Example:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Compute f_x, f_y

Solution: If $(x,y) \neq (0,0)$, then

$$f = \frac{xy}{x^2+y^2}$$

$$f_x = \frac{y}{x^2+y^2} - \frac{xy \cdot 2x}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$f_y = \frac{x}{x^2+y^2} - \frac{xy \cdot 2y}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

If $(x,y) = (0,0)$, then

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0 \cdot h}{h^2+0^2} - 0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\frac{0 \cdot k}{0^2+k^2} - 0}{k} = 0$$

So

$$f_x = \begin{cases} \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_y = \begin{cases} \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Check: f_x, f_y not continuous at $(0,0)$