Introduction to Parabolic Gluing Methods II

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June 11, 2024

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Lecture II: Parabolic Gluing Method-L^2 Case:

an inner-outer gluing scheme

applied to energy-critical Fujita equation

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n \times (0, T)$$

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Theorem 1 del Pino-Musso-Wei (2019)

Assume $n = 5$ and $p = p_S(5) = \frac{7}{3}$. For given point q and any sufficiently small $T > 0$ there is an initial condition u_0 such that the solution $u(x,t)$ blows-up at exactly q

$$u(x,t) \sim \alpha_n \left(\frac{\lambda(t)}{\lambda(t)^2 + |x - \xi(t)|^2} \right)^{\frac{n-2}{2}}, \quad \lambda(t) \to 0$$

with rates type II, where

$$\|u(\cdot,t)\|_{L^\infty} \sim \frac{1}{(\lambda(t))^{\frac{3}{2}}} \sim \frac{1}{(T-t)^3}, \quad \xi(t) \to q, \text{ as } t \to T.$$

$$\lambda(t) \sim (T-t)^2$$

Furthermore the blow-up is 1 co-dimensional stable.

We may assume that $q = 0$.

Building Block

Building Block: Caffarelli,Gidas and Spruck (1989), all the positive solutions to Euler-Lagrange equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \ u > 0 \qquad \text{in } \mathbb{R}^n$$

are Talenti bubbles $u = U[z,\lambda](x) := (n(n-2))^{\frac{n-2}{4}} \left(\frac{\lambda}{1+\lambda^2|x-z|^2}\right)^{\frac{n-2}{2}}$

$$U = U[0,1] = (n(n-2))^{\frac{n-2}{4}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}$$

$$U[z,\lambda](x) = \lambda^{-\frac{n-2}{2}} U(\frac{x-z}{\lambda})$$

Spectral Information of U

$$\Delta\phi + pU^{p-1}\phi = \mu\phi, \qquad \|\phi\|_{L^\infty} < +\infty$$

- Principal eigenvalue $\mu_0 > 0, \phi_0 = Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-c|y|}$

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- $\mu_1 = ... = \mu_{n+1} = 0$, Eigenfunctions

$$Z_j = \frac{\partial U}{\partial z_j}, j = 1, ..., n, Z_{n+1} = \frac{\partial U}{\partial \lambda}$$

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Clearly

$$Z_j \sim <y>^{-(n-1)} \in L^2, j = 1, ..., n$$

$$Z_{n+1} \sim <y>^{-(n-2)} \in L^2, \iff n \geq 5$$

$n \geq 5$: L^2-case
$n = 3, 4$: non-L^2 case

Formal Computations: $n = 5$

We start with the ansatz:

$$u \sim u_0 = \left(\frac{\lambda(t)}{\lambda(t)^2 + |x|^2}\right)^{\frac{n-2}{2}} + \psi_0$$

where ψ_0 is a solution to the linear heat equation

$$\psi_{0,t} = \Delta\psi_0$$
$$\psi_0(x,0) = Z_0$$

so the initial data is

$$u_0 = U_{\lambda(0)} + Z_0$$

Error

$$S(u) = \Delta u + u^p - u_t$$

$$E = S(u) = (U[\lambda;0] + \psi_0)^p - (U[\lambda;0])^p - (U[\lambda;0])_t$$

$$\sim p(U[\lambda;0])^{p-1}\psi_0 - (U[\lambda;0])_t$$

Rescaling $x = \lambda y$:

$$E \sim p\lambda^{-2}U^{p-1}\psi_0(\lambda y) + \lambda^{-\frac{n}{2}}\lambda'[\frac{n-2}{2}U + y\nabla U]$$

The last term is

$$Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

Orthogonal Condition

Now we need an orthogonal condition:

$$0 = \int EZ_{n+1}$$

$$\int EZ_{n+1} \sim \int [p\lambda^{-2} U_0^{p-1} \psi_0(\lambda y) + \lambda^{-\frac{n}{2}} \lambda' Z_{n+1}] Z_{n+1}$$

$$\lambda^{-2} \psi_0(0) \sim C \lambda^{-\frac{n}{2}} \lambda'$$

$$\lambda' \sim C \psi_0(0) \lambda^{\frac{n-4}{2}}$$

If we look for finite time blow-up we set

$$\lambda(t) \sim (T-t)^{\beta}, \beta > 0$$

We will need

$$n < 6$$

$$\psi_0(0) < 0$$

Parabolic Gluing Method

We can make the above computation rigorous by Parabolic Gluing Method.

The main idea is to extend the inner-outer gluing scheme in infinite dimensional reduction method which has been successfully used in many nonlinear elliptic equations to parabolic equation.

Key elements in the elliptic gluing: Fredholm and modulli space theory for elliptic operators

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Development of Parabolic Gluing Method: Singularity Formations for harmonic map flows [Davila-del Pino-Wei, Invent. Math. 2020, 177 pages]

Some key observations

$$u_t = \Delta_x + |u|^{p-1}u$$

- If $|x| << \sqrt{T-t}$, then $u_t << \Delta_x u$ and the problem is in elliptic region:
$$\Delta_x u + |u|^{p-1}u \sim 0$$
- If $|x| \sim \sqrt{T-t}$, this is the parabolic region:
$$u_t \sim \Delta_x u$$
- If $|x| >> \sqrt{T-t}$, then $u_t >> \Delta_x u$ and the problem is in ODE region:
$$u_t = |u|^{p-1}u$$

Parabolic Inner-Outer Gluing Scheme

We are looking for a solution $u(x,t)$ of the equation

$$S(u) = -u_t + \Delta u + u^p = 0$$

which at main order has the form

$$u \sim u_0(x,t) := \left(\frac{\lambda(t)}{(\lambda(t))^2 + |x-\xi(t)|^2}\right)^{\frac{n+2}{n-2}} + O(1)$$

$O(1)$: will be added later.

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$\lambda(t)$, $\xi(t)$ are smooth functions in $[0,T)$ such that $\lambda(T) = 0$, $\xi(T) = q$.

The general strategy: very simple

- Construct a good approximate solution $u_0(x,t)$ that depends on the parameter functions $\lambda(t)$, $\xi(t)$.

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- Construct a good approximate solution $u_0(x,t)$ that depends on the parameter functions $\lambda(t)$, $\xi(t)$.
- Construct a genuine solution of the form $u = u_0 + \varphi$ by linearization: φ is small compared to u_0.
- The equation is

$$\varphi_t = \Delta\varphi + p u_0^{p-1}\varphi + E_0(x,t) + N(\varphi)$$

where

$$E_0(x,t) = -\partial_t u_0 + \Delta u_0 + u_0^p$$

is the error of approximation.

Inner-Outer Gluing

A crucial part of the inner-outer gluing scheme is to decompose the solution into the form

$$u_0 + \eta(\frac{x-\xi}{R\lambda})\Phi(\frac{x-\xi}{\lambda}) + \Psi$$

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$$u_0 + \eta(\frac{x-\xi}{R\lambda})\Phi(\frac{x-\xi}{\lambda}) + \Psi$$

Φ solves the inner problem, which is solved only in

$$|x-\xi| < R(t)\lambda(t)$$

and Ψ solves the outer problem, and η is a suitable cut-off. Both equations form a nonlinear parabolic system.

Formulating the inner–outer gluing system

We fix a point $q \in \Omega$. Let us consider a smooth function Z_0^* with

$$Z_0^*(q) < 0.$$

We let $Z^*(x,t)$ be the unique solution of the initial-boundary value problem

$$\begin{cases} Z_t^* = \Delta Z^* \text{ in } \mathbb{R}^5 \times (0,\infty), \\ Z^*(\cdot,0) = Z_0^* \text{ in } \mathbb{R}^5. \end{cases}$$

We consider functions $\xi(t) \to q$, and parameters $\lambda(t) \to 0$ as $t \to T$. We look for a solution of the form

$$u(x,t) = U_{\lambda(t),\xi(t)}(x) + Z^*(x,t) + \varphi(x,t)$$

with a remainder φ consisting of inner and outer parts

$$\varphi(x,t) = \lambda^{-\frac{n-2}{2}}\phi\,(y,t)\,\eta_R(y) + \psi(x,t), \quad y = \frac{x-\xi(t)}{\lambda(t)}$$

where

$$\eta_R(y) = \eta_0\left(\frac{|y|}{R}\right)$$

and η_0 is a smooth cut-off function.

Error

$$\begin{aligned} S(U_{\lambda,\xi} + Z^* + \varphi) \ &= \\ &- \varphi_t + \Delta\varphi + pU_{\lambda,\xi}^{p-1}(\varphi + Z^*) + \mu^{-\frac{n+2}{2}} E + N(Z^* + \varphi) \\ = \ &\eta_R \lambda^{-\frac{n+2}{2}} \Big[- \lambda^2 \phi_t + \Delta_y \phi + pU(y)^{p-1}[\phi + \lambda^{\frac{n-2}{2}}(Z^* + \psi)] + E \Big] \\ &- \psi_t + \Delta_x \psi + p\lambda^{-2}(1 - \eta_R) U(y)^{p-1}(Z^* + \psi) + A[\phi] \\ &+ B[\phi] + \lambda^{-\frac{n+2}{2}} E(1 - \eta_R) + N(Z^* + \varphi) \end{aligned}$$

$$E(y,t) := \lambda\dot{\lambda}[y\cdot\nabla U(y) + \frac{n-2}{2}U(y)] + \lambda\dot{\xi}\cdot\nabla U(y)$$

$$N_{\lambda,\xi}(Z) := |U_{\lambda,\xi}+Z|^{p-1}(U_{\lambda,\xi}+Z) - U_{\lambda,\xi}^{p} - pU_{\lambda,\xi}^{p-1}Z,$$

$$A[\phi] := \lambda^{-\frac{n+2}{2}}\left\{\Delta_y\eta_R\phi + 2\nabla_y\eta_R\nabla_y\phi\right\},$$

$$B[\phi] := \lambda^{-\frac{n}{2}}$$

$$\dot{\lambda}\left[y\cdot\nabla_y\phi + \frac{n-2}{2}\phi\right]\eta_R + \dot{\xi}\cdot\nabla_y\phi\,\eta_R + \left[\dot{\lambda}y\cdot\nabla_y\eta_R + \dot{\xi}\cdot\nabla_y\eta_R\right]\phi$$

Thus, we will have a solution if the pair $(\phi(y,t), \psi(x,t))$ solves the following inner–outer gluing system

Inner Problem:

$$\lambda^2 \phi_t \;=\; \Delta_y \phi + pU(y)^{p-1}\phi + H(\psi, \lambda, \xi) \text{ in } B_{2R}(0) \times (0,T)$$

where

$$H(\psi,\lambda,\xi)(y,t) \;:=\; \lambda^{\frac{n-2}{2}} pU(y)^{p-1}(Z^*(\xi+\lambda y,t)+\psi(\xi+\lambda y,t))+E(y,t), \quad y$$

Outer Problem:

$$\begin{cases} \psi_t &= \Delta_x \psi + G(\phi, \psi, \mu, \xi) \text{in } \mathbb{R}^2 \times (0,T) \\ \psi(\cdot, 0) &= 0 \qquad\qquad\qquad\qquad \text{in } \mathbb{R}^2 \end{cases}$$

where

$$\begin{aligned} G(\phi, \psi, \lambda, \xi)(x,t) \;&:= p\lambda^{-2}(1-\eta_R)U(y)^{p-1}(Z^* + \psi) + A[\phi] \; + B[\phi] \\ &\quad + \lambda^{-\frac{n+2}{2}} E(1-\eta_R) + N(Z^* + \varphi). \end{aligned}$$

Simplified Inner-Outer

Inner Problem:

$$\partial_t \phi = \Delta_x \phi + p u_0^{p-1} \phi + \eta E_0$$
$$+ p u_0^{p-1} \psi + \text{quadratic terms}$$
$$|x| < 2R\lambda(t)$$

Outer Problem:

$$\psi_t = \Delta \psi + p u_0^{p-1} (1 - \eta) \psi$$
$$+ (1 - \eta) E_0 + 2 \nabla \eta \nabla \phi + \phi \Delta \eta$$
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Question: choose suitable radius R so that the inner-outer problems decouple

Choice of R

- $R(t)\lambda(t) << \sqrt{T-t}$ (self-similar)

- R larger, the outer problem gains more regularity; but the estimates for the inner problem get worse and the nonlinear terms get worse. A suitable balance.

- This is very important in the non-L^2 case. In the L^2 case, any choice of $R(t)\lambda(t) << \sqrt{T-t}$ is fine.

Inner variables and linearized inner equation

Inner variables:

$$y := \frac{x-\xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

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In the inner variables the approximation u_0 is such that

$$u_0(y,\tau) \to U(y) = \Big(\frac{1}{1+|y|^2}\Big)^{\frac{n-2}{2}} \quad \text{as } \tau \to \infty.$$

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The inner equation is

$$\phi_\tau = \Delta\phi + pU^{p-1}\phi$$
$$+ \text{error} + pU^{p-1}\psi + \text{small linear terms}$$

The key observation is that in order that the inner and outer problem are decoupled we only need to estimate

- Outer to Inner

$$pU^{p-1}\psi$$

- Inner to Outer

$$2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$

$$\eta_R = \eta(\frac{x-\xi}{R\lambda}) = \eta(\frac{y}{R})$$

Only the boundary decaying of ϕ near ∂B_R is required.

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For the gluing to work we need to find an inner solution which has fast spatial decay, and fast time decay.

Model Problem

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty$$

$$\phi(y, \tau_0) = \phi_0(y)$$

$$h \sim \tau^{-\nu}(1 + |y|)^{-2-\sigma}$$

$$\nu, \sigma > 0$$

Solutions always exist. The problem is to find a fast-decaying solution in both space and time. Eigenvalues of

$$\Delta_y\phi + pU^{p-1}(y)\phi = \mu\phi$$

$$\mu_0 > 0, \phi = Z_0 \sim e^{-r}$$

$$\mu_1 = ... = \mu_{n+1} = 0, Z_j = \frac{\partial W}{\partial y_j}, j = 1, ..., 5, Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

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$\mu_0 > 0$ corresponds to the instability of the blow-up

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty$$

$$h \sim \tau^{-\nu}(1+|y|)^{-2-\sigma}$$

Question: can we find a solution ϕ with fast decaying in both spatial and time variable (inside the self-similar regime)?

$$\phi \sim \tau^{-\nu}(1+|y|)^{-\sigma}, \quad |y| << \sqrt{\tau}$$

In elliptic theory, this can be achieved by Fredholm Theory (some orthogonality conditions needed). But for parabolic problems, there are no Fredholm Theory.

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In general, No!

Instability

$$\phi_\tau = \Delta\phi + pU^{p-1}(y)\phi + h, \tau_0 < \tau < +\infty$$

$\phi = e^{\mu_0(\tau-\tau_0)}Z_0(y)$ solves the equation with $h = 0$ and the initial condition

$$\phi(y, \tau_0) = Z_0.$$

We have to get rid of this type of initial conditions. This leads to co-dimensional one instability.

Linear Theory

$$\begin{cases} \phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \int hZ_j = 0, j = 1, ..., n+1 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

Result: Let $\nu, \sigma \in (2,3)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{1+|y|^\sigma}$$

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- The initial condition is NOT arbitrary. It is the part of the solution.
- Notice that we need $2 < \sigma < 3$. We need to improve the first error term (only $U^{p-1} \sim < y >^{-4}$).

Estimates for inner-outer coupling

$$|\phi_0(y,\tau)| \lesssim \tau^{-\nu}\frac{1}{1+|y|^\sigma}$$

Near the boundary $B_R, |y| \sim R$

$$|\phi| \lesssim |y|^{-\sigma}, |\nabla\phi| \sim |y|^{-1-\sigma}$$

The inner-outer coupling term

$$2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$
$$\sim R^{-2-\sigma}$$
$$\sim R^{-\frac{\sigma}{2}}|y|^{-2-\frac{\sigma}{2}}$$

Outer Problem:

$$\psi_t = \Delta\psi + pu_0^{p-1}(1-\eta_R)\psi$$
$$+ (1-\eta)E_0 + 2\nabla\eta\nabla\phi + \phi\Delta\eta$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^5, 0 < t < T$$

In the rescaled variable, the operator

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satisfies the Maximum Principle.

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Using Maximum Principle, we obtain that the coupling effect is

$$|\psi| \lesssim \tau^{-\nu} R^{-\frac{\sigma}{2}} < y >^{-\frac{\sigma}{2}}$$

Linear Theory for Outer Problem

$$\begin{cases} \psi_t = \Delta_x \psi + (1-\eta_R) p u_0^{p-1} \psi + g(x,t) \\ \psi(x,0) = 0 \end{cases}$$

Assume that

$$|g| \lesssim K_1 [\frac{1}{\lambda_0^2} \frac{1}{1+|y|^{2+\sigma}} + 1]$$

Then

$$|\psi| \lesssim K_2 [\frac{1}{\lambda_0^2} \frac{1}{1+|y|^{2+\sigma}} + T^{\frac{3}{2}\sigma}]$$

$$\psi = \mathcal{T}^{out}[2\nabla\phi\nabla\eta + \phi\Delta\eta + ...]$$
$$= \mathcal{T}^{out}[G(\phi, \psi, \lambda, \xi)]$$

Going back to inner equation, we measure the effect of Outer-to-Inner:

$$\begin{aligned}\phi_\tau &= \Delta\phi + pU^{p-1}\phi \\ &\quad + \text{error} + pU^{p-1}\psi + \text{small linear terms} \\ &\quad pU^{p-1}\psi \sim \tau^{-\nu} R^{-\frac{\sigma}{2}} < y >^{-4-\frac{\sigma}{2}}\end{aligned}$$

Recall the original error

$$h \sim \tau^{-\nu} < y >^{-2-\sigma}$$

Our problem is reduced to the Inner Problem only.

Construction of Linear Theory

Existence follows from linear parabolic theory. A Priori Estimates??? We shall carry out the construction by means of the blow-up argument.

$$\begin{cases} \phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,t), & y \in \mathbb{R}^5, \tau \in (\tau_0, +\infty) \\ \int hZ_j = 0, j = 1, ..., n+1, & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0. \end{cases}$$

Main Result: Let $\nu \in (0,1), \sigma \in (2,3)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then there exists a solution ϕ such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{(1+|y|)^{\sigma}}$$

Main Result: Let $\nu \in (0,1), \sigma \in (2,3)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then there exists a solution ϕ such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{(1+|y|)^{\sigma}}$$

A general theory for $\sigma > 0$ can also be obtained. See Lecture 3.

Dealing with Z_0

First we can choose suitable initial condition c_0 such that

$$\begin{cases} \phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z_0, & y \in \mathbb{R}^5, \tau \in (\tau_0, +\infty) \\ \int hZ_j = 0, j = 1, ..., n+1 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = 0. \end{cases}$$

where

$$c(\tau) \int Z_0^2 = \int_{\mathbb{R}^5} h(y,\tau) Z_0$$

Choice of Initial Condition c_0

In fact, we let $\phi = \hat{\phi} + p(\tau)Z_0$. Then we choose $p(\tau)$ such that

$$c(\tau) := \frac{dp}{d\tau}(\tau) - \lambda_0 p(\tau) = q(\tau) = \int_{\mathbb{R}^5} h(y,\tau)Z_0(y)dy$$

This ODE has a unique bounded solution

$$p(\tau) = \int_\tau^\infty e^{\lambda_0(\tau - s)} q(s)ds$$

and hence its initial condition is imposed: we need one linear constraint, on the initial value $\phi(y,0)$

$$\int_{\mathbb{R}^n} \phi(y,0)Z_0(y)dy = \langle \ell, h\rangle := \int_{\tau_0}^\infty e^{-\lambda_0 s} \int_{\mathbb{R}^n} h(y,s)Z_0(y)dyds$$

This corresponds to codimension One instability.

We first claim that

$$\int_{\mathbb{R}^5} \phi Z_i = 0 \text{ for all } \tau \in [\tau_0, +\infty),\ i = 0, 1, ..., n+1.$$

Indeed, we multiply the equation by $Z_i \eta_{R'}$, where $\eta_{R'} := \eta(\frac{|y|}{R'})$ and η is the standard cut-off function.

$$\phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z_0$$

Then using the zero initial condition we have

$$\int_{\mathbb{R}^5} \phi(\cdot,\tau)\cdot Z_i\eta_{R'}$$
$$= \int_{\tau_0}^{\tau} ds \int_{\mathbb{R}^5} (\phi(\cdot,s)\cdot L_0[\eta_{R'}Z_i] + hZ_i\eta_{R'} - c(s)Z_0Z_i\eta_{R'}).$$

Further computation gives

$$\begin{aligned}
&\int_{\mathbb{R}^5} [\phi(\cdot,s)\cdot L_0[\eta_{R'}Z_i] + hZ_i\eta_{R'} - c(s)Z_0Z_i\eta_{R'}].\\
&= \int_{\mathbb{R}^5} \phi(\cdot,s)[\eta_{R'}L_0[Z_i] + Z_i\Delta\eta_{R'} + 2\nabla\eta_{R'}\cdot\nabla Z_i]\\
&+ hZ_i\eta_{R'} - c(s)Z_0Z_i\eta_{R'}\\
&= O((R')^{-(\sigma_1-2)})
\end{aligned}$$

By taking $R' \to +\infty$, we get the desired result.

We prove by contradiction. Suppose that there exist sequences $\tau^k \to +\infty$ and $y_k \in \mathbb{R}^5$ such that

$$\max \tau^\nu (1+|y|)^\sigma |\phi(y)| = \tau_k^\nu (1+|y_k|)^\sigma |\phi(y_k)| = 1$$
$$|h^k| \lesssim o(1)\tau^{-\nu}(1+|y|)^{-2-\sigma}$$

$$\begin{cases} \phi_\tau^k = \Delta_y \phi^k + pU^{p-1}(y)\phi^k + h^k(y,t), & \tau \in (\tau_0, +\infty) \\ \int \phi^k Z_j = 0, j = 0, 1, ..., n+1, & \tau \in (\tau_0, +\infty) \\ \phi^k(y, \tau_0) = 0. \end{cases}$$

Case 1. $|y_k| \le M$

In this case, shift τ to $\tau_k + \tau$, up to a subsequence, $\phi^k \to \phi_\infty$ uniformly on compact subsets with $\phi_\infty \neq 0$ and ϕ_∞ is an ancient solution to

$$\begin{cases} \partial_s \phi_\infty = \Delta \phi_\infty + pU^{p-1}(y)\phi_\infty \ \text{ in } \ \mathbb{R}^5 \times (-\infty, 0], \\ \displaystyle\int_{\mathbb{R}^5} \phi_\infty(y,\tau) \cdot Z_j(y) dy = 0 \ \text{ for all } \ \tau \in (-\infty, 0], \ j = 0, 1, ..., 6, \\ |\phi_\infty(y,\tau)| \le \dfrac{1}{1+|y|^\sigma} \ \text{ in } \ \mathbb{R}^5 \times (-\infty, 0] \end{cases}$$

Note that the orthogonality conditions above are well-defined if $\sigma > 2$.

We now claim that $\phi_\infty = 0$. Indeed, by parabolic regularity theory, $\phi_\infty(y,\tau)$ is smooth. By scaling argument, we get

$$\frac{1}{1+|y|}|\nabla\phi_\infty| + |\phi_{\infty,\tau}| + |\Delta\phi_\infty| \lesssim \frac{1}{1+|y|^{2+\sigma}}.$$

Differentiating the equation with respect to τ, we get $\partial_\tau\phi_{\infty,\tau} = \Delta\phi_{\infty,\tau} + pU^{p-1}(y)\phi_{\infty,\tau}$ and

$$\frac{1}{1+|y|}|\nabla\phi_{\infty,\tau}| + |\phi_{\infty,\tau\tau}| + |\Delta\phi_{\infty,\tau}| \lesssim \frac{1}{1+|y|^{4+\sigma}}.$$

Differentiating the equation with respect to τ and integrating, we get

$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^5} |\phi_{\infty,\tau}|^2 + \int_{\mathbb{R}^5} |\nabla\phi_{\infty,\tau}|^2 - pU^{p-1}\phi_{\infty,\tau}^2 = 0,$$

Since $\int_{\mathbb{R}^5} \phi_\infty(y,\tau) \cdot Z_j(y) dy = 0$ for all $\tau \in (-\infty, 0]$, $j = 0, 1, ..., 6$,

$$\int_{\mathbb{R}^5} |\nabla \phi_{\infty,\tau}|^2 - pU^{p-1}\phi_{\infty,\tau}^2 \geq 0$$

$$\partial_\tau \int_{\mathbb{R}^5} |\phi_{\infty,\tau}|^2 \geq 0$$

Hence

$$\int_{\mathbb{R}^5} |\phi_{\infty,\tau}|^2 \leq 0, \tau < 0$$

From above, we get

$$\partial_\tau \int_{\mathbb{R}^5} |\phi_{\infty,\tau}|^2 \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^5} |\phi_{\infty,\tau}|^2 < +\infty.$$

Hence $\phi_{\infty,\tau} = 0$ and $L_0[\phi_\infty] = 0$. Since ϕ_∞ is bounded, by the non-degeneracy of L_0, ϕ_∞ is a linear combination of Z_j, $j = 1, \cdots, 6$, a contradiction.

Case 2: $|y_k| \to +\infty$

Suppose there exists y_k with $|y_k| \to +\infty$ such that

$$(\tau_k)^\nu(1 + |y_k|^\sigma)|\phi_k(y_k, \tau_k)| \geq \frac{1}{2}.$$

Let

$$\tilde{\phi}_k(z, \tau) := (\tau_k)^\nu |y_k|^\sigma \phi_k(|y_k|z, |y_k|^2\tau + \tau_k).$$

Then $\tilde{\phi}_k \to \tilde{\phi} \neq 0$ uniformly on compact subsets of $\mathbb{R}^5 \setminus \{0\} \times (-\infty, 0]$ with $\tilde{\phi}$ satisfying ancient solution of heat equation

$$\begin{cases} \tilde{\phi}_\tau = \Delta \tilde{\phi}, & \text{in } \mathbb{R}^5 \setminus \{0\} \times (-\infty, 0], \\ |\tilde{\phi}(z, \tau)| \leq |z|^{-\sigma}, & \text{in } \mathbb{R}^5 \setminus \{0\} \times (-\infty, 0]. \end{cases}$$

Removable singularities for ancient solutions of heat equation.

Claim: functions $\tilde{\phi} \equiv 0$.
This follows from Maximum Principle: We consider the function

$$\omega(y,\tau) = (|y|^2 + c\tau)^{-\frac{\sigma}{2}} + \epsilon |y|^{-3}$$

for some constant $c > 0$.
Direct computations give us

$$\omega_\tau - \Delta\omega = a_1(|y|^2 + c\tau)^{-\frac{\sigma}{2}-2}\left[(3 - \sigma - \frac{c}{2})\rho^2 + (4c - \frac{c^2}{2})\tau\right].$$

Then we know that if $\sigma < 3$, we can always find $c > 0$ such that $\omega(y, \tau + M)$ is a super-solution when $-M < \tau < 0$, where M is a large constant. Thus, $|\tilde{\phi}| \le 2\omega(\rho, \tau + M)$. By letting $M \to \infty$ and the arbitrariness of ϵ, we get $\tilde{\phi} = 0$, a contradiction.

Proof of Theorem 1: fixed point argument.

With the linear theories we are now ready to carry out the proof of Theorem 1.
We want to find a tuple $\vec{p} = (\phi, \psi, \mu, \xi)$ solving the inner–outer gluing system so that a desired blow-up solution u is constructed. This is achieved by formulating the problem as a fixed point problem for $\vec{p}$ in a small region of a suitable Banach space.