Introduction to Parabolic Gluing Methods III

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Lecture II: Parabolic Gluing Method-L^2 Case:

an inner-outer gluing scheme

applied to energy-critical Fujita equation

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n \times (0, T)$$

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Theorem 1 **del Pino-Musso-Wei (2019)**

Assume $n = 5$ and $p = p_S(5) = \frac{7}{3}$. For given point q and any sufficiently small $T > 0$ there is an initial condition u_0 such that the solution $u(x,t)$ blows-up at exactly q

$$u(x,t) \sim \alpha_n \left(\frac{\lambda(t)}{\lambda(t)^2 + |x - \xi(t)|^2} \right)^{\frac{n-2}{2}}, \quad \lambda(t) \to 0$$

with rates type II, where

$$\|u(\cdot,t)\|_{L^\infty} \sim \frac{1}{(\lambda(t))^{\frac{3}{2}}} \sim \frac{1}{(T-t)^3}, \;\; \xi(t) \to q, \text{ as } t \to T.$$

$$\lambda(t) \sim (T-t)^2$$

Furthermore the blow-up is 1 co-dimensional stable.

We may assume that $q = 0$.

We consider functions $\xi(t) \to q$, and parameters $\lambda(t) \to 0$ as $t \to T$. We look for a solution of the form

$$u(x,t) = U_{\lambda(t),\xi(t)}(x) + Z^*(x,t) + \varphi(x,t)$$

with a remainder φ consisting of inner and outer parts

$$\varphi(x,t) = \lambda^{-\frac{n-2}{2}}\phi\,(y,t)\,\eta_R(y) + \psi(x,t)$$

$$y = \frac{x - \xi(t)}{\lambda(t)}$$

Simplified Inner-Outer

Inner Problem:

$$\partial_t \phi = \Delta_x \phi + p u_0^{p-1} \phi + \eta E_0$$
$$+ p u_0^{p-1} \psi + \text{quadratic terms}$$
$$|x| < 2R\lambda(t)$$

Outer Problem:

$$\psi_t = \Delta \psi + p u_0^{p-1} (1-\eta) \psi$$
$$+ (1-\eta) E_0 + 2 \nabla \eta \nabla \phi + \phi \Delta \eta$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^5, 0 < t < T$$

The key observation is that in order that the inner and outer problem are decoupled we only need to estimate

- Outer to Inner

$$pU^{p-1}\psi$$

- Inner to Outer

$$2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$

$$\eta_R = \eta(\frac{x-\xi}{R\lambda}) = \eta(\frac{y}{R})$$

Only the boundary decaying of ϕ near ∂B_R is required.

For the gluing to work we need to find an inner solution which has fast spatial decay, and fast time decay.

$$
\begin{cases}
\phi_\tau = \Delta_y \phi + pU^{p-1}(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\
\int hZ_j = 0, j = 1, ..., n+1 & \tau \in (\tau_0, +\infty) \\
\phi(y, \tau_0) = c_0 Z_0, & |y| < 2R.
\end{cases}
$$

Result: Let $\nu, \sigma \in (2, 3)$. Assume that

$$
h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}
$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$
|\phi(y,t)| \lesssim \tau^{-\nu} \frac{1}{1+|y|^\sigma}
$$

Proof of Theorem 1: Leray-Schauder fixed point argument.

With the linear theories we are now ready to carry out the proof of Theorem 1.

We want to find a tuple $\vec{p} = (\phi, \psi, \lambda, \xi)$ solving the inner–outer gluing system so that a desired blow-up solution u is constructed. This is achieved by formulating the problem as a fixed point problem for $\vec{p}$ in a small region of a suitable Banach space.

We first set up inner problem. For a function $h(y,t)$ defined in $\mathcal{D}_{2R}$, we write

$$c_j[h](t) = \frac{\int_{B_{2R}} h(y,t)\, Z_j(y)\, dy}{\int_{B_{2R}} |Z_j(y)|^2 dy}$$

so that the function

$$\bar{h}(y,t) = h(y,t) - \sum_{j=1}^{n+1} c_j[h](t)\, Z_j(y)$$

satisfies

$$\int_{B_{2R}} \bar{h}(y,t)\, Z_j(y)\, dy \;=\; 0 \;\text{ for all }\; j = 1, \ldots, n+1, \quad t \in [0,T)$$

which makes the result of linear theory for inner problem applicable to the equation

$$\begin{cases} \lambda^2\phi_t \ = \ \Delta_y\phi + pU(y)^{p-1}\phi + \bar{H}(\psi,\mu,\xi) \ \ \mathcal{D}_{2R} \\ \phi(\cdot,0) = \ell\, Z_0 \text{ in } B_{2R} \end{cases}$$

where

$$\bar{H}(\psi,\lambda,\xi) = H(\psi,\mu,\xi) - \sum_{j=1}^{n+1} c_j[H(\psi,\mu,\xi)]\, Z_j$$

and $H(\psi,\lambda,\xi)$ is defined as before

$$H(\psi,\lambda,\xi)(y,t) \ := \ \lambda^{\frac{n-2}{2}} pU(y)^{p-1}(Z^*(\xi+\lambda y,t) + \psi(\xi+\lambda y,t)) + E(y,t).$$

Using linear theory for inner problem, we find a solution to the inner problem if the following equation is satisfied

$$\phi = \mathcal{T}_\lambda^{in}[\,\bar{H}(\psi, \lambda, \xi)\,] =: \mathcal{F}_1(\phi, \psi, \lambda, \xi).$$

Then the inner equation is satisfied if in addition we have

$$c_j[H(\psi, \lambda, \xi)] = 0 \quad \text{for all} \quad j = 1, \ldots, n+1.$$

In addition, the outer equation is satisfied provided

$$\psi = \mathcal{T}^{out}[\,G(\phi, \psi, \lambda, \xi)\,] =: \mathcal{F}_2(\phi, \psi, \lambda, \xi).$$

We will solve the above inner-outer system using a degree-theoretical argument.

Leading order dynamics of λ, ξ

Leaving aside smaller order terms, the inner problem is approximately an equation of the form

$$\phi_\tau = \Delta_y \phi + pU(y)^{p-1}\phi + h(y,t) \text{ in } \mathbb{R}^n \times (0,T)$$
$$\phi(y,t) \to 0 \text{ as } |y| \to \infty$$

with

$$\begin{aligned} h(y,t) \ &= \lambda\dot{\lambda}\,(U(y) + y\cdot\nabla U(y)) + p\lambda^{\frac{n-2}{2}}U(y)^{p-1}Z_0^*(q) \\ &+ \ \lambda\dot{\xi}\cdot\nabla U(y) + p\lambda^{\frac{n}{2}}U(y)^{p-1}\nabla Z_0^*(q)\cdot y. \end{aligned}$$

The orthogonality conditions

$$\int_{\mathbb{R}^n} h(y,t) Z_i(y)\, dy \;=\; 0 \;\text{ for all }\; i = 1, \ldots, n+1, \quad t \in [0, T).$$

imply the dynamics of the parameters:

$$\int_{\mathbb{R}^n} h(y,t) Z_{n+1}(y)\, dy \;=\; \lambda\dot{\lambda}(t) \int_{\mathbb{R}^n} Z_{n+1}^2 dy - \frac{n-2}{2} \lambda(t)^{\frac{n-2}{2}} Z_0^*(q) \int_{\mathbb{R}^n} U^p dy.$$

$$\dot{\lambda}(t) = -\beta_n |Z_0^*(q)| \lambda(t)^{\frac{n-4}{2}}, \quad \lambda(T) = 0,$$

and thus for $n = 5$

$$\lambda_*(t) = \alpha (T-t)^2, \quad \alpha = \frac{1}{4} \beta_n^2 |Z_0^*(q)|^2.$$

In a similar way, the remaining n relations lead us to $\dot{\xi}(t) = \lambda(t)^{\frac{n-2}{2}} b$ for a certain vector b. Hence $\dot{\xi}(t) = O(T-t)^3$ and

$$\xi^*(t) = q + O(T-t)^2.$$

$$\lambda(t) = \lambda_*(t) + \mu^{(1)}(t), \quad \xi(t) = q + \xi^{(1)}(t), \quad t \in [0, T]$$

For $\mu \in [0,1]$, we define the homotopy

$$H_\mu(\psi,\lambda,\xi)(y,t) \; = \lambda^{\frac{n-2}{2}} pU(y)^{p-1} Z_0^*(q) + \lambda\dot{\lambda} Z_{n+1}(y) + \lambda \sum_{j=1}^{n} \dot{\xi}_j Z_j(y)$$
$$+ \; \mu\, \lambda^{\frac{n-2}{2}} pU(y)^{p-1} (Z^*(\xi+\lambda y,t) - Z_0^*(q) + \psi(\xi+\lambda y,t$$

and consider the system of equations

$$\begin{cases} \phi \; = \mathcal{T}_\lambda^{in}[\, H_\mu(\psi,\lambda,\xi) - \sum_{j=1}^{n+1} c_j[H_\mu(\psi,\lambda,\xi)]\, Z_j \,] \\ c_j[H_\mu(\psi,\lambda,\xi)] \; = \; 0 \;\; \text{for all} \;\; j = 1,\ldots,n+1, \\ \psi \; = \mathcal{T}^{out}[\, \langle G(\phi,\psi,\lambda,\xi)\,]. \end{cases}$$

We solve the problem by a Leray-Schauder fixed point argument for tuple $\vec{p}_1 = (\phi,\psi,\lambda_1,\xi_1)$ in a small region of a suitable Banach space. Compactness on bounded sets of all the operators involved in the above expression is a direct consequence of the Hölder estimate for the operator $\mathcal{T}^{out}$ and Arzela-Ascoli's theorem.

The case of $n = 6$

As in our previous computation,

$$u \sim \left(\frac{\lambda(t)}{\lambda^2 + |x|^2}\right)^{\frac{n-2}{2}} + \psi_0$$

where ψ_0 is a solution to the linear heat equation

$$\psi_{0,t} = \Delta\psi_0$$

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Error

$$E = S(u) = \Delta u + u^p - u_t$$

$$0 = \int E Z_{n+1}$$

$$\lambda' \sim \psi_0(0)\lambda$$

If we want to construct finite time blow-up

$$\lambda \sim (T-t)^{\beta}$$

then we need

$$\psi(0) \sim \frac{1}{T-t}$$

Outer-solution

We now choose the outer problem to be the self-similar solution of the equation

$$\psi_{0,t} = \Delta\psi_0 + \psi_0^2$$

$$\psi_0 = \frac{1}{T-t}\Psi(\frac{x}{\sqrt{T-t}}), \tau = -\log(T-t)$$

$$\Psi_\tau = \Delta\Psi + y\nabla\Psi - \Psi + \Psi^2$$

$$\psi_0 = \frac{1}{T-t}(1 + \frac{1}{\tau}Z_0(\frac{x}{\sqrt{T-t}}) + ...)$$

Throwing this into the orthogonality condition, one can find (Harada (AIHP 2020)

$$\lambda \sim (T-t)^{-\frac{5}{2}}|\log(T-t)|^{-\frac{15}{4}}$$

The rest of the gluing argument is exactly the same as before.

Lecture III: Parabolic Gluing Method- Non-L^2 Case

An inner-outer gluing scheme

applied to energy-critical Fujita equation

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n \times (0,T)$$

Non $- L^2 -$ case : $n = 2, 3, 4$

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Type II blow-up for $n = 4, p = 3$ in general domains

Theorem 1 **del Pino-Musso-Wei-Zhou (2019)** Let $n = 4, p = p_S(4) = 3$. For each $T > 0$ sufficiently small there exists an initial condition u_0 such that the solution of Problem (P) blows-up at time T exactly at q. It looks at main order like

$$u(x,t) \sim \frac{2\sqrt{2}\lambda(t)}{\lambda^2 + |x - \xi(t)|^2}$$

where

$$\|u(\cdot,t)\|_{L^\infty} \sim \frac{1}{\lambda(t)} \sim \frac{|\log(T-t)|^2}{T-t}, \xi(t) \to q, \text{ as } t \to T.$$

This blow-up is co-dimensional 1 stable.

Type II blow-up for $n = 4, p = 3$ in general domains

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This blow-up is co-dimensional 1 stable.

We only prove for $\xi = 0$ (radially symmetric case)

Going back to formal computation

We start with the ansatz:

$$u \sim \left(\frac{\lambda(t)}{\lambda^2 + |x|^2}\right)^{\frac{n-2}{2}} + \psi_0$$

where ψ_0 is a solution to the linear heat equation

$$\psi_{0,t} = \Delta \psi_0$$

Error

$$E = S(u) = (U[\lambda;0] + \psi_0)^p - (U[\lambda;0])^p - (U[\lambda;0])_t$$
$$\sim p(U[\lambda;0])^{p-1}\psi_0 - (U[\lambda;0])_t$$

Rescaling $x = \lambda y$:

$$E \sim p\lambda^{-2}U^{p-1}\psi_0(\lambda y) + \lambda^{-\frac{n}{2}}\lambda'[\frac{n-2}{2}U + y\nabla U]$$

The last term is

$$Z_{n+1} = \frac{n-2}{2}U + y\nabla U$$

What is wrong with $n = 4$?

$$n = 4, p = 3, U, Z_{n+1} \sim \frac{1}{|y|^2}.$$

- Slow-decaying error; (not even in L^2!)

$$E \sim p\lambda^{-2} U^{p-1} \psi_0(\lambda y) + \frac{\lambda'}{\lambda^2} Z_{n+1}$$

The last term decays only $\frac{1}{|y|^2}$! It is not in $L^2(\mathbb{R}^4)$. This will destroy any solvability theory in parabolic setting.

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- Slowing-decaying kernels; (not even in L^2 so orthogonality does not make sense!)

$$\Delta \phi + 3U^2 \phi = 0, \phi = \sum_{j=1}^{4} c_j \frac{\partial U}{\partial y_j} + c_5 (U + y \nabla U)$$

$$Z_5 = U + y \nabla U \sim \frac{1}{1 + |y|^2} \notin L^2(\mathbb{R}^4)$$

Dealing with Difficulty I: Slow-decaying error

Inside the first error the term

$$\frac{\lambda^{'}}{\lambda^2} Z_{n+1} \quad \sim \frac{\lambda^{'}}{\lambda^2} \left(\frac{1}{1+|y|^2}\right) \notin L^2$$

has the slowest decay.

We go back to the original variable x. This term becomes

$$\lambda^{'} \frac{1}{\lambda^2 + |x|^2}$$

We have to correct it with a global term.

Global Correction Term

We use the heat equation to build a solution to

$$\phi_t = \Delta\phi + \frac{\lambda^{'}(t)}{r^2 + \lambda^2}$$

by means of the formula

$$\phi(r,t) := \int_{-T}^{t} \dot{\lambda}(s)\, \phi_0(r, t-s)\, ds$$

where ϕ_0 is in self-similar form:

$$\phi_0(r,t) := \frac{1 - e^{-\frac{r^2}{4t}}}{r^2}$$

We regularize it so that

$$\phi(r,t) := \int_{-T}^{t} \dot{\lambda}(s)\, \phi_0(\sqrt{r^2 + \lambda^2}, t-s)\, ds$$

We add this term to the initial approximation

$$u_0 = U[\lambda; 0] + \phi[\dot{\lambda}] + \psi_0$$

The new error now has fast decay:

$$E_1 = S[u_0] \sim \lambda^{-2} U^2 [\phi[\dot{\lambda}] + \psi_0(\lambda y)]$$

The problem is now to compute the global nonlocal term $\phi[\dot{\lambda}]$.

If we write

$$PU = U_\lambda + \psi$$

then

$$\begin{cases} \psi_t = \Delta\psi + \frac{\partial}{\partial t}(U_{\lambda(t)}) \\ \psi(x,0) = u_0 - U_{\lambda(0)} \end{cases}$$

When $n = 4$, we can not drop $\frac{\partial}{\partial t}(U_{\lambda(t)})$

$$\frac{\partial}{\partial t}(U_{\lambda(t)}) \sim \frac{\lambda'}{\lambda^2 + |x|^2}$$

The global term is precisely used to cancel the slow-decaying part of the projection.

Dealing with Slow-decaying Kernel

Inner variables:

$$y := \frac{x - \xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

Dealing with Slow-decaying Kernel

Inner variables:

$$y := \frac{x-\xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

In the inner variables the approximation u_0 is such that

$$u_0(y,\tau) \to U(y) = \frac{2\sqrt{2}}{1+|y|^2} \quad \text{as } \tau \to \infty.$$

Dealing with Slow-decaying Kernel

Inner variables:

$$y := \frac{x-\xi}{\lambda}, \quad \tau := \tau_0 + \int_0^t \frac{1}{\lambda^2} \to \infty \quad \text{as } t \uparrow T.$$

In the inner variables the approximation u_0 is such that

$$u_0(y,\tau) \to U(y) = \frac{2\sqrt{2}}{1+|y|^2} \quad \text{as } \tau \to \infty.$$

The new inner equation is

$$\phi_\tau = \Delta\phi + 3U^2\phi$$
$$+ \text{error} + 3U^2(\phi^0[\lambda] + \psi) + \text{small linear terms}$$

$$h = \text{error} + 3U^2(\phi^0[\lambda] + \psi) + \text{small linear terms}$$

Outer Problem

Outer Problem:

$$\psi_t = \Delta\psi + p u_0^{p-1}(1-\eta)\psi$$
$$+ (1-\eta)E_1 + 2\nabla\eta_R\nabla\phi + \phi\Delta\eta_R$$
$$+ \text{quadratic terms}$$
$$\text{in } \mathbb{R}^4, 0 < t < T$$

The key observation is that in order that the inner and outer problem are decoupled we only need to estimate

- Outer to Inner
$$3U^2\psi$$
- Inner to Outer
$$2\nabla\eta_R\nabla\phi + \phi\Delta_R$$
  Only the boundary decaying of ϕ near ∂B_R is required.

This is the key, as the inner solution we found grows large in the interior but decays near the boundary.

Linear Theory for Mode Zero: $\phi = \phi_0(r)$

$$\begin{cases} \phi_\tau = \Delta_y \phi + 3U^2(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = \phi(r) & \\ \int hZ_5 = 0 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

Result: Let $\nu, \sigma \in (0,1)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma}(\log R)}{1+|y|^4}$$

Estimates for inner-outer coupling

$$|\phi_0(y,\tau)| \lesssim \tau^{-\nu}\frac{R^{4-\sigma}\log R}{1+|y|^4}$$

In the inner region $|y|\sim 1$,

$$|\phi| \lesssim R^{4-\sigma}\log R$$

Near the boundary $B_R, |y| \sim R$

$$|\phi| \lesssim R^{-\sigma}\log R$$

The latter part is good, however the first part creates huge problem for nonlinear terms since we have to take cut-off far away to gain regularity

$$R\sim \tau^{\frac{1}{3}}$$

Step 1: Solving the elliptic part

By the elliptic finite-dimensional reduction method

$$L_0[H] := \Delta H + 3U^2(y)H = -h \text{ in } \mathbb{R}^4,$$

$$H(y) \to 0 \quad \text{as} \quad |y| \to \infty$$

$$\int hZ_5 = 0$$

admits a solution $H =: L_0^{-1}[h]$ satisfies the estimate

$$|H(y,\tau)| \lesssim \frac{\tau^{-\nu}}{(1+|y|)^{\sigma}}$$

Step 2: Linear Theory Without Orthogonality Condition

We claim that for all sufficiently large $R > 0$ there exists $\Phi = \Phi(y, \tau)$ and $c = c(\tau)$ which solve Problem

$$\begin{cases} \Phi_\tau = \Delta_y \Phi + 3U^2(y)\Phi + H - c(\tau)Z_0, & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \Phi = \Phi(r) & \\ \Phi(R, \tau) = 0 & \tau \in (\tau_0, +\infty) \\ \int_{B_R} \Phi Z_0 = 0 & \tau \in (\tau_0, +\infty) \\ \Phi(y, \tau_0) = 0 & |y| < 2R \end{cases}$$

and satisfy the estimates

$$|\Phi(y, \tau)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma} \log R}{1 + |y|^2}$$

Step 3

Finally we take

$$\phi = L_0[\Phi]$$

Recall

$$\Phi_\tau = L_0[\Phi] + H - cZ_0$$

$$L_0[\Phi_\tau] = (L_0[\Phi])_\tau = L_0[L_0[\Phi]] + L_0[H] - cL_0[Z_0]$$

Then ϕ satisfies

$$\phi_\tau = \Delta\phi + 3U^2\phi + h - c\mu_0 Z_0$$

and

$$|\phi| \lesssim \frac{R^{4-\sigma}\log R}{1+|y|^4}$$

This is exactly the solution we have constructed for mode zero.

Proof of Step 2: Linear Theory Without Orthogonality Condition

This is a linear theory without any orthogonality condition. The estimates are very bad!!!

$$\begin{cases} \Phi_\tau = \Delta_y \Phi + 3U^2(y)\Phi + H - c(\tau) Z_0, & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \Phi = \Phi(r) & \\ \Phi(R, \tau) = 0 & \tau \in (\tau_0, +\infty) \\ \int_{B_R} \Phi Z_0 = 0 & \tau \in (\tau_0, +\infty) \\ \Phi(y, \tau_0) = 0 & |y| < 2R \end{cases}$$

$$|H| \lesssim \tau^{-\nu} \frac{1}{(1+|y|)^\sigma}$$

$$|\Phi(y,\tau)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma} \log R}{1+|y|^2}$$

Step 2.1: Concentrating the error

Let $\eta(s)$ be the smooth cut-off function, and consider $\eta_\ell(y) = \eta(|y| - \ell)$, for a large but fixed number ℓ independently of R. By standard parabolic theory, there exists a unique solution Φ_1 to

$$\Phi_{1,\tau} = \Delta\Phi_1 + 3U(r)^2(1-\eta_\ell)\Phi_1 + H(y,\tau) - c(\tau)Z_0 \text{ in } B_{2R} \times (\tau_0, \infty)$$
$$\Phi_1 = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \text{ in } B_{2R},$$

Key observation: $\Delta + 3U^2(1-\eta_\ell)$ satisfies Maximum Principle (the kernel Z_5 only changes sign once).
By barrier method, the function Φ_1 satisfies the bound

$$|\Phi_1| \lesssim \tau^{-\nu} R^{2-\sigma}$$

Setting $\Phi = \Phi_1 + \Phi_2$. Then Φ_2 satisfies

$$\Phi_{2,\tau} = \Delta\Phi_2 + 3U(r)^2\Phi_2 + 3U^2\Phi_1(1-\eta) - c(\tau)Z_0 \text{ in } B_{2R} \times (\tau_0, \infty)$$
$$\Phi_2 = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0, \infty), \quad \Phi_2(\cdot, \tau_0) = 0 \text{ in } B_{2R}.$$

Now the new error $3U^2\Phi_1(1-\eta) - c(\tau)Z_0$ has fast decay: $O(e^{-|y|})$
We can adjust c such that

$$\int_{B_{2R}} \Phi_2(\cdot, \tau)\, Z_0 \;=\; 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty).$$

Setting $\Phi = \Phi_1 + \Phi_2$. Then Φ_2 satisfies

$$\Phi_{2,\tau} = \Delta\Phi_2 + 3U(r)^2\Phi_2 + 3U^2\Phi_1(1-\eta) - c(\tau)Z_0 \text{ in } B_{2R} \times (\tau_0, \infty)$$
$$\Phi_2 = 0 \quad \text{on} \quad \partial B_{2R} \times (\tau_0, \infty), \quad \Phi_2(\cdot, \tau_0) = 0 \text{ in } B_{2R}.$$

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$$\int_{B_{2R}} \Phi_2(\cdot, \tau)\, Z_0 \;=\; 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty).$$

Existence follows from linear parabolic theory. A Priori Estimates???

Testing the equation against Φ_2 and integrating in space, we obtain the relation

$$\partial_\tau \int_{B_{2R}} \Phi_2^2 + \int_{B_R} \left[|\nabla \Phi_2|^2 - 3U^2|\phi_2|^2\right] = \int_{B_{2R}} (3U^2(1-\eta)\Phi_1)\Phi_2.$$

Since dimension is 4 and $\Phi_2 = 0$ on ∂B_{2R}, there exists $C > 0$ such that, for any $\phi \in H_0^1(B_{2R})$ with $\int \phi Z_0 = 0$, the following inequality holds

$$\int_{B_R} \left[|\nabla \phi|^2 - 3U^2|\phi|^2\right] \geq \frac{C}{R^2 \log R} \int \phi^2.$$

Using the fact that $\Phi_2(\cdot,\tau_0) = 0$ and Gronwall's inequality, we readily get the L^2-estimate

$$\|\Phi_2(\cdot,\tau)\|_{L^2(B_{2R})} \lesssim \tau^{-\nu} R^2 \log R \|3U^2(1-\eta)\Phi_1)\|_{L^2(B_{2R})}$$

Now, using standard parabolic estimates in the equation satisfied by Φ_2 we obtain then that on any large fixed radius $M > 0$,

$$\|\Phi_2(\cdot,\tau)\|_{L^\infty(B_M)} \lesssim \tau^{-\nu} R^{4-\sigma} \log R \quad \text{for all} \quad \tau > \tau_0.$$

Since the right hand side has a fast decay at infinity and taking into account that we are in dimension 4, outside B_ℓ we can dominate the solution by a barrier of the order $\tau^{-\nu}|y|^{-2}$. As a conclusion, also using local parabolic estimates for the gradient, we find that

$$|\Phi_2(y,\tau)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma}\log R}{1+|y|^2}$$

Combining the estimates of Φ_1 and Φ_2:

$$|\Phi| \le |\Phi_1| + |\Phi_2| \lesssim \frac{R^{4-\sigma}\log R}{1+|y|^2}$$

Remark

The estimates in mode zero:

$$\begin{cases} \phi_\tau = \Delta_y \phi + 3U^2(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = \phi(r) & \\ \int hZ_5 = 0 & \tau \in (\tau_0, +\infty) \\ \phi(y,\tau_0) = c_0 Z_0, & |y| < 2R \\ h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}} & \end{cases}$$

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma}(\log R)}{1+|y|^4}$$

is barely enough for 4D critical problem

$$u_t = \Delta u + u^3$$

It losses too many R!!!

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Question: can we have a linear theory with less loss of R?

Key idea: Secondary Inner-Outer Gluing

Secondary Inner-Outer Gluing

In a new paper, Davila-del Pino-Lai-Wei-Zhou (2021) designed a secondary inner-outer gluing and obtained the following:

$$\begin{cases} \phi_\tau = \Delta_y \phi + 3U^2(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = \phi(r) & \\ \int hZ_5 = 0 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

Let $a, \sigma \in (0,1)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ^0, c_0) such that

$$|\phi^0(y,\tau)| \lesssim \begin{cases} \tau^{-\nu} \frac{R^{a(4-\sigma)}(\log R)}{1+|y|^4}, & |y| < 2R^a; \\ \tau^{-\nu} \frac{\log R}{1+|y|^\sigma}, & 2R^a < |y| < 2R \end{cases}$$

Decompose

$$\phi^0 = \eta(\frac{|y|}{R^a})\phi_{in} + \phi_{out}$$

where ϕ_{in} and ϕ_{out} solve the coupled system

$$\begin{cases} \partial_\tau \phi^0_{in} = \Delta_y \phi^0_{in} + 3U^2(y)\phi^0_{in} + 3U^2(y)\phi^0_{out} + h^0 + \tilde{c}^0 Z_5 - cZ_0, \quad \text{in } B_{R^a} \times \\ \phi^0_{in}(y,0) = 0, \quad \text{in } \ B_{R^a} \end{cases}$$

$$\begin{cases} \partial_\tau \phi^0_{out} = \Delta_y \phi^0_{out} + 3(1-\eta_{R^a})U^2(y)\phi^0_{out} + \mathsf{C}[\phi^0_{in}] + (1-\eta_{R^a})h^0, \quad \text{in } B_{2R} \\ \phi^0_{out}(y,0) = 0, \quad \text{in } B_{2R(0)} \\ \phi^0_{out} = 0, \quad \text{on } \partial B_{2R} \times (0,T) \end{cases}$$

Final step: the reduced equations for $\lambda(t)$

By the linear theory, approximately, orthogonality conditions

$$\int_{\mathbb{R}^4} hZ_5(y)dy \approx 0 \quad t \in (0,T)$$

should be satisfied.

Final step: the reduced equations for $\lambda(t)$

By the linear theory, approximately, orthogonality conditions

$$\int_{\mathbb{R}^4} hZ_5(y)dy \approx 0 \quad t \in (0,T)$$

should be satisfied.
What is h?

$$h \sim \eta E_1 \sim 3U^2(\phi^0[\lambda] + \psi)$$

The orthogonality conditions above imply that

$$\int_{-T}^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds \sim \psi_0(0) + o(1)$$

Equation for λ

Direct computations next yield

$$\int_0^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s}ds = Z_0^*(q) + \text{h.o.t.}$$

Strategy: the nonlocal operator can be in fact approximated by

$$\int_0^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s}ds =$$

$$\int_0^{t-(T-t)} \frac{\dot\lambda(s)}{t-s}ds + \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)}{t-s}ds - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)-\dot\lambda(s)}{t-s}ds$$

$$= \int_0^{t-(T-t)} \frac{\dot\lambda(s)}{t-s}ds + \dot\lambda(t)[\log(T-t) - 2\log\lambda(t)] - \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(t)-\dot\lambda(s)}{t-s}ds$$

$$\approx \int_0^t \frac{\dot\lambda(s)}{T-s}ds - \dot\lambda(t)\log(T-t) := \beta(t).$$

Notice that

$$\dot{\beta}\log(T-t) = \frac{\log(T-t)\dot{\lambda}(t)}{T-t} - \dddot{\lambda}(t)\log^2(T-t) + \frac{\dot{\lambda}(t)\log(T-t)}{T-t}$$
$$= \frac{d}{dt}[-\dot{\lambda}(t)\log^2(T-t)].$$

If we choose

$$\dot{\lambda}(t) = -\frac{c}{\log^2(T-t)}, \quad c > 0,$$

then $\beta \equiv const \approx Z_0^*(q)$. Normalizing and imposing $\lambda(T) = 0$, we get

$$\lambda(t) \sim \frac{|\log T|(T-t)}{\log^2(T-t)}.$$

The full solvability of the reduced problem is rather delicate since we need to control the remainder, and its form

$$\int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot{\lambda}(t)-\dot{\lambda}(s)}{t-s} ds$$

suggests that Hölder continuity in $\dot{\lambda}$ is needed, which roughly means that we need to take the cut-off region large ($\eta_R \phi_{\rm in}$) to gain enough regularity. In reality, we take $R = R(t) = \lambda^{-\beta}$ for some $\beta \in (0, 1/2)$ such that $|x-\xi| \lesssim \lambda R = \lambda^{1-\beta} << \sqrt{T-t}$ (inside the self-similar region as $\lambda \sim \frac{T-t}{\log^2(T-t)}$).

Finite time blow-up in dimension 3

Theorem (del Pino-Musso-Wei-Zhang-Zhou)

Let $n = 3$ *and* $k \in \mathbb{N}$. *For each* $T > 0$ *sufficiently small there exists an initial condition* u_0 *such that the solution of*

$$u_t = \Delta u + u^5$$

blows up at time T *which looks like at main order*

$$u(x,t) = \eta\left(\frac{x}{r\sqrt{T-t}}\right)\left[\mu^{-\frac{1}{2}}(t)U\left(\frac{x}{\mu(t)}\right) + 2\partial_t\mu_0(t)\,\mu^{\frac{1}{2}}(t)J\left(\frac{x}{\mu(t)}\right)\right]$$
$$+\left(1-\eta\left(\frac{x}{r\sqrt{T-t}}\right)\right)A^{\frac{1}{2}}(T-t)^k\frac{1}{x}C_kH_{2k}\left(\frac{1}{2}\frac{x}{\sqrt{T-t}}\right)+\theta(x,t$$

where

$$H_{2k}(\frac{z}{2}) = \frac{(-1)^k(2k)!}{k!}\left(1 - k\frac{z^2}{2} + ... + a_k z^{2k}\right), \qquad C_k = \frac{(-1)^k k!\sqrt{3}}{(2k)!},$$

Moreover, the blow-up rate $\mu(t)$ *satisfies*

$$\mu(t) \sim \mu_0(t) = 3^{\frac{1}{2}}A(T-t)^{2k}, \quad k \in \mathbb{N}$$

Ansatz

For the inner part, we use

$$U[\mu; 0] + \phi[\mu](r)$$

where $\phi[\mu]$ is the global correction term.
For the outer part we use self-similar solutions of the heat equation.
In the self-similar variable

$$z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t), \quad \Phi(z,\tau) = (T-t)^{\frac{1}{4}} u(x,t),$$

$$\Phi_\tau = \Delta\Phi - \frac{1}{2}(z \cdot \nabla\Phi) - \frac{\Phi}{4} + \mid \Phi \mid^4 \Phi.$$

We next find good approximate solution for the above equation in both inner and outer regimes.

For the outer part, the first profile will be chosen as the solution to the linear problem

$$\partial_\tau \Phi_{out} = \Delta \Phi_{out} - \frac{1}{2}(z \cdot \nabla \Phi_{out}) - \frac{\Phi_{out}}{4}.$$

Writing $\Phi_{out} = e^{\gamma\tau} m(z)$, we look for radially symmetric solutions to the following ODE

$$m'' + (\frac{2}{z} - \frac{1}{2}z)m' - (\gamma + \frac{1}{4})m = 0,$$

which turns out to be an eigenvalue problem for Hermite polynomials. In order to get even solutions with polynomial growth rate in the outer regime, we take $\gamma = \frac{1}{4} - k,\ k \in \mathbb{N}$, and then

$$m(z) = A^{\frac{1}{2}} C_k \frac{1}{z} H_{2k}(\frac{z}{2}),$$

where H_{2k} is Hermite polynomial

$$H_{2k}(\frac{z}{2}) = \frac{(-1)^k (2k)!}{k!}(1 - k\frac{z^2}{2} + \cdots + a_k z^{2k}),\ C_k = \frac{(-1)^k k! \sqrt{3}}{(2k)!},$$

Linear Theory in Dimension 3

$$\begin{cases} \phi_\tau = \Delta_y \phi + 5U^4(y)\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi = \phi(r) & \\ \int hZ_4 = 0 & \tau \in (\tau_0, +\infty) \\ \phi(y, \tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

Result (del Pino-Musso-Wei (Analysis & PDE 2019)): Let $\nu, \sigma \in (0,2)$. Assume that

$$h \sim \frac{\tau^{-\nu}}{(1+|y|)^{2+\sigma}}$$

Then for sufficiently large R there exists a solution (ϕ, c_0) such that

$$|\phi(y,t)| \lesssim \tau^{-\nu} \frac{R^{4-\sigma}}{1+|y|^3}$$

Need $\sigma > 1$.

Reduced Equation for λ

The reduced equation for λ becomes

$$\int_0^t \frac{\mu(s)}{(t-s)^{1/2}} ds = \sum_{j=0}^{k-1} c_j Z_j(0,t) + h(t),$$

Using Laplace transform one can solve this problem precisely.