$$I = \iint_D (1-x^2)^{\frac{3}{2}}\, dx\, dy, \qquad \text{where } D = \{(x,y) \mid x^2+y^2 \le 1\}$$

$$= \left\{(r,\theta) \;\middle|\; \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$I = \int_0^{2\pi}\int_0^1 (1-r^2\cos^2\theta)^{\frac{3}{2}}\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi}\int_0^1 (1-r^2\cos^2\theta)^{\frac{3}{2}}\, r\, dr\, d\theta \qquad (\text{symmetry of } \cos^2\theta)$$

Note:

$$\int_0^1 (1-r^2\cos^2\theta)^{\frac{3}{2}}\, r\, dr \overset{r^2=t}{=\!=\!=} \frac{1}{2}\int_0^1 (1-t\cos^2\theta)^{\frac{3}{2}}\, dt$$

$$= \frac{1}{2}\times\frac{2}{5}\times(1-t\cos^2\theta)^{\frac{5}{2}}\times\frac{1}{-\cos^2\theta}\Bigg|_{t=0}^{t=1}$$

$$= \frac{1}{5}\left(\frac{1}{\cos^2\theta} - \frac{(1-\cos^2\theta)^{\frac{5}{2}}}{\cos^2\theta}\right)$$

$$= \frac{1}{5}\left(\sec^2\theta - \frac{\sin^5\theta}{\cos^2\theta}\right)$$

$$\int \sec^2\theta\, d\theta = \tan\theta + C$$

$$\int \frac{\sin^5\theta}{\cos^2\theta}\, d\theta \underset{du=-\sin\theta d\theta}{\overset{u=\cos\theta}{=\!=\!=}} -\int \frac{\sin^4\theta}{\cos^2\theta}\, d(\cos\theta) = -\int \frac{(1-u^2)^2}{u^2}\, du$$

$$= -\int\left(\frac{1}{u^2} - 2 + u^2\right) du = \frac{1}{u} + 2u - \frac{1}{3}u^3$$

$$= \frac{1}{\cos\theta} + 2\cos\theta - \frac{1}{3}\cos^3\theta$$

$$I = 2\int_0^{\pi}\frac{1}{5}\left[\sec^2\theta - \frac{\sin^5\theta}{\cos^2\theta}\right] d\theta = \frac{2}{5}\left[\tan\theta - \frac{1}{\cos\theta} - 2\cos\theta + \frac{1}{3}\cos^3\theta\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{5}\left[0 + 2 + 4 - \frac{2}{3}\right] = \frac{32}{15}$$