Example 1: Find the points on the surface $xy - z^2 + 1 = 0$ that are closest to the origin.

Solution: min $\quad w = x^2 + y^2 + z^2$

subject to $\quad xy - z^2 + 1 = 0$

Lagrange multiplier:

$xy - z^2 + 1 = 0$ — ①

$2x = \lambda y$ — ②

$2y = \lambda x$ — ③

$2z = -\lambda 2z$ — ④

From ② & ③ $\Rightarrow x^2 = y^2 \Rightarrow$ either $x = y$ or $x = -y$

Case 1: $x = y$

From ④ $\Rightarrow$ either $z = 0$ or $\lambda = 1$

If $z = 0$, then $xy - z^2 + 1 = 0 \Rightarrow x^2 + 1 = 0$, not possible

If $\lambda = 1$, then go back to ② $\Rightarrow 2x = x \Rightarrow x = 0$. So $y = x = 0$

go back to ① $\Rightarrow z = \pm 1$

We obtain two critical points $(0, 0, \pm 1)$

Case 2. $x = -y$

From ④ $\Rightarrow$ $z=0$ or $\lambda = 1$

If $z=0$, then $xy - z^2 + 1 = 0 \Rightarrow -x^2+1 = 0 \Rightarrow x = \pm 1$

two critical points $(\pm 1, \mp 1, 0)$

If $\lambda = 1$, then go back to ② $\Rightarrow$ $2x = -x \Rightarrow x = 0$. So $y=0$

go back to ① $\Rightarrow$ $z = \pm 1$

two critical points $(0, 0, \pm 1)$

Altogether we get 4 critical points

$(0,0,\pm 1)$, $(\pm 1, \mp 1, 0)$

so $\min w = \min \left(w(0,0,\pm 1), w(\pm 1, \mp 1, 0) \right)$

$= \min (1, 2) = 2$

min distance is $d = \sqrt{\min w} = \sqrt{1}$.

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