Introduction to Parabolic Gluing Methods

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Summer School on Calculus of Variations

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Brezis-Nirenberg Problem

[Brezis, Haim; Nirenberg, Louis Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Comm. Pure Appl. Math. 36 (1983), no. 4, 437-477]

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the following problem is considered

$$\begin{cases} \Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0, u > 0 \text{ in } \Omega; \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

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Theorem (Brezis-Nrenberg (1983)): If $n \geq 4, 0 < \lambda < \lambda_1$; $n = 3, 0 < \lambda_* < \lambda < \lambda_1$, then the least energy solution exists.

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$n = 3, 0 < \lambda_* < \lambda < \lambda_1$, then the least energy solution exists.

Dimension 3 is different!

Analysis of bubbling solutions (Han (1989), Bahri-Li-Rey (1995), Druet, Musso, Pistoia, Premoselli, Robert,, ...); Construction of bubbling solutions (del Pino, Musso, Pistoia, Premoselli, Robert, Rey, Wei, ...)

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Extensions to Yamabe problems, sign-changing solutions, p-Laplacian, Hardy-Sobolev terms, Henon equations, nonlinear Schrodinger equations, nonlinear Dirac, $curl \times curl$, Moser-Trudinger equation, bi-harmonic, nonlinear elliptic systems, fractional laplacians, Schrodinger-Poisson, Kirchoff, Hartree-Choquard, ...

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What about Parabolic Brezis-Nirenberg?

$$\Delta u - u_t + |u|^{\frac{4}{n-2}} u = 0 \text{ in } \Omega \times (0, T)$$

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- The aims of my talk are two-folds: First we want to develop a parabolic gluing methods for constructing bubbling solutions to parabolic Brezis-Nirenberg problem
- Second, we want to develop analytical tools to analyze possible Loss of Compactness for parabolic blow-up problems.
  Elliptic: R. Schoen,..., YY Li, Shafrir, L.Zhang, M.Zhu, Druet, Premoselli, Robert, Hebey, ...

Outline of Lecture Series

- 1. An Overview of Blow-up for Fujita Equation

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$$p = \frac{n+2}{n-2}, n = 5, 6$$

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- 4. Parabolic Gluing Method III—Distorted Fourier Transforms

$$u_t = \Delta u + e^u \text{ in } \mathbb{R}^2$$

References for Parabolic Gluing Method

Three basic references for parabolic gluing method:

1. L^2 case: Carmen Cortazar, Manuel del Pino and Monica Musso, Greens function and infinite-time bubbling in the critical nonlinear heat equation, Journal of the European Mathematical Society, 22(1):283344, 2020.

2. (L^2 and) Non-L^2 case: Manuel del Pino, Juan Davila and Juncheng Wei, Singularity formation for the two-dimensional harmonic map flow into S^2, Inventione Mathematicae 219 (2020), no.2, 345466.

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www.math.ubc.ca/ jcwei/teachings.html

Lecture I. An overview of blow-up for nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u, p > 1, x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \in L^\infty(\mathbb{R}^n). \end{cases} \quad (0.1)$$

- Local well-posedness: standard parabolic theory. (Optimal space $u_0 \in L^{\frac{n(p-1)}{2}}$ Mizoguchi-Souplet (Adv in Math. 2019))

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- Local well-posedness: standard parabolic theory. (Optimal space $u_0 \in L^{\frac{n(p-1)}{2}}$ Mizoguchi-Souplet (Adv in Math. 2019))
- Blow up in finite time T:

$$\lim_{t \to T} \limsup \|u(\cdot,t)\|_{L^\infty} = +\infty$$

.

Fujita (1966), Quittner-Souplet ...

$$p_{FJ} = \frac{n+2}{n}$$

$$\lim_{t\to T} \limsup \|u(\cdot,t)\|_{L^\infty} = +\infty$$

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- $u_t = |u|^{p-1}u$ blows up at finite time

$$u = A_p(T-t)^{\frac{1}{p-1}}$$

- **Type I** if there exists a constant C such that for any $t < T$,

$$C^{-1}(T-t)^{-\frac{1}{p-1}} \le \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C(T-t)^{-\frac{1}{p-1}},$$

(this is like $u_t = |u|^{p-1}u$,)

- Otherwise it is called **Type II**:

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$

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- Otherwise it is called **Type II**:
$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = +\infty.$$
- If $T = +\infty$, blow-up at infinity.

Classification of Blow-ups

$$\partial_t u - \Delta u = |u|^{p-1}u, p > 1, x \in \mathbb{R}^n$$

- When $p < \frac{n+2}{n-2}$ (Sobolev subcritical), all blow ups are Type I. (Giga-Kohn (CPAM 1987) (positive solutions), Giga-Matsui-Sasayama (IUMJ 2014) (sign-changing solutions)...)

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$$\tau = -\log(T-t), y = \frac{x}{\sqrt{T-t}}$$

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$$\tau = -\log(T-t), y = \frac{x}{\sqrt{T-t}}$$
- $p = \frac{n+2}{n-2}$, energy-critical. More difficult.

Classification of Blow-ups (Critical Case)

$$\partial_t u - \Delta u = |u|^{\frac{4}{n-2}} u, x \in \mathbb{R}^n$$

- Filippas-Herrero-Velázquez [2000]: All blow-ups are of Type I, for positive radially decreasing solutions, $n \geq 3$

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$$\Delta U + U^{\frac{n+2}{n-2}} = 0, U > 0$$
- (Wang-Wei 2021-2024 (200 pages)): If $n \geq 7$, $p = \frac{n+2}{n-2}$ and $u_0 \geq 0$, or $n \geq 5, p = \frac{n+2}{n-2}, u_0 = u_0(|x|)$, then all (finite-time) blow ups are Type I.

Examples of Type II for $p \geq \frac{n+2}{n-2}$

$$\partial_t u - \Delta u = |u|^{\frac{4}{n-2}} u, x \in \mathbb{R}^n$$

Type II blow-up for $n \leq 6, p = \frac{n+2}{n-2}$

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- del Pino-Musso-Wei-Zhou (DCDS-A 2019): $n = 4, p = 3$, nonradial case, multiple bubbles

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  Inner-outer gluing approach

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- del Pino-Musso-Wei-Zhou-Zhang (2020): $n = 3, p = 5$, nonradial case

$$\|u\|_{L^\infty} \sim (T-t)^{-k}, k = 1, 2, 3, ...$$

Type II Blow-up for $p > \frac{n+2}{n-2}$

- Herrero-Velazquez (1997), $p > p_{JL}(n)$, radial case; Mizoguchi, Seki, ...
- Collot (MAMS 2017), $p > p_{JL}(n), p = 2m+1$, nonradial case; Collot-Merle-Raphael (JAMS 2020), $p > p_{JL}(n-1), n \geq 13$;
- Matano-Merle (JFA 2004, CPAM 2009), No Type II blow-up for $u = u(r)$, $\frac{n+2}{n-2} < p < p_{JL}(n)$;
- del Pino-Musso-Wei (JFA 2020), Type II blow-up for $\frac{n+2}{n-2} < p = \frac{n+1}{n-3} < p_{JL}(n)$;
- Lai-del Pino-Musso-Zhou-Wei (2021), Type II blow-up for $p = 3, 5 \leq n \leq 8$

Classification of Blow-ups when $p \geq \frac{n+2}{n-2}$

Based on the examples of Type II blow-ups, it is reasonable to

Conjecture 1: If $p = \frac{n+2}{n-2}$, all positive blow-ups are Type I, for $n \geq 3$

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Conjecture 2: If $p = \frac{n+2}{n-2}$, all solutions (sign-changing) blow-ups are Type I, for $n \geq 7$

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Conjecture 3: If $\frac{n+2}{n-2} < p < p_{JL}(n)$ and $p \neq \frac{m+2}{m-2}$, all solutions (sign-changing) blow-ups are Type I.

Collot-Raphael-Merle (JAMS 2020)

Blow-up at infinity when $p = p_S = \frac{n+2}{n-2}$

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- bubble towers at forward and backward time infinity: del Pino-Musso-Wei (Analysis PDE 2021) ($t = +\infty$), Wei-Sun-Zhang (CVPDE 2022) ($t = -\infty$)

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- Ageno-del Pino 2023: general domain, $n = 3$, non-radial positive solutions

Fila-King Program

Fila-King (2012) considered the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{0.2}$$

They first introduce the concept of the threshold solution. For any nonnegative, smooth function $u_0(x)$ with $u_0 \not\equiv 0$, let us define

$$\alpha^* = \alpha^*(u_0) := \sup\{\alpha > 0 : \ T_{\max}(\alpha u_0) = \infty\},$$

and $u^* := u(x, t; \alpha^* u_0)$ is called the threshold solution associated with u_0. Roughly speaking, the threshold solution lies on the borderline between global solutions and those that blow up in finite time since for $\alpha \gg \alpha^*$, the nonlinearity dominates the Laplacian and vice versa. At the threshold level, the dynamics for u^* in the pointwise sense might be global and bounded, global and unbounded, or blow up in finite time. Any of these might happen depending on the power nonlinearity and the domain.

Fila-King program concerns the threshold solution to above equation:
Conjecture (Fila-King 2012) For radial initial value u_0 satisfying

$$\lim_{r\to\infty} r^\gamma u_0(r) = A \quad \text{for some} \quad A > 0 \quad \text{and} \quad \gamma > \frac{n-2}{2}. \tag{0.3}$$

The threshold solution u of (0.2) with initial value u_0 should satisfy

$$\lim_{t\to\infty} \frac{\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}}{\varphi(t;n,\gamma)} = C$$

for some positive constant C depending on n and u_0, where $\varphi(t;n,\gamma)$ is given as:

	$\frac{n-2}{2} < \gamma < 2$	$\gamma = 2$	$\gamma > 2$
$n = 3$	$t^{\frac{\gamma-1}{2}}$ $1 < \gamma < 2$ partially [1]	$t^{\frac{1}{2}}(\ln t)^{-1}$ partially [1]	$t^{\frac{1}{2}}$ partially [1]
$n = 4$	$t^{-\frac{2-\gamma}{2}} \ln t$ trichotomy dynamics [2]	1 trichotomy dynamics [2]	$\ln t$ partially [3]
$n = 5$	$t^{-\frac{3(2-\gamma)}{2}}$ partially [4]	$(\ln t)^{-3}$ partially [4]	1 partially [4]

For $n \geq 6$ and $\gamma > \frac{n-2}{2}$, $\varphi(t; n, \gamma) = 1$.

[1] del Pino-Musso-Wei (Analysis PDE 2021)
[2] Wei-Zhang-Zhou (JDE 2024)
[3] Li-Wei-Zhang-Zhou (NA 2024)
[4] Wei-Zhang-Zhou (preprint, 2023)

Inner-Outer Parabolic Gluing Method

In the rest of the lectures, I shall introduce the Inner-Outer Parabolic Gluing Method.
The main idea is to extend the inner-outer gluing scheme in infinite dimensional reduction method which has been successfully used in many nonlinear elliptic equations to parabolic equation.

The Inner-Outer Gluing Scheme was first developed by del Pino-Kowalczyk-Wei (Annals of Mathematics 2011): Counterexample to De Giorgi's Conjecture for the Allen-Cahn

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^n$$

for dimensions $n \geq 9$.

Inner-Outer Gluing Scheme

Inner-Outer Gluing Scheme for elliptic problems: the idea is to "Zoom In" the interface region and decouple the whole nonlinear PDE problem into two problems: the inner problem, which is only solved near the interfaces, captures the essential geometric information of the interfaces; the outer problem, which is solved in the whole space, sums all global and external effects.

Key elements in the gluing: Fredholm and modulli space theory for elliptic operators

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Inner-Outer Gluing Scheme for parabolic problems: the idea is to "Zoom In" the singularity region and decouple the whole nonlinear parabolic problem into two problems: the inner problem, which is only solved near the singularities; the outer problem, which is solved in the whole space-time, sums all global and external effects. The reduced dynamics captures the local and nonlocal effects of the singularities.

Key elements in the gluing: linear theory for linearized parabolic operators.

Applications of Inner-Outer Parabolic Gluing Methods

in Singularities Formations of

Geometric

and Physical Flows

1. Harmonic Map Flows

Singularity formations in Harmonic Map Flows

$$u_t = \Delta u + |\nabla u|^2 u$$

for a function u defined on a subset of the plane with values in S^2:

$$u : \Omega \to S^2, \quad |u| = 1$$

where $\Omega \subset \mathbb{R}^2$ or two-dimensional Riemann surface.

Juan Davila, Manuel del Pino, Juncheng Wei, Singularity formation for the two-dimensional harmonic map flow into S^2. Inventione Mathematicae 219 (2020), no.2, 345466
Forward/backward blow-ups, spontaneous blow-ups.

2. Half-Harmonic Maps

2.1. Harmonic Map Flows with Free Boundary

$$\sqrt{u_t - \Delta}\, u \perp T_u \mathbb{S}^1$$

Y. Sire, J. Wei and Y. Zheng. Singularity formation in the harmonic map flow with free boundary American Journal of Mathematics 145 (2023), no. 4, 12731314.

Finite-time blowup

2.2. Half Harmonic Map Flow

$$u_t + \sqrt{-\Delta}\; u \perp T_u\mathbb{S}^1$$

Y. Sire, J. Wei and Y. Zheng, Infinite time blow-up for half-harmonic map flow from $\mathbb{R}$ into $\mathbb{S}^1$ American Journal of Mathematics 143(2021), no.4, 1261-1335.

Infinite time blowup

3.Yang-Mills Flows

Let $E \to M$ be a vector bundle with structure Lie group G over a four dimensional Riemannian manifold, for a time dependent connection $A = A(t)$, the Yang-Mills heat flow (YMH) is:

$$\frac{\partial A}{\partial t} = -D_A^* F_A.$$

- F_A is the curvature and D_A^* denotes the adjoint of the covariant differential on $\mathfrak{g}_E$-valued forms.
- (YMH) is the gradient of Yang-Mills functional

$$YM(A) = \frac{1}{2} \int_M |F_A|^2 dV.$$

Yang-Mills heat flow: Global well-posedness and infinite time blowups

Dimension 4:

- Schlatter (1996) proved the global existence of four dimensional Yang-Mills heat flow for small data.

- The global well-posedness for any initial data was established by Waldron (Invent Math 2019).
- Walderon's solutions may well have infinite time singularities: Yannick Sire, J. Wei and Y. Zheng, Infinite time bubbling for the $SU(2)$ Yang-Mills heat flow on $\mathbb{R}^4$, arXiv: 2023

Nematic Liquid Crystal Flow (NLCF):

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla P = \Delta v - \epsilon_0 \nabla \cdot (\nabla d \odot \nabla d) & \text{in } \Omega \times (0,T) \\ \nabla \cdot v = 0 & \text{in } \Omega \times (0,T) \\ \partial_t d + v \cdot \nabla d = \Delta d + |\nabla d|^2 d & \text{in } \Omega \times (0,T) \\ |d| = 1 & \text{in } \Omega \times (0,T) \end{cases}$$

C. Lai, F. Lin, C. Wang, J. Wei and Y. Zhou, Finite time blow-up for the nematic liquid crystal flow in dimension two. Comm. Pure Appl. Math 75(2022), no.1, 128-196.

5. The heat flow of the H-system

We consider a geometric flow that describes the evolution of parametric surfaces with constant mean curvature. It is the associated heat flow of the H-system

$$\begin{cases} u_t = \Delta u - 2u_{x_1} \wedge u_{x_2} & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{0.4}$$

where $u(x,t) = u(x_1, x_2, t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$, and $u_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a given smooth map.

Yannick Sire, J. Wei, Y. Zheng and Y. Zhou, Finite-time singularity formation for the heat flow of the H-system, preprint 2023.

6. Keller-Segel system in $\mathbb{R}^2$.

Keller-Segel system:

$$(KS) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\ v = (-\Delta)^{-1} u := \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z,t)\, dz \\ u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

- Infinite time blowup at critical mass
  Juan Davila, Manuel del Pino, Jean Dolbeault, Monica Musso, J. Wei, Existence and stability of infinite time blow-up in the Keller-Segel system, to appear in Arch. Rational Mech. Analysis
- Finite time blowup
  Federico Buseghin, Juan Davila, Manuel del Pino, Monica Musso, Existence of finite time blow-up in Keller-Segel system, arXiv 20023.

Part II– Parabolic Gluing Method—L^2 Case

I now introduce the parabolic gluing method applied to the critical Fujita equation

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u$$

This is an inner-outer gluing scheme applied to parabolic equation. This method contains two approaches

- L^2 case: dimensions $n \geq 5$
- Non-L^2 case: dimensions $n = 2, 3, 4$

Building Block

Building Block: Caffarelli,Gidas and Spruck (1989), all the positive solutions to Euler-Lagrange equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0,\ u > 0 \qquad \text{in } \mathbb{R}^n$$

are Talenti bubbles $u = U[z,\lambda](x) := (n(n-2))^{\frac{n-2}{4}} \left(\frac{\lambda}{1+\lambda^2|x-z|^2}\right)^{\frac{n-2}{2}}$

$$U = U[0,1] = (n(n-2))^{\frac{n-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

$$U[z,\lambda](x) = \lambda^{-\frac{n-2}{2}} U(\frac{x-z}{\lambda})$$

Spectral Information of U

$$\Delta\phi + pU^{p-1}\phi = \mu\phi, \qquad \|\phi\|_{L^\infty} < +\infty$$

- Principal eigenvalue $\mu_0 > 0, \phi_0 = Z_0(y) \sim e^{-c|y|}$

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- Principal eigenvalue $\mu_0 > 0, \phi_0 = Z_0(y) \sim e^{-c|y|}$
- $\mu_1 = ... = \mu_{n+1} = 0$, Eigenfunctions

$$Z_j = \frac{\partial U}{\partial z_j}, j = 1, ..., n, Z_{n+1} = \frac{\partial U}{\partial \lambda}$$

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Clearly

$$Z_j \sim < y >^{-(n-1)} \in L^2, j = 1, ..., n$$

$$Z_{n+1} \sim < y >^{-(n-2)} \in L^2, \iff n \geq 5$$

$n \geq 5$: L^2-case
$n = 3, 4$: non-L^2 case

2.1–Finite time blow-up in $\mathbb{R}^5$

Theorem 1 **del Pino-Musso-Wei (2019)**

Assume $n = 5$ and $p = p_S(5) = \frac{7}{3}$. For given points $q_1, q_2, ..., q_k$ and any sufficiently small $T > 0$ there is an initial condition u_0 such that the solution $u(x,t)$ blows-up at exactly those k points

$$u(x,t) \sim \alpha_n \sum_{j=1}^{k} \left(\frac{\lambda_j(t)}{\lambda_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}, \quad \lambda_j(t) \to 0$$

with rates type II, where

$$\|u(\cdot,t)\|_{L^\infty} \sim \frac{1}{(\lambda_j(t))^{\frac{3}{2}}} \sim \frac{1}{(T-t)^3}, \quad \xi_j(t) \to q_j, \text{ as } t \to T.$$

$$\lambda_j(t) \sim (T-t)^2$$

Furthermore the blow-up is k co-dimensional stable.

Proof of Theorem 1–an introduction to parabolic gluing method

We consider energy-critical heat equation in $\mathbb{R}^n$:

$$S(u) = -u_t + \Delta u + |u|^{p-1}u = 0,\ (x,t) \in \mathbb{R}^n \times (0,T)$$

$$p = \frac{n+2}{n-2}$$

We want to use the parabolic gluing method to construct the Type II blow-up of the type

$$u(x,t) \approx \sum_{j=1}^{k} \left(\frac{\lambda_j(t)}{\lambda_j^2 + |x-\xi_j|^2} \right)^{\frac{n-2}{2}}$$

To focus on the idea, we let $k = 1$ and $\xi_j = 0$.

Let us first do a formal computation.

A formal computation

Let us recall what we do in the elliptic case:

$$-\Delta u - \lambda u = |u|^{\frac{4}{n-2}} u \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

For single bubble, our initial ansatz is the standard Aubin-Talenti bubble:

$$u \sim U_{\mu,\xi} := \left(\frac{\mu}{\mu^2 + |x-\xi|^2}\right)^{\frac{n-2}{2}}$$

But we have a linear term $-\lambda u$ and Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

A convenient way is to project the bubble:

$$-\Delta P_\Omega U - \lambda P_\Omega U = (U_{\mu,\xi})^{\frac{n+2}{n-2}} \text{ in } \Omega; P_\Omega U = 0 \text{ on } \partial\Omega$$

and then write

$$P_\Omega U = U + \varphi$$

For parabolic Brezis-Nirenberg problem,

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{n-2}} u \\ u(x,0) = u_0(x) \end{cases}$$

we do the same. We project

$$U_\lambda := \Big(\frac{\lambda(t)}{\lambda(t)^2 + |x|^2}\Big)^{\frac{n-2}{2}}$$

into

$$\begin{cases} (PU)_t = \Delta PU + U_\lambda^{\frac{n+2}{n-2}} \\ PU(x,0) = u_0 \end{cases}$$

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u_0 is free

If we write

$$PU = U_\lambda + \psi$$

then

$$\begin{cases} \psi_t = \Delta\psi + \frac{\partial}{\partial t}(U_{\lambda(t)}) \\ \psi(x,0) = u_0 - U_{\lambda(0)} \end{cases}$$

We will drop $\frac{\partial}{\partial t}(U_{\lambda(t)})$ for the moment (when $n = 5$).
We rewrite with the ansatz:

$$u \sim u_0 = \left(\frac{\lambda(t)}{\lambda(t)^2 + |x|^2}\right)^{\frac{n-2}{2}} + \psi_0$$

where ψ_0 is a solution to the linear heat equation

$$\psi_{0,t} = \Delta\psi_0$$

$$\psi_0(x,0) = Z_0$$

so the initial data is

$$u_0 = U_{\lambda(0)} + Z_0$$

Error

$$S(u) = \Delta u + u^p - u_t$$

$$E = S(u) = (U[\lambda;0] + \psi_0)^p - (U[\lambda;0])^p - (U[\lambda;0])_t$$
$$\sim p(U[\lambda;0])^{p-1}\psi_0 - (U[\lambda;0])_t$$

Rescaling $x = \lambda y$:

$$E \sim p\lambda^{-2}U^{p-1}\psi_0(\lambda y) + \lambda^{-\frac{n}{2}}\lambda^{'}[\frac{n-2}{2}U + y\nabla U]$$

The last term is

$$Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

Orthogonal Condition

Now we need an orthogonal condition:

$$0 = \int E Z_{n+1}$$

$$\int E Z_{n+1} \sim \int [p\lambda^{-2} U_0^{p-1} \psi_0(\lambda y) + \lambda^{-\frac{n}{2}} \lambda' Z_{n+1}] Z_{n+1}$$

$$\lambda^{-2} \psi_0(0) \sim C \lambda^{-\frac{n}{2}} \lambda'$$

$$\lambda' \sim C \psi_0(0) \frac{1}{6-n} \lambda^{\frac{n-4}{2}}$$

If we look for finite time blow-up we set

$$\lambda(t) \sim (T-t)^{\beta}, \beta > 0$$

We will need

$$n < 6$$

$$\psi_0(0) < 0$$

- The above computation shows that there should be No positive Type II blow-ups. All finite time blow-ups are Type I.
- The above computation breaks down if $n = 3, 4$ since $Z_{n+1} \notin L^2$
- When $n = 6$, it is a borderline case. This suggests that we need to adjust the outer solution ψ_0. (One can glue a Type-I blow-up outer solution (Harada 2020).)
- If $n \geq 7$, then $\frac{n-4}{2} > 1$ and the above computations show that there should be no finite time blow-up even for sign-changing solutions
- When $n = 5$, we formally get

$$\beta = 2, \lambda \sim (T-t)^2$$

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$$\beta = 2, \lambda \sim (T - t)^2$$

Rigorous Proofs???

Parabolic Gluing Method

We can make the above computation rigorous by Parabolic Gluing Method.

The main idea is to extend the inner-outer gluing scheme in infinite dimensional reduction method which has been successfully used in many nonlinear elliptic equations to parabolic equation.

Key elements in the elliptic gluing: Fredholm and modulli space theory for elliptic operators

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Key elements in the elliptic gluing: Fredholm and modulli space theory for elliptic operators
Parabolic gluing: NO Fredholm theory for parabolic problems!

Development of Parabolic Gluing Method: Singularity Formations for harmonic map flows [Davila-del Pino-Wei, Invent. Math. 2020, 177 pages]

Some key observations

$$u_t = \Delta_x + |u|^{p-1}u$$

- If $|x| << \sqrt{T-t}$, then $u_t << \Delta_x$ and the problem is in elliptic region:
$$\Delta_x + |u|^{p-1}u \sim 0$$
- If $|x| \sim \sqrt{T-t}$, this is the parabolic region:
$$u_t \sim \Delta_x u$$
- If $|x| >> \sqrt{T-t}$, then $u_t >> \Delta_x$ and the problem is in ODE region:
$$u_t = |u|^{p-1}u$$

Parabolic Inner-Outer Gluing Scheme

We explain the scheme in the case of one point, $k = 1$, when

$$p = \frac{n+2}{n-2}, n = 5$$

We mainly follow the idea in
Manuel del Pino, Monica Musso, J. Wei, Y. Zheng: Sign-changing blowing-up solutions for the critical nonlinear heat equation Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 21 (2020), 569641.

In the above paper, the building block is an nonradial sign-changing solution to Yamabe problem

$$\Delta Q + |Q|^{\frac{4}{n-2}} Q = 0$$

We only use the nondegeneracy property of Q. (Proved in Musso-Wei (CMP2015).)

We are looking for a solution $u(x,t)$ of the equation

$$S(u) = -u_t + \Delta u + u^p = 0$$

which at main order has the form

$$u \sim u_0(x,t) := \left(\frac{\lambda(t)}{(\lambda(t))^2 + |x-\xi(t)|^2}\right)^{\frac{n+2}{n-2}} + O(1)$$

$O(1)$: will be added later.

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$O(1)$: will be added later.

$\lambda(t)$, $\xi(t)$ are smooth functions in $[0,T)$ such that $\lambda(T)=0$, $\xi(T)=q$.

The general strategy: very simple

- Construct a good approximate solution $u_0(x,t)$ that depends on the parameter functions $\lambda(t)$, $\xi(t)$.

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The general strategy: very simple

- Construct a good approximate solution $u_0(x,t)$ that depends on the parameter functions $\lambda(t)$, $\xi(t)$.
- Construct a genuine solution of the form $u = u_0 + \varphi$ by linearization: φ is small compared to u_0.
- The equation is

$$\varphi_t = \Delta\varphi + p u_0^{p-1}\varphi + E_0(x,t) + N(\varphi)$$

where

$$E_0(x,t) = -\partial_t u_0 + \Delta u_0 + u_0^p$$

is the error of approximation.

Inner-Outer Gluing

A crucial part of the inner-outer gluing scheme is to decompose the solution into the form

$$u_0 + \eta\left(\frac{x-\xi}{R\lambda}\right)\Phi\left(\frac{x-\xi}{\lambda}\right) + \Psi$$

Inner-Outer Gluing

A crucial part of the inner-outer gluing scheme is to decompose the solution into the form

$$u_0 + \eta(\frac{x-\xi}{R\lambda})\Phi(\frac{x-\xi}{\lambda}) + \Psi$$

Φ solves the inner problem, which is solved only in

$$|x-\xi| < R(t)\lambda(t)$$

and Ψ solves the outer problem, and η is a suitable cut-off. Both equations form a nonlinear parabolic system.

Formulating the inner–outer gluing system

We fix a point $q \in \Omega$. Let us consider a smooth function Z_0^* with
$$Z_0^*(q) < 0.$$
We let $Z^*(x,t)$ be the unique solution of the initial-boundary value problem
$$\begin{cases} Z_t^* = \Delta Z^* \ \mathbb{R}^5 \times (0,\infty), \\ Z^*(\cdot,0) = Z_0^* \text{in } \mathbb{R}^5. \end{cases}$$

We consider functions $\xi(t) \to q$, and parameters $\mu(t) \to 0$ as $t \to T$. We look for a solution of the form

$$u(x,t) = U_{\lambda(t),\xi(t)}(x) + Z^*(x,t) + \varphi(x,t)$$

with a remainder φ consisting of inner and outer parts

$$\varphi(x,t) = \lambda^{-\frac{n-2}{2}} \phi\,(y,t)\, \eta_R(y) + \psi(x,t), \quad y = \frac{x-\xi(t)}{\mu(t)}$$

where

$$\eta_R(y) = \eta_0\left(\frac{|y|}{R}\right)$$

and η_0 is a smooth cut-off function.

Error

$$
\begin{aligned}
S(U_{\lambda,\xi} + Z^* +) \ &= \\
&- \varphi_t + \Delta\varphi + pU_{\lambda,\xi}^{p-1}(\varphi + Z^*) + \mu^{-\frac{n+2}{2}} E + N(Z^* + \varphi) \\
= \ &\eta_R \lambda^{-\frac{n+2}{2}} \Big[- \lambda^2 \phi_t + \Delta_y \phi + pU(y)^{p-1}[\phi + \lambda^{\frac{n-2}{2}}(Z^* + \psi)] + E \Big] \\
&- \psi_t + \Delta_x \psi + p\lambda^{-2}(1 - \eta_R)U(y)^{p-1}(Z^* + \psi) + A[\phi] \\
&+ B[\phi] + \mu^{-\frac{n+2}{2}} E(1 - \eta_R) + N(Z^* + \varphi)
\end{aligned}
$$

$$E(y,t) \; := \lambda\dot\lambda[y\cdot\nabla U(y) + \frac{n-2}{2}U(y)] + \lambda\dot\xi\cdot\nabla U(y)$$

$$\begin{aligned} N_{\lambda,\xi}(Z) &:= |U_{\lambda,\xi}+Z|^{p-1}(U_{\lambda,\xi}+Z) - U^p_{\lambda,\xi} - pU^{p-1}_{\lambda,\xi}Z, \\ A[\phi] &:= \lambda^{-\frac{n+2}{2}}\left\{\Delta_y\eta_R\phi + 2\nabla_y\eta_R\nabla_y\phi\right\}, \\ B[\phi] &:= \\ &\lambda^{-\frac{n}{2}}\left\{\dot\lambda\big[y\cdot\nabla_y\phi + \frac{n-2}{2}\phi\big]\eta_R + \dot\xi\cdot\nabla_y\phi\,\eta_R + \big[\dot\lambda y\cdot\nabla_y\eta_R + \dot\xi\cdot\nabla_y\eta_R \right.\end{aligned}$$

Thus, we will have a solution if the pair $(\phi(y,t), \psi(x,t))$ solves the following inner–outer gluing system

Inner:

$$\lambda^2\phi_t \;=\; \Delta_y\phi + pU(y)^{p-1}\phi + H(\psi,\lambda,\xi) \text{ in } B_{2R}(0)\times(0,T)$$

Outer:

$$\begin{cases} \psi_t &= \Delta_x\psi + G(\phi,\psi,\mu,\xi) \text{in } \mathbb{R}^2\times(0,T) \\ \psi(\cdot,0) &= 0 \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^2 \end{cases}$$

where

$$\begin{aligned} H(\psi,\lambda,\xi)(y,t) \;&:= \lambda^{\frac{n-2}{2}} pU(y)^{p-1}(Z^*(\xi+\lambda y,t)+\psi(\xi+\lambda y,t)) + E(y,t) \\ G(\phi,\psi,\lambda,\xi)(x,t) \;&:= p\lambda^{-2}(1-\eta_R)U(y)^{p-1}(Z^*+\psi) + A[\phi] \;+ B[\phi] \\ &\qquad + \lambda^{-\frac{n+2}{2}} E(1-\eta_R) + N(Z^*+\varphi), \quad y = \frac{x-\xi}{\lambda}. \end{aligned}$$

Simplified Inner-Outer

Inner Problem:

$$\partial_t \phi = \Delta \phi + pU_0^{p-1}\phi + \eta E_0$$
$$+ pU_0^{p-1}\psi + \text{quadratic terms}$$
$$|x| < 2R\lambda(t)$$

Outer Problem:

$$\psi_t = \Delta \psi + pU_0^{p-1}(1-\eta)\psi$$
$$+ (1-\eta)E_0 + 2\nabla\eta\nabla\phi + \phi\Delta\eta$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^5, 0 < t < T$$

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$$\text{in } \mathbb{R}^5, 0 < t < T$$

Question: choose suitable radius R so that the inner-outer problems decouple

Choice of R

- $R(t)\lambda(t) << \sqrt{T-t}$ (self-similar)
- R larger, the outer problem gains more regularity; but the estimates for the inner problem get worse and the nonlinear terms get worse. A suitable balance.
- This is very important in the non-L^2 case. In the L^2 case, any choice of $R(t)\lambda(t) << \sqrt{T-t}$ is fine.

Inner variables and linearized inner equation

Inner variables:

$$y := \frac{x - \xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

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In the inner variables the approximation u_0 is such that

$$u_0(y,\tau) \to U(y) = \Big(\frac{1}{1+|y|^2}\Big)^{\frac{n-2}{2}} \quad \text{as } \tau \to \infty.$$

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The inner equation is

$$\begin{aligned} \phi_\tau &= \Delta\phi + pU^{p-1}\phi \\ &\quad + \text{error} + pU^{p-1}\psi + \text{small linear terms} \end{aligned}$$

The key observation is that in order that the inner and outer problem are decoupled we only need to estimate

- Outer to Inner

$$pU^{p-1}\psi$$

- Inner to Outer

$$2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$

$$\eta_R = \eta(\frac{x-\xi}{R\lambda}) = \eta(\frac{y}{R})$$

Only the boundary decaying of ϕ near ∂B_R is required.

This is the key, as the inner solution we found may grow large in the interior but decays near the boundary.

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Only the boundary decaying of ϕ near ∂B_R is required.

This is the key, as the inner solution we found may grow large in the interior but decays near the boundary.
For the gluing to work we need to find an inner solution which has fast spatial decay, and fast time decay.

Model Problem

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty, |y| < 2R$$

$$\phi(y, \tau_0) = \phi_0(y)$$

$$h \sim \tau^{-\nu}(1 + |y|)^{-2-\sigma}$$

$$\nu, \sigma > 0$$

Solutions always exist. The problem is to find a fast-decaying solution in both space and time. Eigenvalues of

$$\Delta_y\phi + pU^{p-1}(y)\phi = \mu\phi$$

$$\mu_0 > 0, \phi = Z_0 \sim e^{-r}$$

$$\mu_1 = ... = \mu_{n+1} = 0, Z_j = \frac{\partial W}{\partial y_j}, j = 1, ..., 5, Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

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$$h \sim \tau^{-\nu}(1 + |y|)^{-2-\sigma}$$

$$\nu, \sigma > 0$$

Solutions always exist. The problem is to find a fast-decaying solution in both space and time. Eigenvalues of

$$\Delta_y\phi + pU^{p-1}(y)\phi = \mu\phi$$

$$\mu_0 > 0, \phi = Z_0 \sim e^{-r}$$

$$\mu_1 = ... = \mu_{n+1} = 0, Z_j = \frac{\partial W}{\partial y_j}, j = 1, ..., 5, Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

$\mu_0 > 0$ corresponds to the instability of the blow-up

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty$$

$$h \sim \tau^{-\nu}(1+|y|)^{-2-\sigma}$$

Question: can we find a solution ϕ with fast decaying in both spatial and time variable (inside the self-similar regime)?

$$\phi \sim \tau^{-\nu}(1+|y|)^{-\sigma}, \quad |y| << \sqrt{\tau}$$

In elliptic theory, this can be achieved by Fredholm Theory (some orthogonality conditions needed). But for parabolic problems, there are no Fredholm Theory.

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In general, No!

Instability

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty$$

$\phi = e^{\mu_0(\tau-\tau_0)}Z_0(y)$ solves the equation with $h = 0$ and the initial condition

$$\phi(y, \tau_0) = Z_0.$$

We have to get rid of this type of initial conditions. This leads to co-dimensional instability.

Linear Theory

$$\begin{cases} \phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = 0 \text{ on } \partial B_{2R}(\tau_0, +\infty) & \\ \int hZ_j = 0, j = 1, ..., n+1 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

Result: Let $\nu, \sigma \in (2,3)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{1+|y|^\sigma}$$

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- The initial condition is NOT arbitrary. It is the part of the solution.
- Notice that we need $2 < \sigma < 3$. We need to improve the first error term (only $U^{p-1} \sim < y >^{-4}$).

Estimates for inner-outer coupling

$$|\phi_0(y,\tau)| \lesssim \tau^{-\nu}\frac{1}{1+|y|^\sigma}$$

Near the boundary $B_R, |y| \sim R$

$$|\phi| \lesssim |y|^{-\sigma}, |\nabla\phi| \sim |y|^{-1-\sigma}$$

The inner-outer coupling term

$$2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$
$$\sim R^{-2-\sigma}$$
$$\sim R^{-\frac{\sigma}{2}}|y|^{-2-\frac{\sigma}{2}}$$

Outer Problem:

$$\psi_t = \Delta\psi + p u_0^{p-1}(1-\eta)\psi$$
$$+ (1-\eta)E_0 + 2\nabla\eta\nabla\phi + \phi\Delta\eta$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^5, 0 < t < T$$

In the rescaled variable, the operator

$$\Delta\psi + p u_0^{p-1}(1-\eta_R)\psi$$

satisfies the Maximum Principle.

Outer Problem:

$$\begin{aligned}\psi_t = \Delta\psi + pu_0^{p-1}(1-\eta)\psi \\ + (1-\eta)E_0 + 2\nabla\eta\nabla\phi + \phi\Delta\eta \\ + \text{quadratic terms} \\ \text{in } \mathbb{R}^5, 0 < t < T\end{aligned}$$

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Using Maximum Principle, we obtain that the coupling effect is

$$|\psi| \lesssim \tau^{-\nu} R^{-\frac{\sigma}{2}} < y >^{-\frac{\sigma}{2}}$$

Linear Theory for Outer Problem

$$\begin{cases} \psi_t = \Delta_x \psi + (1-\eta_R) p u_0^{p-1} \psi + g(x,t) \\ \psi(x,0) = 0 \end{cases}$$

Assume that

$$|g| \lesssim K_1 [\frac{1}{\lambda_0^2} \frac{1}{1+|y|^{2+\sigma}} + 1]$$

Then

$$|\psi| \lesssim K_2 [\frac{1}{\lambda_0^2} \frac{1}{1+|y|^{2+\sigma}} + T^{\frac{3}{2}\sigma}]$$

$$\psi = \mathcal{T}^{out} [2 \nabla \phi \nabla \eta + \phi \Delta \eta + ...]$$

Going back to inner equation, we measure the effect of Outer-to-Inner:

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi$$
$$+ \text{error} + pU^{p-1}\psi + \text{small linear terms}$$
$$pU^{p-1}\psi \sim \tau^{-\nu} R^{-\frac{\sigma}{2}} < y >^{-4-\frac{\sigma}{2}}$$

Recall the original error

$$h \sim \tau^{-\nu} < y >^{-2-\sigma}$$

Our problem is reduced to the Inner Problem only.