Introduction to Parabolic Gluing Methods IV

Juncheng Wei

Department of Mathematics

Chinese University of Hong Kong

Summer School on Calculus of Variations

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wei@math.cuhk.edu.hk

Lecture IV: Parabolic Gluing Method-L^2 Case:

an inner-outer gluing scheme

applied to Liouville heat equation

$$u_t = \Delta u + e^u \quad \text{in } \mathbb{R}^2 \times (0, T)$$

June 11, 2024

Summary

In the last three lectures we considered inner-outer parabolic gluing method for energy-critical Fujita equation:

$$u_t = \Delta u + |u|^{\frac{4}{n-2}} u \ \text{ in } \mathbb{R}^n \times (0, T)$$

Lecture I-II: $n = 5, 6$(L^2 case)
Lecture III: $n = 3, 4(non{-}$L^2 case)

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Lecture I-II: $n = 5, 6(\mathsf{L}^2$ case)
Lecture III: $n = 3, 4(non{-}\mathsf{L}^2$ case)
Today: $n = 2$
Linear theory for linearized heat Liouville

$$\Delta U + e^U = 0 \int e^U < +\infty, \text{ in } \mathbb{R}^2$$

$$e^U = \frac{8}{(1+|y|^2)^2}$$

$$\phi_\tau = \Delta\phi + e^U\phi + h \text{ in } \mathbb{R}^2 \times (\tau_0, +\infty)$$

Heat flow of the H-system

We consider a geometric flow that describes the evolution of parametric surfaces with constant mean curvature. It is the associated heat flow of the H-system

$$\begin{cases} u_t = \Delta u - 2u_{x_1} \wedge u_{x_2} & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^2, \end{cases} \tag{0.1}$$

where $u(x,t) = u(x_1, x_2, t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^3$, and $u_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a given smooth map.

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Open Question: Are there finite time blow-up for $H-$system?

Unlike harmonic map flow (Chang-Ding-Ye (1993)), this is not known even in the equivariant case!!!

H-Bubbles: classification & non-degeneracy

$H-$bubbles:

$$\Delta u - 2u_{x_1} \wedge u_{x_2} = 0 \ \text{ in } \ \mathbb{R}^2.$$

• Brezis-Coron 1985: all the finite-energy solutions must be the H-bubble

$$u(z) = \Pi\left(\frac{P(z)}{Q(z)}\right) + C, \quad z = (x, y) = x + iy,$$

where $\Pi : \mathbb{C} \to \mathbb{S}^2$

$$\Pi(z) = \frac{1}{1+|z|^2}\begin{bmatrix} 2z \\ |z|^2 - 1 \end{bmatrix},$$

P, Q are polynomials and $C \in \mathbb{R}^3$.

Steady-State problem in $\mathbb{R}^2$: classification & non-degeneracy

As a consequence, a typical class of solutions: $W^{(m)}(z) = \Pi\left(z^m\right)$ for $m \in \mathbb{Z}^+$, where m is the degree of the map.

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Natural question: Is $W^{(m)}(z)$ L^∞-nondegenerate? In other words, is the bounded kernel to the linearized equation around $W^{(m)}$

$$\Delta\phi = 2\left(W^{(m)}_{x_1} \wedge \phi_{x_2} + \phi_{x_1} \wedge W^{(m)}_{x_2}\right)$$

finite-dimensional?

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finite-dimensional?

Chanillo and Malchiodi 2005: L^∞-nondegeneracy for the H-bubble with $m = \pm 1$. They conjectured that non-degeneracy holds for higher degree $|m| \geq 2$.

Notations

Frenet basis:

$$W^{(m)}(x) = W^{(m)}(r,\theta) = \begin{bmatrix} \frac{2r^m\cos(m\theta)}{r^{2m}+1} \\ \frac{2r^m\sin(m\theta)}{r^{2m}+1} \\ \frac{r^{2m}-1}{r^{2m}+1} \end{bmatrix}, \quad x = (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2,$$

$$E_1^{(m)} = E_1^{(m)}(r,\theta) = \begin{bmatrix} \frac{r^{2m}-1}{r^{2m}+1}\cos(m\theta) \\ \frac{r^{2m}-1}{r^{2m}+1}\sin(m\theta) \\ \frac{-2r^m}{r^{2m}+1} \end{bmatrix},$$

$$E_2^{(m)} = E_2^{(m)}(r,\theta) = \begin{bmatrix} -\sin(m\theta) \\ \cos(m\theta) \\ 0 \end{bmatrix}.$$

Non-degeneracy: $|m| \geq 2$

Theorem I ([Sire-W.-Zheng-Zhou, 2023 preprint]): $W^{(m)}(x)$ is non-degenerate in the sense that all bounded solutions of the linearized equation are linear combinations of $4m+5$ functions:

$$\frac{r^{2m}-1}{r^{2m}+1}W^{(m)}, \frac{r^m}{1+r^{2m}}\cos(m\theta)W^{(m)}, \frac{r^m}{(1+r^{2m})}\sin(m\theta)W^{(m)},$$

$$\frac{r^{m-k}}{1+r^{2m}}\left(\cos(k\theta)E_1^{(m)}+\sin(k\theta)E_2^{(m)}\right),$$

$$\frac{r^{m-k}}{1+r^{2m}}\left(\sin(k\theta)E_1^{(m)}-\cos(k\theta)E_2^{(m)}\right),$$

$$\frac{r^{m+l}}{1+r^{2m}}\left(\cos(l\theta)E_1^{(m)}-\sin(l\theta)E_2^{(m)}\right),$$

$$\frac{r^{m+l}}{1+r^{2m}}\left(\sin(l\theta)E_1^{(m)}+\cos(l\theta)E_2^{(m)}\right), k=0,\cdots,m, \quad l=1,\cdots,m.$$

Here $m \geq 1$.

Nondegeneracy

NLS:

$$\Delta w - w + w^p = 0 \text{ in } \mathbb{R}^n, \quad \text{kernel is } n - \text{dimensional}$$

Liouville or Sobolev bubble:

$$\Delta u + e^u = 0 \text{ or} \Delta u + u^{\frac{n+2}{n-2}} = 0, \quad \text{kernel is } (n+1) - \text{dimensional}$$

Harmonic Maps:

$$\Delta u + |\nabla u|^2 u = 0, deg(u) = m, \; \text{kernel is } (4m+2) - \text{dimensional}$$

H-bubbles

$$\Delta u = 2u_{x_1} \wedge u_{x_2}, deg(u) = m, \text{kernel is } (4m+5) - \text{dimensional}$$

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$$m = 1, 4m+5 = 9; \quad m = 2, 4m+5 = 13$$

Finite-time blow-up: $m = 1$

- Blow-up profile: degree 1 H-bubble

$$W(x) = \frac{1}{1+|x|^2} \begin{bmatrix} 2x \\ |x|^2 - 1 \end{bmatrix}, \quad x \in \mathbb{R}^2.$$

- Frenet basis: W, E_1, E_2. $W^{\perp} = \mathrm{span}\{E_1, E_2\}$.
- 1-equivariant form:

$$W(x) = \begin{bmatrix} e^{i\theta} \sin w(r) \\ \cos w(r) \end{bmatrix}, \; w(r) = \pi - 2\arctan(r), \; x = re^{i\theta}.$$

- Q_γ: γ-rotation matrix around z-axis.

Type II finite-time blow-up

Theorem II ([Sire-W.-Zheng-Zhou, 2023 preprint]): For any point $q \in \mathbb{R}^2$ and sufficiently small $T > 0$, there exists u_0 such that $\nabla_x u(x,t)$ with $u(x,t)$ solving (0.1) blows up at q as $t \to T$. More precisely, there exist $\kappa \in \mathbb{R}_+$, $\gamma_* \in \mathbb{R}$, and a map $u_* \in H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ such that

$$u(x,t) - u_*(x) - Q_{\gamma_*}\left[W\left(\frac{x-\xi(t)}{\lambda(t)}\right) - W(\infty)\right] \to 0 \quad \text{as} \quad t \to T$$

in $H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ with

$$\lambda(t) \sim \kappa \frac{T-t}{|\log(T-t)|^2}, \quad \xi(t) \sim q \quad \text{as} \quad t \to T.$$

In particular, we have

$$|\nabla u(\cdot,t)|^2 dx \overset{*}{\rightharpoonup} |\nabla u_*|^2 dx + 8\pi\delta_q \quad \text{as} \quad t \to T$$

in the sense of Radon measures.

General strategy

Construction by parabolic inner–outer gluing method

A motivational example with symmetry: Assume that the solution to the heat flow takes the equivariant form:

$$u(x,t) = \psi(r,t) \begin{bmatrix} e^{i\theta} \sin\varphi(r,t) \\ \cos\varphi(r,t) \end{bmatrix}.$$

Then (0.1) becomes

$$\begin{cases} \varphi_t = \varphi_{rr} + \dfrac{\varphi_r}{r} + \dfrac{(-\sin\varphi + 2r\varphi_r)\psi_r}{r\psi} - \dfrac{\sin(2\varphi)}{2r^2}, \\ \psi_t = \psi_{rr} + \dfrac{\psi_r}{r} - \dfrac{2\psi^2\varphi_r \sin\varphi}{r} + \dfrac{-1+\cos(2\varphi) - 2r^2\varphi_r^2}{2r^2}\psi. \end{cases}$$

Notice that $(\pi - 2\arctan(r), 1)$ is a stationary solution, and the linearization around it is given by:

A motivational example with symmetry

$$\begin{cases} (\phi_1)_t = (\phi_1)_{rr} + \dfrac{(\phi_1)_r}{r} - \dfrac{r^4 - 6r^2\lambda^2 + \lambda^4}{r^2(r^2+\lambda^2)^2}\phi_1 - \dfrac{6\lambda}{r^2+\lambda^2}(\phi_2)_r, \\ (\phi_2)_t = (\phi_2)_{rr} + \dfrac{(\phi_2)_r}{r} + \dfrac{8\lambda^2}{(r^2+\lambda^2)^2}\phi_2 \end{cases}$$

for perturbations ϕ_1 and ϕ_2 of φ_0 and ψ_0, respectively.

Note the second equation is the linearization for the 2D Liouville equation

$$\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2$$

around the bubble $\log \frac{8\lambda^2}{(r^2+\lambda^2)^2}$.

Linearized Liouville Heat Equation

Among the 4m+5=9 modes, the most difficult part is

$$\begin{cases} \phi_\tau = \Delta_y \phi + \frac{8}{(1+|y|^2)^2}\phi + h(y,t), & |y| < 2R, \tau \in (\tau_0, +\infty) \\ \phi(y,\tau_0) = c_0 Z_0, & |y| < 2R. \end{cases}$$

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Kernel of Liouville equations:

$$\Delta\phi + \frac{8}{(1+y|^2)^2}\phi = 0, \text{in } \mathbb{R}^2$$

$$Z_j(y) = \frac{y_j}{1+|y|^2}, j = 1,2, Z_0 = \frac{|y|^2-1}{|y|^2+1}$$

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Key to the inner-outer gluing scheme: obtain a solution ϕ with both fast space and time decay!

Dealing with modes

- For translation modes
$$Z_j = \frac{y_j}{1+|y|^2} \notin L^2(\mathbb{R}^2)$$
the method of non-L^2 (Lecture III) works.
- For scaling mode 0,
$$Z_0 = \frac{|y|^2 - 1}{|y|^2 + 1}$$
the Non-L^2 method does not work because the operator far away is 2D Laplace operator:
$$\phi_\tau \sim \Delta_{\mathbb{R}^2}\phi$$
We cannot expect any spatial decay by the convolution of $2D$ heat kernel. Estimates come with a logarithmic loss or even worse!
- To overcome this problem, we use a new method Distorted Fourier Transform (DFT).

Why distorted Fourier transform

Consider

$$u_t = \Delta u + h$$

To compute the fundamental solution we take Fourier transform

$$(\mathcal{F}u)_t = -|\xi|^2 \mathcal{F}(u), \mathcal{F}(u) = e^{-|\xi|^2 t}$$
$$S = \mathcal{F}^{-1}[\mathcal{F}(u)]$$

Duhammel's formula

$$u = \int_0^t \int S(x-y, t-s) h(y,s) dy ds$$

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What about

$$u_t = \Delta u + V(r)u + h?$$

Develop a Fourier transform corresponding to the operator $L_0 = \Delta + V(r)$.

Distorted Fourier transform (DFT): brief intro

- General theory on the spectral analysis of (half-line) Schrdinger operator: Gesztesy-Zinchenko 2006.
- Applications of (DFT) in singularity formations for wave equations and wave maps: Krieger-Schlag-Tataru (Invent Math 2008) & (Duke 2009), Krieger-Miao (Duke 2020), Krieger-Miao-Schlag (preprint 2020 (360 pages)).
- Applications of (DFT) in asymptotic stability near solitons/kinks for Klein-Gordon, sine-Gordon, NLS, integrable systems: Germain-Pusateri CPAM 2022, Lührmann-Schlag CMP 2021, etc.

(DFT): brief intro

Consider the half-line (radial) Schrödinger operator

$$\mathcal{T} = -\frac{d^2}{d\rho^2} + V(\rho)$$

Potential V: real valued, decaying as $\rho \to \infty$, singular as $\rho \to 0$. Let $\phi(\cdot, \xi)$, $\theta(\cdot, \xi)$ be a fundamental system to

$$\mathcal{T} f = \xi f.$$

Spectral measure of $\mathcal{T}$:

$$\rho((\lambda_1, \lambda_2]) = \frac{1}{\pi} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \mathrm{Im}\ m(\lambda + i\epsilon) d\lambda,$$

where

$$m(z) = \frac{W(\theta(\cdot, \xi), \psi(\cdot, \xi))}{W(\psi(\cdot, \xi), \phi(\cdot, \xi))}.$$

(DFT): brief intro

Here $W(\cdot,\cdot)$ is the Wronskian, and ψ is the generalized Weyl-Titchmarsh m function $C\psi(\cdot,\xi) = \theta(\cdot,\xi) + m(\xi)\phi(\cdot,\xi)$ with some constant $C \neq 0$.

Distorted Fourier transform:

$$\hat{f}(\xi) = \int_0^\infty \phi(\rho,\xi) f(\rho) d\rho$$

and its inverse

$$f(\rho) = \int_{\mathbb{R}} \phi(\rho,\xi)\hat{f}(\xi)\rho(d\xi).$$

Two concrete examples in mind:

- $V(\rho) = 0$: (DFT) $\iff$ Fourier transform
- $V(\rho) = c/\rho^2$: (DFT) $\iff$ Hankel transform

DFT for L_0: Estimates for eigenfunctions and density of spectral measure

We consider the operator

$$L_0 := \partial_{rr} + \frac{1}{r}\partial_r + \frac{8}{(1+r^2)^2},$$

which has kernels

$$K_1 = \frac{r^2-1}{r^2+1}, \quad K_2 = \frac{(r^2-1)\log r - 2}{r^2+1}.$$

Let $u(\rho) = r^{-1/2}v(r)$. Then

$$L_0(u) = r^{-1/2}\mathcal{L}_0 v,$$

with

$$\mathcal{L}_0 v := \partial_{rr}v + \frac{v}{4r^2} + \frac{8v}{(1+r^2)^2}.$$

The new operator $\mathcal{L}_0$ has kernel

$$\Phi_0^0(r) = \frac{r^{1/2}(r^2-1)}{r^2+1},$$

and the other one is given by

$$\begin{aligned}\Theta_0^0(r) = & -\Phi_0^0(r)\int \frac{1}{(\Phi(r))^2}dr \\ = & -\frac{r^{1/2}(r^2-1)}{r^2+1}\left(\log r - \frac{2}{r^2-1}\right)\end{aligned}$$

for which

$$W[\Theta_0^0, \Phi_0^0] = 1.$$

The spectrum of $\mathcal{L}_0$ equals

$$spec(-\mathcal{L}_0) = \{\xi_d\} \cup [0, \infty),$$

where the only negative eigenvalue $\xi_d = -\mu_0$ is negative and simple, and its corresponding eigenfunction ϕ_d has exponential decay.

$$\Delta Z_0 + \frac{8}{(1+r^2)^2} Z_0 = \mu_0 Z_0$$

The operator $\mathcal{L}_0$ has a resonance at zero since $\mathcal{L}_0[\Phi_0^0] = 0$ and $\Theta_0^0 \notin L^2(dr)$.

We now consider the fundamental system of solutions $\phi(r, z)$ and $\theta(r, z)$ to

$$-\mathcal{L}_0 y = zy$$

for all $z \in \mathbb{C}$ so that

$$W[\theta(\cdot, z), \phi(\cdot, z)] = 1.$$

Notice that these functions are entire in z, $\phi(r, 0) = c\Phi_0^0(r)$ for some normalization constant c. Let $\psi(r, z)$ be a Weyl-Titchmarsh solution. The generalized Weyl-Titchmarsh m function is defined as

$$C\psi(\cdot, z) = \theta(\cdot, z) + m(z)\phi(\cdot, z)$$

for some constant $C \neq 0$. Then

$$m(z) = \frac{W[\theta(\cdot, z), \psi(\cdot, z)]}{W[\psi(\cdot, z), \phi(\cdot, z)]}.$$

A spectral measure of $\mathcal{L}_0$ is obtained as

$$\rho((\lambda_1, \lambda_2]) = \frac{1}{\pi} \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} \mathrm{Im}\ m(\lambda + i\epsilon) d\lambda.$$

Lemma: The m function is

$$d\rho = \delta_{\xi_d} + \rho(\xi)d\xi, \quad \rho(\xi) = \pi^{-1}\text{Im } m(\xi + i0^+).$$

The distorted Fourier transform (DFT) defined as

$$\mathcal{F} : f \to \hat{f},$$

$$\hat{f}(\xi_d) = \int_0^\infty \phi_d(r)f(r)dr, \quad \hat{f}(\xi) = \lim_{b\to\infty} \int_0^b \phi(r,\xi)f(r)dr, \quad \xi \geq 0,$$

is a unitary operator from $L^2(\mathbb{R}^+)$ to $L^2(\{\xi_d\} \cup \mathbb{R}^+, \rho)$, and its inverse is given by

$$\mathcal{F}^{-1} : \hat{f} \to f(r) = \hat{f}(\xi_d)\phi_d(r) + \lim_{\mu\to\infty} \int_0^\mu \phi(r,\xi)\hat{f}(\xi)\rho(\xi)d\xi.$$

(DFT): eigenfunction estimates for $\tilde{\mathcal{L}}_0$

Next, we want to give the asymptotic expansion of distorted basis $\Phi^0(r,\xi)$ satisfying $-\mathcal{L}_0\Phi^0(r,\xi) = \xi\Phi^0(r,\xi)$. We derive these following Krieger-Miao-Schlag-Tataru:

Proposition I (Local expansion): For any $\xi \in \mathbb{C}$, $\Phi^0(r,\xi)$ admits the asymptotic expansion

$$\Phi^0(r,\xi) = \Phi^0_0(r) + r^{1/2}\sum_{j=1}^{\infty}(-r^2\xi)^j\Phi^0_j(r^2)$$

for $r^2\xi$ bounded. Here $\Phi^0_0(r) = \frac{r^{1/2}(r^2-1)}{r^2+1}$,

$$\Phi^0_1(u) = \frac{u-1}{12u(u+1)^2}\Bigg[(3u-2\pi^2)(1+u) + 6(1+u)\log(1+u)[2+\log(1+$$

$$+ 12(1+u)\mathrm{Polylog}\left(2, \frac{1}{1+u}\right)\Bigg] + \frac{(u+3)\log(u+1)-3u}{u(u+1)},$$

$$|\Phi^0_j(u)| \le C, \quad j \ge 2.$$

Next we estimate the Weyl-Titchmarsh function $\Psi_0^+(r,\xi)$ with

$$-\mathcal{L}_0\Psi_0^+(r,\xi) = \xi\Psi_0^+(r,\xi), \quad \xi > 0,$$

namely

$$\partial_{rr}\Psi_0^+(r,\xi) + \frac{\Psi_0^+(r,\xi)}{4r^2} + \xi\Psi_0^+(r,\xi) = -\frac{8\Psi_0^+(r,\xi)}{(1+r^2)^2}.$$

Lemma

A Weyl-Titchmarsh function is of the form

$$\Psi_0^+(r,\xi) = \xi^{-1/4} e^{ir\xi^{1/2}} \sigma(r\xi^{1/2}, r) \quad \text{for} \quad r\xi^{1/2} \gtrsim 1$$

where σ admits the asymptotic series approximation about q with a fixed r,

$$\sigma(q,r) \sim \sum_{j=0}^{\infty} q^{-j} \psi_j^+(r), \quad \psi_0^+(r) = 1,$$

$$\psi_1^+(r) = -\frac{i}{8} + i\left(\frac{2r^2}{r^2+1} + 2r\arctan r - r\pi\right)$$

with

$$\sup_{r>0} |(r\partial_r)^k \psi_j^+| \le d_{jk},$$

and

$$|(r\partial_r)^\alpha (q\partial_q)^\beta (\sigma(q,r) - \sum_{j=0}^{j_0} q^{-j}\psi_j^+(r))| \le e_{\alpha,\beta,j_0} q^{-j_0-1}.$$

(DFT): density estimate

Proposition II (Remote region): For $r^2\xi \gtrsim 1$, $\xi > 0$,

$$\left|\Phi^0(r,\xi)\right| \lesssim \xi^{-\frac{1}{4}}.$$

Proposition III (Density estimate): The spectrum measure $\rho_0(\mathrm{d}\xi)$ of $-\tilde{\mathcal{L}}_0$ is absolutely continuous on $\xi \geq 0$ with density

$$\frac{\mathrm{d}\rho_0(\xi)}{\mathrm{d}\xi} \simeq 1.$$

Duhamel via (DFT)

Consider

$$\begin{cases} \partial_\tau \phi_0(\rho,\tau) = \mathcal{L}_0 \phi_0(\rho,\tau), \\ \phi_0(\rho,\tau_0) = g(\rho), \end{cases}$$

where $\tau_0 \geq 1$, g is a Schwartz function. Set $\phi_0(\rho,\tau) = \rho^{-\frac{1}{2}} A_0(\rho,\tau)$, then

$$\begin{cases} \partial_\tau A_0(\rho,\tau) = \tilde{\mathcal{L}}_0 A_0(\rho,\tau), \\ A_0(\rho,\tau_0) = \rho^{\frac{1}{2}} g(\rho). \end{cases}$$

By the distorted Fourier transform,

$$A_0(\rho,\tau) = \int_0^\infty \int_0^\infty e^{-\xi\tau} \Phi^0(\rho,\xi) \Phi^0(x,\xi) \rho_0(\,\mathrm{d}\xi) x^{\frac{1}{2}} g(x)\,\mathrm{d}x.$$

Duhamel via (DFT)

Then by Duhamel's principle

$$\phi_0(\rho,\tau) = \int_{\tau_0}^{\tau}\int_0^{\infty}\int_0^{\infty} e^{-\xi(\tau-s)}\rho^{-\frac{1}{2}}\Phi^0(\rho,\xi)\Phi^0(x,\xi) \times x^{\frac{1}{2}}h_0(x,s)\rho_0(d\xi)dxds$$

gives a solution to the non-homogeneous equation with RHS h_0 and zero initial data.

Linear theory at mode 0 (no orthogonality)

Linear theory at mode 0 without orthogonality: Consider

$$\begin{cases} \partial_\tau \phi_0(\rho,\tau) = \mathcal{L}_0^W \phi_0(\rho,\tau) + h_0(\rho,\tau), \\ \phi_0(\rho,\tau_0) = 0. \end{cases}$$

where $\tau_0 \geq 1$, $|h_0(\rho,\tau)| \leq \frac{\tau^{-\nu}}{1+|\rho|^\ell}$, where $2 < \ell < \frac{5}{2}$. Then

$$|\phi_0| \lesssim \begin{cases} \tau^{-1} & \text{if } \nu > 1, \\ \tau^{-\nu} \log \tau & \text{if } \nu \leq 1. \end{cases}$$

Linear theory at mode 0 (with orthogonality)

Linear theory at mode 0 with orthogonality: If, in addition, h_0 satisfies the orthogonality condition

$$\int_{\mathbb{R}^2} h_0(y,\tau)\mathcal{Z}_0(y)\,\mathrm{d}y = 0, \quad \text{for all} \ \ \tau_0 \le \tau,$$

then we have the following better estimate

$$|\phi_0(\rho,\tau)| \lesssim \begin{cases} \tau^{-\nu}\langle\rho\rangle^{2-\ell} + \tau^{-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} s^{-\nu} ds & \text{if} \ \ \rho \le \tau^{\frac{1}{2}} \\ \rho^{-\frac{1}{2}}\left(\tau^{\frac{5}{4}-\frac{\ell}{2}-\nu} + \tau^{\frac{1}{4}-\frac{\ell}{2}} \int_{\frac{\tau_0}{2}}^{\frac{\tau}{2}} s^{-\nu} ds\right) & \text{if} \ \ \rho > \tau^{\frac{1}{2}} \end{cases}.$$

Remarks

- Only a loss of log without orthogonality, and orthogonality removes the loss of log. This seems to be the best estimates that one can expect.
- We will not impose orthogonalities at mode 0 since this will further complicate interactions. Without orthogonality, the log is controlled by using Hölder properties inherited from the outer problem.
- One can also use distorted Fourier transform to deal with other modes, but estimates for mode 1 can be very bad via DFT (so our previous L^2 + non-L^2-methods are better).

Reverse
Parabolic
Gluing
Method (L^2 case)

Reverse Parabolic Gluing Method

In the last part of Lecture 4, I will discuss a reverse parabolic gluing method to prove the following:

Theorem

(Wang-Wei 2021) If $n \geq 7$, $p = \frac{n+2}{n-2}$ and u is positive, then all blow ups are Type I.

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(Wang-Wei 2021) If $n \geq 7$, $p = \frac{n+2}{n-2}$ and u is positive, then all blow ups are Type I.

We have seen that the method of inner-outer parabolic gluing is very powerful in constructing Type II blowup solutions. Just as in the study of Yamabe or Brezis-Nirenberg problem, we can now also use this method to study the Type II blowup behavior. This is a reverse inner-outer parabolic gluing.

Reverse inner-outer elliptic gluing: Wang-Wei, CPAM 2019: Finite Morse index solutions of

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^2$$

have finite ends.

Key Ideas of Proofs

- Energy concentration: Tangent flow analysis and Lin-Wang blown-down argument;
  No energy bounds

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- Energy concentration: Tangent flow analysis and Lin-Wang blown-down argument;
  No energy bounds
- Parabolic second order estimates: Reverse Inner-outer parabolic gluing mechanism to exclude Multiplicity One case
- Exclusion of Higher Multiplicity case (bubbling towering and bubbling clustering), parabolic compactness argument for Yamabe (Schoen, Khuri-Marques-Schoen, Y. Li, etc.)

Energy concentration phenomena

Theorem

Given a sequence of suitable weak solutions u_i *in* Q_1, *satisfying* $\sup_i \int_{Q_1} |\nabla u_i|^2 + |u_i|^{p+1} < +\infty$.

- *A subsequence converges to a weak solution* u_∞*;*
- *there exists a defect measure* μ *such that*
$$|\nabla u_i|^2 dxdt \rightharpoonup |\nabla u_\infty|^2 dxdt + \mu, \quad |u_i|^{p+1} dxdt \rightharpoonup |u_\infty|^{p+1} dxdt + \mu;$$
- $\mu = \mu_t dt$ *and* μ_t *lives on a rectifiable set;*
- (u_∞, μ_t) *is a generalized Brakke flow.*

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- Motivated by Lin-Wang's work (blow-down argument) on harmonic map heat flow.
- Main tools: ε-regularity theorem and monotonicity formula of Giga-Kohn.

Giga-Kohn Monotonicity Formula

For each $(x,t) \in B_{3/4} \times (-3/4, 1]$ and $s \in (0, 1/4)$, define

$$\Theta_s(x,t) := s^{\frac{p+1}{p-1}} \int_{B_1} \left[\frac{|\nabla u(y,t-s)|^2}{2} - \frac{|u(y,t-s)|^{p+1}}{p+1} \right] G(y-x,s)\psi(y$$

$$+\frac{1}{2(p-1)} s^{\frac{2}{p-1}} \int_{B_1} u(y,t-s)^2 G(y-x,s)\psi(y)^2 dy + Ce^{-cs^{-1}}.$$

$$G(x,t) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

Localized monotonicity formula: If C is universally large and c is universally small, then for any $(x,t) \in B_{3/4} \times (-3/4, 1]$ and $0 < s_1 < s_2 < 1/4$,

$$\Theta_{s_2}(x,t) - \Theta_{s_1}(x,t) \geq$$

$$\int_{s_1}^{s_2} \tau^{\frac{2}{p-1}-1} \int_{B_1} \left| (t-\tau)\partial_t u(y,t-\tau) + \frac{u(y,t-\tau)}{p-1} + \frac{y}{2} \cdot \nabla u(y,t-\tau) \right|^2$$

$$\times G(y-x,\tau)\psi(y)^2 dy d\tau.$$

Energy concentration in the critical case: bubbling

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Energy concentration in the critical case: bubbling

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- By Struwe's global compactness theorem, for a.e. t, the following bubble tree convergence holds for $u_i(t)$:

Theorem

There exist $N(t)$ points $\xi_{ik}^*(t)$, positive constants $\lambda_{ik}^*(t)$, $k = 1, \cdots, N(t)$, all converging to 0 as $i \to +\infty$, and $N(t)$ bubbles W^k, such that in $H^1(B_1)$,

$$u_i(x,t) = \sum_{k=1}^{N(t)} W^k_{\xi_{ik}^*(t), \lambda_{ik}^*(t)}(x) + o_i(1).$$

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- A bubble is an entire solution of the stationary equation with finite energy.
- If u_i is positive, all bubbles arising in this process are standard ones, thanks to Caffarelli-Gidas-Spruck.

Bubble clustering and towering

Bubble towering: bubbles are located at almost the same point (w.r.t. the bubble scales), but the height of one bubble is far larger than the other one's :

$$\limsup_{i\to+\infty} \frac{|\xi^*_{ik}(t)-\xi^*_{i\ell}(t)|}{\max\{\lambda^*_{ik}(t),\lambda^*_{i\ell}(t)\}} < +\infty, \qquad \frac{\lambda^*_{ik}(t)}{\lambda^*_{i\ell}(t)}+\frac{\lambda^*_{i\ell}(t)}{\lambda^*_{ik}(t)} \to +\infty.$$

Bubble clustering: if for some $k \neq \ell$,

$$\lim_{i\to+\infty} |\xi^*_{ik}(t)-\xi^*_{i\ell}(t)| = 0$$

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Bubbling-towerings do exist: del Pino-Musso-Wei (2019) $(n \geq 7, T = +\infty)$

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- Modulate parameters to get an optimal approximation:

$$\phi_i(t) := u_i(t) - \sum_k W_{\xi_{ik}(t),\lambda_{ik}(t)}.$$

  This gives an orthogonal condition (note that the linearized equation has nontrivial kernels), which leads to reduction equations (here some ODEs) for the modulated parameters.

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  This gives an orthogonal condition (note that the linearized equation has nontrivial kernels), which leads to reduction equations (here some ODEs) for the modulated parameters.
- Because now we are looking at the next order term (i.e. the error function ϕ_i), we need to take care of
  - interaction between bubbles and the background;
  - interaction between different bubbles.

Inner-outer decomposition

In previous lecture, we used the inner-outer gluing method to construct Type II blow up solutions. Here we take a reverse view, using this decomposition to analyse the behavior of the error function,

$$\phi_i = \phi_{i,inn} + \phi_{i,out}.$$

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In the following we look at the one bubble case more closely.

The standard bubble

By Caffarelli-Gidas-Spruck, all entire positive solutions to the stationary equation are given by Aubin-Talenti bubbles

$$W_{\xi,\lambda}(x) := \left(\frac{\lambda}{\lambda^2 + \frac{|x-\xi|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}, \qquad \lambda > 0, \quad \xi \in \mathbb{R}^n.$$

They have finite energy, which are always equal to

$$\Lambda := \int_{\mathbb{R}^n} |\nabla W_{\xi,\lambda}|^2 = \int_{\mathbb{R}^n} W_{\xi,\lambda}^{p+1}.$$

One bubble case: blow up profile

Assume u_i is a sequence of smooth, positive solutions in Q_1 s.t.

$$|\nabla u_i|^2 dxdt \sim |\nabla u_\infty|^2 dxdt + \Lambda \delta_0 dt,$$

where u_∞ is a smooth solution, Λ is the energy of the standard bubble.

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Lemma (Blow up profile)

For any $t \in [-81/100, 81/100]$, there exists a unique maxima point of $u_i(\cdot, t)$ in the interior of $B_1(0)$. Denote this point by $\xi_i^*(t)$ and let $\lambda_i^*(t) := u_i(\xi_i^*(t), t)^{-\frac{2}{n-2}}$.
As $i \to +\infty$,

$$\lambda_i^*(t) \to 0, \quad \xi_i^*(t) \to 0, \quad \text{uniformly in } C([-81/100, 81/100]),$$

and the function

$$u_i^t(y, s) := \lambda_i^*(t)^{\frac{n-2}{2}} u_i\left(\xi_i^*(t) + \lambda_i^*(t) y, t + \lambda_i^*(t)^2 s\right),$$

converges to the standard bubble $W(y)$ in $C^\infty_{loc}(\mathbb{R}^n \times \mathbb{R})$.

One bubble case: orthogonal decomposition

Lemma

For any t*, there exists a unique* $(a_i(t), \xi_i(t), \lambda_i(t)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^+$ *with*

$$\frac{|\xi_i(t) - \xi_i^*(t)|}{\lambda_i(t)} + \left|\frac{\lambda_i(t)}{\lambda_i^*(t)} - 1\right| + \left|\frac{a_i(t)}{\lambda_i(t)}\right| = o(1),$$

such that for each $k = 0, \cdots, n+1$*,*

$$\int_{B_1} \left[u_i(x,t) - W_{\xi_i(t),\lambda_i(t)}(x) - a_i(t) Z_{0,\xi_i(t),\lambda_i(t)}(x)\right]$$

$$\times \eta\left(\frac{x-\xi_i(t)}{K\lambda_i(t)}\right) Z_{k,\xi_i(t),\lambda_i(t)}(x) dx = 0.$$

Here we take a cut-off at $K\lambda_i(t)$-scale and a scaling preserving the L^2 norm:

$$Z_{k,\xi,\lambda}(x) := \lambda^{-\frac{n}{2}} Z_k\left(\frac{x-\xi}{\lambda}\right).$$

Reduction procedure

The error function ϕ_i satisfies

$$\partial_t \phi_i - \Delta \phi_i = p W_i^{p-1} \phi_i + \left(-a_i' + \mu_0 \frac{a_i}{\lambda_i^2}, \xi_i', \lambda_i' \right) \cdot Z_i + \text{h.o.t.} \qquad (0.2)$$

Together with the orthogonal condition

$$\int_{B_1} \phi_i(x,t) \eta \left(\frac{x - \xi_i(t)}{K \lambda_i(t)} \right) Z_{k,\xi_i(t),\lambda_i(t)}(x) dx = 0,$$

we can (and we need to) get at the same time

- equations for λ_i' ... $\Longrightarrow$ blow up rate;
- a priori estimates on ϕ_i.

Inner-outer decomposition

Keep K as the large constant used in the orthogonal decomposition. Take another constant L satisfying $1 \ll L \ll K$. Denote

$$\eta_{i,in}(x,t) := \eta\left(\frac{x-\xi_i(t)}{K\lambda_i(t)}\right), \quad \eta_{i,out}(x,t) := \eta\left(\frac{x-\xi_i(t)}{L\lambda_i(t)}\right).$$

Set

$$\phi_{i,in}(x,t) := \phi_i(x,t)\eta_{i,in}(x,t), \quad \phi_{out}(x,t) := \phi(x,t)\left[1-\eta_{i,out}(x,t)\right].$$

We analyse the inner and outer equations separately.

Reverse Inner-outer Gluing Scheme

- Inner problem estimate:

$$\mathcal{I} \le A\mathcal{O} + \text{higher order terms from scaling parameters etc.}$$

  where $\mathcal{I}$ is a quantity measuring the inner component, $\mathcal{O}$ is a quantity measuring the outer component.

- Outer problem estimate:

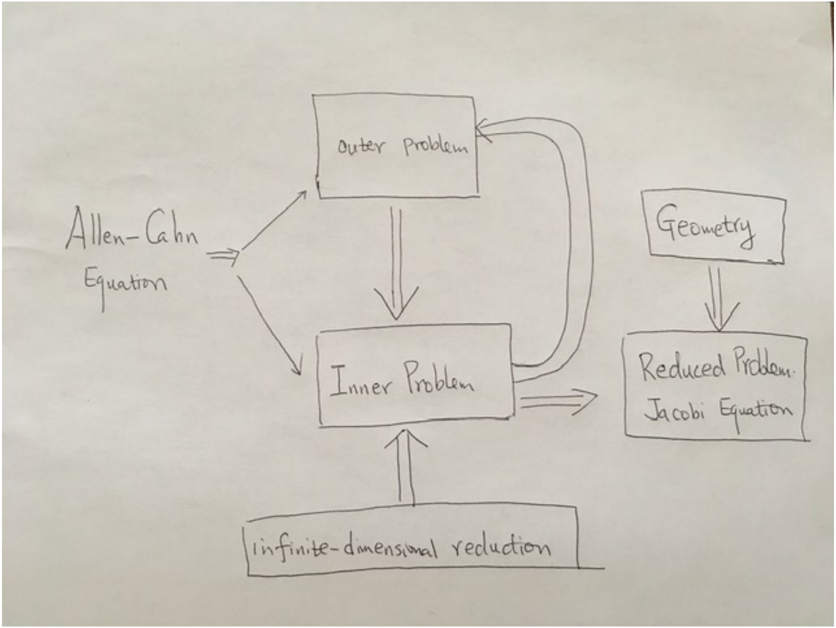
$$\mathcal{O} \le B\mathcal{I} \ + \text{effect from initial-boundary value}$$
$$+ \text{ higher order terms from scaling parameters etc.}$$

The inner-outer gluing mechanism works thanks to the fact that

$$A\ B < 1$$

This follows from a fast decay estimate away from the bubble domains, where we mainly rely on a Gaussian bound on heat kernels associated to a parabolic operator with small Hardy term

Feedback between inner and outer components



Inner problem

Introduce an inner coordinate system around $\xi_i(t)$ by

$$y := \frac{x - \xi_i(t)}{\lambda_i(t)}, \qquad \tau = \tau(t),$$

where

$$\tau'(t) = \lambda_i(t)^{-2}, \qquad \tau(0) = 0.$$

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Let $\varphi_{i,K}(y,\tau) := \lambda_i(t)^{\frac{n-2}{2}} \phi_i(x,t) \eta_{i,K}(x,t)$. Then

$$\begin{cases} \partial_\tau \varphi_K - \Delta_y \varphi_K = pW^{p-1}\varphi_K + \lambda^{-1}\left(-\dot{a} + \mu_0 a, \dot{\xi}, \dot{\lambda}\right) \cdot Z + E_K, \\ \int_{\mathbb{R}^n} \varphi_K(y,\tau) Z_i(y) dy = 0, \quad \forall \tau. \end{cases}$$

Non-degeneracy of the linearized operator $\Longrightarrow$ exponential decay in τ.

Outer problem

The outer component satisfies

$$
\begin{aligned}
\partial_t \phi_{i,out} - \Delta \phi_{i,out} &= O\left(\frac{\delta}{|x-\xi_i(t)|^2}\right)\phi_{i,out} + \\
&+ \text{ terms from inner component and } \lambda_i' \ldots
\end{aligned}
$$

where $\delta \ll 1$, thanks to the fast decay away from bubble point.

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$$+ \text{ terms from inner component and } \lambda_i' \ldots$$

where $\delta \ll 1$, thanks to the fast decay away from bubble point.
Key Heat Kernel Estimates: The heat kernel $G(x, y; t, s)$ of the operator $\partial_t - \Delta - (\frac{\delta}{|x-\xi(t)|^2} + C) + \xi'(t)\nabla$

$$G(x, y; t, s) \leq C(t-s)^{-\frac{n}{2}} e^{-c\frac{|x-y|^2}{t-s}} (1 + \frac{\sqrt{t-s}}{|x|})^\gamma (1 + \frac{\sqrt{t-s}}{|y|})^\gamma$$

$$\gamma = \frac{n-2}{2} - \sqrt{(\frac{n-2}{2})^2 - 4\delta}$$

(Saloff-Coste (2012), Moschini-Tesei (2007))

Then $\phi_{i,out} = \phi_{i1} + \phi_{i2} + \phi_{i3} + \cdots$, where

- ϕ_{i1} solves the Cauchy-Dirichlet problem, and it is almost regular in the interior;
- ϕ_{i2} is determined by $\phi_{i,inn}$, ϕ_{i3} is determined by λ_i' ..., all enjoying a fast decay away from bubble point;

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- ϕ_{i2} is determined by $\phi_{i,inn}$, ϕ_{i3} is determined by λ_i' ..., all enjoying a fast decay away from bubble point;
- the last property gives us a small transmission coefficient (inner to outer), which closes our estimates on ϕ_i.

Reduction equation

- ϕ_i are uniformly bounded in L^∞, and satisfy some uniform weighted gradient estimates (blowing up near bubble points).
- By linearzing local Pohozaev identity, or equivalently, by substituting the decomposition $u_i = W_i + \phi_i$ into the stationary condition for u_i,

$$\Longrightarrow \begin{cases} \lambda_i'(t) = c_1(n)\left[\phi_i(\xi_i(t),t) + o_i(1)\right]\lambda_i^{\frac{n-4}{2}}, \\ \xi_i'(t) = -c_2(n)\left[\nabla\phi_i(\xi_i(t),t) + o_i(1)\right]\lambda_i^{\frac{n-2}{2}}. \end{cases}$$

Here, due to some technical issues, we need to take a new orthogonal decomposition at scale $\lambda_i^{1/2}$.

Corollary

For positive solutions of Fujita critical equations, bubble towering is unstable.

Proof: By a rescaling, we can choose $u_\infty = W$. Then λ_i increases, which forces $\xi_i(t)$ to move to infinity, so bubble towering will be transformed into bubble clustering.

Bubble clustering

Setting: $\forall t$, there are exactly N (≥ 2) bubbles located at $\xi^*_{ij}(t)$, with height $\lambda^*_{ij}(t)^{-\frac{n-2}{2}}$, satisfying for some large M,

$$|\xi^*_{ij}(t) - \xi^*_{ik}(t)| \geq M \max\left\{\lambda^*_{ij}(t) + \lambda^*_{ik}(t)\right\}.$$

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- We can still take localized orthogonal and inner-outer decomposition for each bubble.
- A new term describing the interaction between different bubbles: $\sum_{k\neq j} W_{ij}^{p-1} W_{ik}$.
- The interaction term is positive ($\sim$ repulsive force between bubbles) $\Longrightarrow$

$$\lambda'_{ij} \geq -C\lambda_{ij}^{\frac{n-4}{2}}.$$

Unstable mechanism in bubble clustering

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Hence eventually there is only one bubble near the blow up time.