}\big\langle -i\alpha\cdot\nabla v + (1+V_0)\beta v - W_0|v|^{p-2}v,\, \varphi\big\rangle,\end{aligned}$$

which implies that v solves

$$-i\alpha\cdot\nabla v + (1+V_0)\beta v = W_0|v|^{p-2}v \tag{5.13}$$

with the energy

$$\Gamma_{V_0W_0}(v) = \frac{p-2}{2p}\int_{\mathbb{R}^3} W_0|v|^p \ge \gamma_{V_0W_0}.$$

By a Fatou's Lemma,

$$\int_{\mathbb{R}^3} W_0|v|^p \le \liminf_{j\to\infty}\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)|v_j|^p$$

which, jointly with Lemma 2.5, implies that

$$\Gamma_{V_0W_0}(v) \le \liminf_{j\to\infty} c_{\varepsilon_j} \le \gamma_{V_0W_0}$$

(recalling that $\hat{\Phi}_{\varepsilon_j}$ and Φ_{ε_j} have the same least energy c_{ε_j}). Therefore,

$$\lim_{j\to\infty} c_{\varepsilon_j} = \Gamma_{V_0W_0}(v) = \gamma_{V_0W_0}. \tag{5.14}$$

Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \le 1$, $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{v}_j(x) = \eta(2|x|/j)v(x)$. One has

$$\|v - \tilde{v}_j\| \to 0 \quad\text{and}\quad |v - \tilde{v}_j|_q \to 0 \quad\text{as } j \to \infty \tag{5.15}$$

for $q \in [2,3]$. Setting $z_j = v_j - \tilde{v}_j$, it is not difficult to verify by applying a Brézis-Lieb's argument ([36]) that along a subsequence,

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)\big(|v_j|^p - |z_j|^p - |\tilde{v}_j|^p\big)\right| = 0 \tag{5.16}$$

and

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)\big(|v_j|^{p-2}v_j-|z_j|^{p-2}z_j-|\tilde{v}_j|^{p-2}\tilde{v}_j\big)\varphi\right|=0 \tag{5.17}$$

uniformly in $\varphi\in E$ with $\|\varphi\|\le 1$. Using the exponential decay of v, (5.15), and the facts that $\hat{V}_{\varepsilon_j}(x)\to V_0$, $\hat{W}_{\varepsilon_j}(x)\to W_0$ as $j\to\infty$ uniformly on any bounded set of x, one checks easily the following

$$\int_{\mathbb{R}^3}\hat{V}_{\varepsilon_j}(x)v_j\tilde{v}_j\to\int_{\mathbb{R}^3}V_0|v|^2 \tag{5.18}$$

and

$$\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)|\tilde{v}_j|^p\to\int_{\mathbb{R}^3}W_0|v|^p, \tag{5.19}$$

consequently,

$$\begin{aligned}\hat{\Phi}_{\varepsilon_j}(z_j)=&\,\hat{\Phi}_{\varepsilon_j}(v_j)-\Gamma_{V_0W_0}(v)\\&+\frac{1}{p}\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)\big(|v_j|^p-|z_j|^p-|\tilde{v}_j|^p\big)+o(1)\\=&\,o(1)\end{aligned}$$

as $j\to\infty$, which implies that

$$\hat{\Phi}_{\varepsilon_j}(z_j)\;\to\;0 \tag{5.20}$$

as $j\to\infty$. Similarly, by (5.17),

$$\hat{\Phi}'_{\varepsilon_j}(z_j)\varphi=\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}(x)\big(|v_j|^{p-2}v_j-|z_j|^{p-2}z_j)-|\tilde{v}_j|^{p-2}\tilde{v}_j\big)\varphi+o(1)=o(1)$$

as $j\to\infty$ uniformly in $\|\varphi\|\le 1$, which implies that

$$\hat{\Phi}'_{\varepsilon_j}(z_j)\;\to\;0 \tag{5.21}$$

as $j\to\infty$. Now one has

$$\hat{\Phi}_{\varepsilon_j}(z_j)-\frac{1}{2}\hat{\Phi}'_{\varepsilon_j}(z_j)z_j=\Big(\frac{1}{2}-\frac{1}{p}\Big)\int_{\mathbb{R}^3}\hat{W}_{\varepsilon_j}|z_j|^p$$

which, jointly with (5.20) and (5.21), shows that $\|z_j\|\to 0$ as $j\to\infty$. This, together with (5.15), implies $v_j\to v$ in E. Note that by (5.12) and (5.13)

$$Az_j=\hat{W}_{\varepsilon_j}(x)f(|v_j|)v_j-W_0f(|u|)u-\big(\hat{V}_{\varepsilon_j}(x)\beta v_j-V_0\beta u\big).$$

It yields $|Az_j|_2 \to 0$. Therefore $v_j \to u$ in H^1.

Step 2) $v_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$. Assume by contradiction that the conclusion does not hold. This, jointly with the sub-solution estimate (5.8), implies that there exist $\sigma > 0$, $x_j \in \mathbb{R}^3$ with $|x_j| \to \infty$, and $C_0 > 0$ independent of j such that $\sigma \le |v_j(x_j)| \le C_0\big(\int_{B_1(x_j)} |v_j|^2\big)^{1/2}$. Since $v_j \to v$ in H^1 one gets

$$\sigma \le C_0\Big(\int_{B_1(x_j)} |v_j|^2\Big)^{1/2} \to 0,$$

a contradiction.

Step 3) $\{\varepsilon_j y'_j\}_j$ is bounded. Assume by contradiction that $\varepsilon_j|y'_j| \to \infty$. Then $V_0 \ge \tau_w$ and $W_0 < \kappa$ so $\gamma_{V_0W_0} > \gamma_{\tau_w\kappa}$ by Lemma 4.3. However, by (5.14) and (5.10), $c_{\varepsilon_j} \to \gamma_{V_0W_0} \le \gamma_{\tau_w\kappa}$, a contradiction. Therefore, we can assume $\varepsilon_j y'_j \to y_0$, $V_0 = V(y_0)$ and $W_0 = W(y_0)$ which, together with (5.13), implies that v is a least energy solution of (1.7). Now by *Step 2* it is easy to see that one can assume that $y_j = y'_j$ is a maximum point of $|u_j|$.

Step 4) $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{A}) = 0$. It is sufficient to check that $y_0 \in \mathscr{A}$. Assume indirectly that $y_0 \notin \mathscr{A}$. Then it is easy to check by the definition of $\mathscr{A}$ that $\gamma_{V(y_0)W(y_0)} > \gamma_{\tau_w\kappa}$, which, together with (5.14) and (5.10), implies

$$\lim_{\varepsilon\to 0} c_\varepsilon = \gamma_{V(y_0)W(y_0)} > \gamma_{\tau_w\kappa} = \lim_{\varepsilon\to 0} c_\varepsilon,$$

a contradiction. Finally, assuming in addition that $\mathscr{V} \cap \mathscr{W} \ne \emptyset$, one has $\mathscr{A} = \mathscr{V} \cap \mathscr{W}$, so $\lim_{\varepsilon\to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{V} \cap \mathscr{W}) = 0$ and v_ε converges in H^1 to a least energy solution of (1.8).

Step 5) There is $C > 0$ such that for all ε small

$$|u_\varepsilon(x)| \le Ce^{-\sqrt{\sigma/4}|x-y_\varepsilon|} \qquad \forall\, x \in \mathbb{R}^3. \tag{5.22}$$

It suffices to verify this for sequences. By *Step 2* and (5.9) we may take $\delta > 0$ and $R > 0$ such that $|v_j(x)| \le \delta$ and

$$\Delta|v_j| \ge \frac{\sigma}{4}|v_j|$$

for all $|x| \ge R, j \in \mathbb{N}$. Let $\Gamma(y) = \Gamma(y, 0)$ be a fundamental solution to $-\Delta + \sigma/4$. Using the uniform boundedness, one may choose Γ so that $|v_j(y)| \le \frac{\sigma}{4}\Gamma(y)$ holds on $|y| = R$, all $j \in \mathbb{N}$. Let $\tilde{z}_j = |v_j| - \frac{\sigma}{4}\Gamma$. Then we obtain

$$\Delta\tilde{z}_j = \frac{\sigma}{4}\tilde{z}_j.$$

By the maximum principle we can conclude that $\tilde{z}_j(y) \le 0$ on $|y| \ge R$. It is well known that there is $C' > 0$ such that $\Gamma(y) \le C' \exp(-\sqrt{\sigma/4}|y|)$ on $|y| \ge 1$. We see that

$$|v_j(y)| \le C \exp(-\sqrt{\sigma/4}|y|)$$

for all $y \in \mathbb{R}^3$ and all $j \in \mathbb{N}$, that is,

$$|u_j(x)| \le C \exp(-\sqrt{\sigma/4}|x - y_j|)$$

for all $x \in \mathbb{R}^3$ and all $j \in \mathbb{N}$.

The proof is hereby complete. □

6 Proofs of main results: the critical case

Setting $u(x) = w(\varepsilon x)$, the equation (1.4) is equivalent to the following

$$(6.1) \qquad -i\alpha \cdot \nabla u + \beta u + M_\varepsilon(x)u = W_{1\varepsilon}(x)|u|^{p-2}u + W_{2\varepsilon}(x)|u|u.$$

Proof of Theorem 1.4. We may assume that $0 \in \mathscr{V}$, $x_v = 0$ and $\tau = V(0)$, $\kappa_{jv} = W_j(0)$. Solutions of (6.1) are critical points of the functional $\Phi_\varepsilon^*(u) := \Phi_\varepsilon^{\tau\vec{\kappa}_v}(u)$ with $\vec{\kappa}_v = (\kappa_{1v}, \kappa_{2v})$. Denote $\Phi_0^*(u) = \gamma_{\tau\vec{\kappa}_v}$. We will adopt an argument different from that of Theorem 1.1. Let $u \in \mathscr{R}_{\tau\vec{\kappa}_v}$ be a solution of (4.2) with $\mu = \tau$ and $\vec{\nu} = \vec{\kappa}_v$. Define u_r, $u_{rj}, j = 0, ..., m-1$, and set E_m as before.

By virtue of Lemma 2.5, let $t_{\varepsilon rj} > 0$ be such that $t_{\varepsilon rj}u_{rj}^+ \in \mathscr{N}_\varepsilon$. Observe that

$$\lim_{\varepsilon\to0}\lim_{r\to\infty} t_{\varepsilon rj} = \lim_{\varepsilon\to0} t_{\varepsilon j} = 1,$$

$$\lim_{\varepsilon\to0}\lim_{r\to\infty} h_\varepsilon(t_{\varepsilon rj}u_{rj}^+) = \lim_{\varepsilon\to0} h_\varepsilon(t_{\varepsilon j}u_{rj}^+) = h_0(u^+) = u^-,$$

$$\lim_{\varepsilon\to0}\lim_{r\to\infty} \|h_\varepsilon(t_{\varepsilon rj}u_{rj}^+) - t_{\varepsilon rj}u_{rj}^-)\| = \lim_{\varepsilon\to0} \|h_\varepsilon(t_{\varepsilon j}u^+) - t_{\varepsilon j}u^-\| = 0.$$

One checks easily the following

$$\begin{aligned}
\max_{w\in E_m} \Phi_\varepsilon^*(w) &= \Phi_\varepsilon^*\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)\Big)\\
&= \Phi_\varepsilon^*\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-\Big) + o(1_r)\\
&= \Phi_\varepsilon^*\Big(\sum_{j=0}^{m-1} t_{\varepsilon j}u_{rj}\Big) + o(1_r)\\
&= \sum_{j=0}^{m-1} \Phi_\varepsilon^*(t_{\varepsilon j}u_{rj}) + o(1_r)\\
&= \sum_{j=0}^{m-1} \Phi_\varepsilon^*(t_{\varepsilon j}u_{rj}^+ + t_{\varepsilon j}u_{rj}^-) + o(1_r)\\
&= \sum_{j=0}^{m-1} \Phi_\varepsilon^*(t_{\varepsilon j}u_{rj}^+ + h_\varepsilon(t_{\varepsilon j}u_{rj}^+)) + o(1_r)\\
&= \sum_{j=0}^{m-1} \Phi_0^*(t_{0j}u_{rj}^+ + h_0(t_{0j}u_{rj}^+)) + o(1_{r\varepsilon})\\
&= \sum_{j=0}^{m-1} \Phi_0^*(u) + o(1_{r\varepsilon})\\
&= m\,\gamma_{\tau\vec{\kappa}_v} + o(1_{r\varepsilon})
\end{aligned}$$

where $o(1_r)$ means arbitrary small as $r \to \infty$, and $o(1_{r\varepsilon})$ means arbitrary small as r sufficiently large and ε sufficiently small.

Now, by assumptions and Lemma 4.10, for any $0 < \delta < \gamma_{\vec{\infty}} - m\,\gamma_{\tau\vec{\kappa}_v}$, one may choose $r > 0$ large and then $\mathcal{E}_m > 0$ small such that, for all $\varepsilon \le \mathscr{E}_m$, $\max_{w\in E_m} \Phi_\varepsilon^*(w) \le \gamma_{\vec{\infty}} - \delta$. Theorem 2.6 applies. □

Proof of Theorem 1.5. Assume $0 \in \mathscr{W} := \mathscr{W}_1 \cap \mathscr{W}_2$, $x_w = 0$ and $\tau_w = V(0)$, $\vec{\kappa} = (\kappa_1, \kappa_2) = (W_1(0), W_2(0))$. Solutions of (6.1) are critical points of the functional $\Phi_\varepsilon^*(u) := \Phi_\varepsilon^{\tau_w\vec{\kappa}}(u)$. Denote $\Phi_0^*(u) = \gamma_{\tau_w\vec{\kappa}}$. Let $u \in \mathscr{R}_{\tau_w\vec{\kappa}}$ be a solution of (4.2) with $\mu = \tau_w$ and $\vec{\nu} = \vec{\kappa}$. Define u_r, $u_{rj}, j = 0, ..., m-1$, and set E_m as before. Then one finds

$$\max_{w\in E_m} \Phi_\varepsilon^*(w) = m\,\gamma_{\tau_w\vec{\kappa}} + o(1_{r\varepsilon}).$$

By assumptions and Lemma 4.10, for any $0 < \delta < \gamma_{\vec{\infty}} - m\,\gamma_{\tau_w\vec{\kappa}}$, one may choose $r > 0$ large and then $\mathcal{E}_m > 0$ small such that, for all $\varepsilon \le \mathscr{E}_m$, $\max_{w\in E_m} \Phi_\varepsilon^*(w) \le \gamma_{\vec{\infty}} - \delta$. Theorem 2.6 applies. $\square$

Proof of Theorem 1.6. We deal with again the case (2) of (P_1) in (Q_3). The proof will be carried out along the lines of proof of Theorem 1.3. So we are sketchy.

Given arbitrarily a sequence $\varepsilon_j \to 0$ as $j \to \infty$, let $u_j \in \mathscr{K}_j \equiv \mathscr{K}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. It is not difficult, by (Q_1), to check that $\{u_j\}$ is non-varnishing. Therefore there exist a sequence $\{y_j'\} \subset \mathbb{R}^3$ and constants $r > 0, \delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B_r(y_j')} |u_j|^2 \ge \delta.$$

Set $v_j(x) = u_j(x + y_j')$. Then v_j solves, denoting $\hat{V}_{\varepsilon_j}(x) = V(\varepsilon_j(x + y_j'))$ and $\hat{W}_{k\varepsilon_j}(x) = W_k(\varepsilon_j(x + y_j'))$ for $k = 1, 2$,

$$-i\alpha \cdot \nabla v_j + (1 + \hat{V}_{\varepsilon_j}(x))\beta v_j = \hat{W}_{1\varepsilon_j}(x)|v_j|^{p-2}v_j + \hat{W}_{2\varepsilon_j}(x)|v_j|v_j \tag{6.2}$$

with the associated energy functional (denoted by $\hat{\Phi}_{\varepsilon_j}^*$) and least energy

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}^*(v_j) = \int_{\mathbb{R}^3} \bar{H}(\varepsilon_j x, |v_j|)$$

where $\bar{H}(\varepsilon_j x, |u|) = \frac{p-2}{2p}\hat{W}_{1\varepsilon_j}(x)|u|^p + \frac{1}{6}\hat{W}_{2\varepsilon_j}(x)|u|^3$. Plainly,

$$\hat{c}_{\varepsilon_j} = \hat{\Phi}_{\varepsilon_j}^*(v_j) = \Phi_{\varepsilon_j}^*(u_j) = c_{\varepsilon_j}.$$

Additionally, $v_j \rightharpoonup u \ne 0$ in E. We can assume that $V(\varepsilon_j y_j') \to V_0$ and $W_K(\varepsilon_j y_j') \to W_{k0}$ as $j \to \infty$. One sees easily that u solves

$$-i\alpha \cdot \nabla u + (1 + V_0)\beta u = W_{10}|u|^{p-2}u + W_{20}|u|u \tag{6.3}$$

with the energy $\Gamma_{V_0\vec{W}_0}(u) = \int_{\mathbb{R}^3} \bar{H}(0, |u|) := \int_{\mathbb{R}^3} \frac{p-2}{2p}\hat{W}_{10}|u|^p + \frac{1}{6}\hat{W}_{20}|u|^3 \ge \gamma_{V_0\vec{W}_0}$. The Fatou lemma yilds

$$\lim_{j\to\infty} c_{\varepsilon_j} = \Gamma_{V_0\vec{W}_0}(u) = \gamma_{V_0\vec{W}_0} \tag{6.4}$$

and

$$\lim_{j\to\infty} \int_{\mathbb{R}^3} \bar{H}(\varepsilon_j x, |v_j|) = \int_{\mathbb{R}^3} \bar{H}(0, |u|) = \gamma_{V_0\vec{W}_0}.$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(s) = 1$ if $s \le 1$, $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{v}_j(x) = \eta(2|x|/j)u(x)$. One has $\|u - \tilde{v}_j\| \to 0$

and $|u-\tilde{v}_j|_s \to 0$ as $j\to\infty$ for $s\in[2,3]$. Setting $z_j = v_j - \tilde{v}_j$, one checks easily that, along a subsequence (see [13, 36]),

$$\hat{\Phi}^*_{\varepsilon_j}(z_j) = \hat{\Phi}^*_{\varepsilon_j}(v_j) - \Gamma_{V_0\vec{W}_0}(u) + o(1)$$

as $j\to\infty$, that is, $\hat{\Phi}^*_{\varepsilon_j}(z_j)\to 0$. Similarly, $\hat{\Phi}^{*\prime}_{\varepsilon_j}(z_j)\to 0$. Now a standard argument yields $v_j\to u$ in E as $j\to\infty$. Observe that by (6.2) and (6.3)

$$\begin{aligned} A_0 z_j = &-\big(\hat{V}_{\varepsilon_j}(x)\beta v_j - V_0\beta u\big)\\ &+\big(\hat{W}_{1\varepsilon_j}(x)|v_j|^{p-2}v_j - W_{10}|u|^{p-2}u\big) + \big(\hat{W}_{2\varepsilon_j}(x)|v_j|v_j - W_{20}|u|u\big). \end{aligned}$$

By the exponential decay of u and Lemma 2.2, it is easy to show that $|A_0 z_j|_2 \to 0$. Therefore $v_j\to u$ in H^1.

As before, one checks that $v_j(x)\to 0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$, and $\{\varepsilon_j y_j'\}_j$ is bounded. Clearly, one may assume that $y_j = y_j'$ is a maximum point of $|u_j|$.

In order to show $\lim_{\varepsilon\to 0}\operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{A}) = 0$, assuming $y_j\to y_0$ it is sufficient to check that $y_0\in\mathscr{A}$. Denote $\vec{W} = (W_1, W_2)$, By (P_3), $\gamma_{\tau_w\vec{\kappa}}$ is archived, it hence follows that

$$\lim_{j\to\infty} c_{\varepsilon_j} = \lim_{j\to\infty} c^{\tau_w\vec{\kappa}}_{\varepsilon_j} \le \gamma_{V(0)\vec{W}(0)} = \gamma_{\tau_w\vec{\kappa}},$$

which, together with (6.4), shows

$$\gamma_{V(y_0)\vec{W}(y_0)} \le \gamma_{\tau_w\vec{\kappa}}.$$

Since $\vec{W}(y_0)\le\vec{\kappa}$ one has $V(y_0)\le\tau_w$. If $\vec{\kappa} = \vec{W}(y_0)$, i.e., $y_0\in\mathscr{W}$, there must be $V(y_0)=\tau_w$. If $\vec{W}(y_0)<\vec{\kappa}$ then we must have $V(y_0)<\tau_w$. In conclusion, $y_0\in\mathscr{A}_w$.

Finally, the argument of *Step 5)* of proof of Theorem 1.3 yields that there exists $C>0$ such that for all $j\in\mathbb{N}$

$$|u_j(x)| \le Ce^{-\sqrt{\omega/2}|x-y_j|}$$

where $\omega = a^2-\tau^2$.

The proof is completed. □

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References

[1] N. Ackermann, A nonlinear superposition principle and multibump solutions of peridic Schrödinger equations. *J. Funct. Anal.* **234** (2006) 423-443.

[2] N. Ackermann, On a periodic Schrą̆godinger equation with nonlocal superlinear part. *Math. Z.* **248** (2014), 423-443.

[3] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schödinger equations. *Arch. Rat. Mech. Anal.* **140** (1997) 285-300.

[4] A. Ambrosetti and A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbb{R}^n$. (Birkhäuser Verlag, Basel, 2006).

[5] M. Balabane, T. Cazenave, A. Douady and F. Merle, Existence of excited states for a nonlinear Dirac field. *Comm. Math. Phys.* **119** (1988) 153-176.

[6] T. Bartsch and Y.H. Ding, Solutions of nonlinear Dirac equations. *J. Differential Equations* **226** (2006) 210-249.

[7] T. Bartsch, Y.H. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory. *Math. Nachr.* **279** (2006), 1267-1289.

[8] J. Byeon and L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity. *Arch. Ration. Mech. Anal.* **185** (2007) 185–200.

[9] J. Byeon and Z.Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations, II. *Calc. Var. Partial Differential Equations* **18** (2003) 207-219.

[10] T. Cazenave and L. Vazquez, Existence of local solutions for a classical nonlinear Dirac field. *Comm. Math. Phys.* **105** (1986) 35-47.

[11] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology.* Vol. 3. (Springer-Verlag, Berlin, 1990).

[12] M. del Pino and P. Felmer, Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* **324** (2002) 1-32.

[13] Y.H. Ding, *Variational Methods for Strongly Indefinite Problems*. Interdisciplinary Mathematical Sciences, 7. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[14] Y.H. Ding, Semi-classical ground states concerntrating on the nonlinear potential for a Dirac equation. *J. Differential Equations* **249** (2010) 1015-1034.

[15] Y.H. Ding and X.Y. Liu, Semi-classical limits of ground states of a nonlinear Dirac equation. *J. Differential Equations* **252** (2012), 4962-4987.

[16] Y.H. Ding and B. Ruf, Solutions of a nonlinear Dirac equation with external fields. *Arch. Ration. Mech. Anal.* **190** (2008) 1007-1032.

[17] Y.H. Ding and B. Ruf, Existence and concentration of semiclassical solutions for Dirac equations with critical nonlinearities. *SIAM J. Math. Anal.* **44** (2012), 3755-3785.

[18] Y.H. Ding and J.C. Wei, Stationary states of nonlinear Dirac equations with general potentials. *Rev. Math. Phys.* **20** (2008) 1007-1032.

[19] Y.H. Ding and J.C. Wei, Multiplicity of semiclassical solutions to nonlinear Schrą̆odinger equations. *J. Fixed Point Theory Appl.* DOI 10.1007/s11784-017-0410-8 (2017)

[20] Y.H. Ding and T. Xu, Localized concentration of semi-classical states for nonlinear Dirac equations. *Arch. Ration. Mech. Anal.* **216** (2015), no. 2, 415 - 447.

[21] M.J. Esteban and E. Séré, Stationary states of the nonlinear Dirac equation: a variational approach. *Comm. Math. Phys.* **171** (1995) 323-350.

[22] M.J. Esteban, M. Lewin and E. Séré, Variational methods in relativistic quantum mechanics. *Bull. Amer. Math. Soc.* **45** (2008) 535-593.

[23] R. Finkelstein, C.F. Fronsdal and P. Kaus, Nonlinear spinor field theory. *Phys. Rev.* **103** (1956) 1571-1579.

[24] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **69** (1986) 397-408.

[25] D. Gilbarg and M.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. (Springer, Berlin, 1983).

[26] C. Gui, Existence of milti-bump solutions for nonlinear Schrödinger equations via variational method. *Comm. Part. Diff. Eqs.* **21** (1996) 787-820.

[27] E. H. Lieb; M. Loss, *Analysis*, Graduate Studies in Mathematics, AMS, Providence, RI, 2001.

[28] F. Merle, Existence of stationary states for nonlinear Dirac equations. *J. Differential Equations* **74** (1988) 50-68.

[29] W.M. Ni and J.C. Wei, On positive solutions concentration on spheres for the Gierer-Meinhardt system. *J. Differential Equations* **221** (2006) 158-189.

[30] Y.G. Oh, On positive multi-lumb bound states of nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* **131** (1990) 223-253.

[31] A. Pankov, Periodic nonlinear Schrödinger equations with application to photonic crystals. *Milan J. Math.* **73** (2005) 259-287.

[32] P. Rabinowitz, On a class of nonlinear Schrödinger equations. *Z. Angew Math. Phys.* **43** (1992) 270-291.

[33] A.F. Ranada, *Classical Nonlinear Dirac Field Mmodels of Extended Particles. In Quantum Theory, Groups, Fields and Particles* (ed. Barut, A.O.). (Reidel, Amsterdam, 1982).

[34] A. Szulkin and T. Weth, Ground state solutions for some indefinite problems. *J. Funct. Anal.* **257** (2009) 3802-3822.

[35] B. Thaller, *The Dirac Equation. Texts and Monographs in Physics*, (Springer, Berlin, 1992).

[36] M. Willem, *Minimax Theorems.* (Birkhäuser, 1996).