QUANTITATIVE STABILITY OF THE TOTAL Q-CURVATURE NEAR MINIMIZING METRICS

JOÃO HENRIQUE ANDRADE, TOBIAS KÖNIG*, JESSE RATZKIN, AND JUNCHENG WEI

Abstract. Under appropriate positivity hypotheses, we prove quantitative estimates for the total k-th order Q-curvature functional near minimizing metrics on any smooth, closed n-dimensional Riemannian manifold for every integer $1 \leq k < \frac{n}{2}$. More precisely, we show that on a generic closed Riemannian manifold the distance to the minimizing set of metrics is controlled quadratically by the Q-curvature energy deficit, extending recent work by Engelstein, Neumayer and Spolaor [17] in the case $k = 1$. Next we prove, for any integer $1 \leq k < \frac{n}{2}$, the existence of an n-dimensional Riemannian manifold such that the k-th order Q-curvature deficit controls a higher power of the distance to the minimizing set. We believe that these degenerate examples are of independent interest and can be used for further development in the field.

Contents

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* Corresponding author.

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1. Introduction and main results

1.1. The k-th order Q-curvature problem. We consider a compact Riemannian manifold (M,g) of dimension n without boundary. In 1992, Graham, Jenne, Mason and Sparling [26] constructed a conformally invariant operator $P_{g,k}$ whose leading term is $(-\Delta_g)^k$ for each integer $1 \le k < \frac{n}{2}$, where Δ_g is the Laplace–Beltrami operator of g (see Appendix A for more details).

The operator $P_{g,k}$ is now known as the GJMS operator of order $2k$. It is naturally constructed from curvature quantities of g and satisfies the transformation law

$$P_{\widetilde{g},k}(\phi) = u^{-\frac{n+2k}{n-2k}} P_{g,k}(\phi u) \quad \text{when} \quad \widetilde{g} = u^{\frac{4}{n-2k}} g. \tag{1.1}$$

Subsequently, the authors defined the (scalar-valued) curvature quantity $Q_{g,k} = \frac{2}{n-2k} P_{g,k}(1)$. Substituting $\phi = 1$ into (1.1), we see

$$Q_{\widetilde{g},k} = \frac{2}{n-2k} u^{-\frac{n+2k}{n-2k}} P_{g,k}(u) \quad \text{when} \quad \widetilde{g} = u^{\frac{4}{n-2k}} g. \tag{1.2}$$

Motivated by this conformal invariance and the analysis of the classical Yamabe problem, one poses the kth–order Yamabe problem: given a compact Riemannian manifold (M,g) of dimension n and an integer $1 \le k < \frac{n}{2}$, we seek a conformal metric $\widetilde{g} = u^{4/(n-2k)} g$ such that $Q_{\widetilde{g},k}$ is constant. (Here and below we fix the background metric g and identify the conformal metric $\widetilde{g} = u^{4/(n-2k)} g$ with its conformal factor u.) By (1.2) this is equivalent to solving the PDE

$$P_{g,k}(u) = c u^{\frac{n+2k}{n-2k}} \quad \text{on} \quad M, \tag{1.3}$$

where c is a constant. We denote the set of solutions by

$$\begin{aligned} \mathcal{CQC}_{g,k} &= \left\{ \widetilde{g} \in [g] : Q_{\widetilde{g},k} \text{ is constant} \right\} \\ &= \left\{ u \in W^{k,2}(M) : u > 0 \text{ a.e. and } P_{g,k}(u) = \lambda u^{\frac{n+2k}{n-2k}} \text{ for some } \lambda \in \mathbb{R} \right\}. \end{aligned}$$

Since $2_k^* := \frac{2n}{n-2k}$, Eq. (1.3) has critical growth in the sense of the embedding $W^{k,2}(M) \hookrightarrow L^{2_k^*}(M)$.

To establish a variational setting, we introduce the functional

$$\mathcal{Q}_{g,k}(u) = \frac{\int_M Q_{\widetilde{g},k} \, \mathrm{d}\mu_{\widetilde{g}}}{\mathrm{vol}_{\widetilde{g}}(M)^{\frac{n-2k}{n}}} = \frac{2}{n-2k} \frac{\int_M u P_{g,k}(u) \, \mathrm{d}\mu_g}{\left(\int_M u^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g \right)^{\frac{n-2k}{n}}}. \tag{1.4}$$

We show below in Lemma 3.2 that $\mathcal{Q}_{g,k}$ is a $\mathcal{C}^2$-functional on $W^{k,2}(M)$. Furthermore, it follows from this proof that $\widetilde{g}$ is a critical point of $\mathcal{Q}_{g,k}$ if and only if $Q_{\widetilde{g},k}$ is constant, which is in turn equivalent to u solving (1.3) with

$$c = \mathcal{Q}_{g,k}(u) \left(\|u\|_{L^{2_k^*}(M)} \right)^{\frac{4k}{n-2k}}.$$

Observe that $\mathcal{Q}_{g,k}$ is scale-invariant, so we will often restrict our attention to unit-volume metrics in the conformal class $[g]$, or equivalently

$$\mathcal{B} = \left\{ u \in W^{k,2}(M) : u > 0 \text{ a.e. and } \|u\|_{L^{2_k^*}(M)} = 1 \right\}.$$

We denote this restricted solution set by $\mathcal{CQC}_{g,k}^* = \mathcal{CQC}_{g,k} \cap \mathcal{B}$.

The variational setting suggests that we seek solutions in $\mathcal{CQC}_{g,k}$ by studying a sequence minimizing the quotient $\mathcal{Q}_{g,k}$, and so we naturally define the k-order Yamabe invariant

$$\mathcal{Y}_{k,+}(M,[g]) = \inf \left\{ \mathcal{Q}_{g,k}(u) : u \in W^{k,2}(M) \text{ and } u > 0 \text{ a.e.} \right\}$$

and the minimizing set

$$\mathcal{M}_{g,k} = \left\{ u \in W^{k,2}(M) \,:\, u > 0 \text{ a.e. and } \mathcal{Q}_{g,k}(u) = \mathcal{Y}_{k,+}(M,[g]) \right\}.$$

Observe that $\mathcal{M}_{g,k} \subset \mathcal{CQC}_{g,k}$. Once again, we often restrict to minimizing solutions with unit volume, *i.e.* $\|u\|_{L^{2^*_k}(M)} = 1$ and denote this restricted set by $\mathcal{M}^*_{g,k}$.

The plus sign in the definition of $\mathcal{Y}_{k,+}$ signifies it is the infimum over functions in $W^{k,2}(M)$ that are positive almost everywhere. If $k = 1$, one can use the maximum principle to show the minimizer over all functions in $W^{k,2}(M)$ is automatically positive almost everywhere. So we may examine the infimum over all functions in $W^{k,2}(M)$. If $k \geq 2$, we lack the maximum principle. Hence, in general, there is no guarantee a minimizing function is an admissible conformal factor.

To place these curvature quantities in a more familiar setting, we remind the reader that $Q_{g,1}$ is (up to multiplication by a constant depending only on n) the scalar curvature R_g and $P_{g,1}$ is the usual conformal Laplacian

$$P_{g,1} = -\Delta_g + \frac{n-2}{4(n-1)} R_g.$$

In addition, we have

$$Q_{g,2} = -\frac{1}{2(n-1)} \Delta_g R_g - \frac{2}{(n-2)^2} |\operatorname{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 \tag{1.5}$$

is Branson's (fourth-order) Q-curvature and $P_{g,2}$ is the Paneitz operator, given by

$$P_{g,2}(u) = (-\Delta_g)^2 u + \operatorname{div}\left(\frac{4}{n-2} \operatorname{Ric}_g(\nabla u, \cdot) - \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g \nabla u \right) + \frac{n-4}{2} Q_{g,2}, \tag{1.6}$$

where Ric_g is the Ricci tensor. Thus (1.3) reduces to

$$P_{g,1}(u) = -\Delta_g(u) + \frac{4(n-1)}{n-2} R_g u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}$$

in the second order case, which is one of the most well-studied partial differential equations in geometric analysis, and

$$P_{g,2}(u) = \frac{n(n-4)(n^2-16)}{16} u^{\frac{n+4}{n-4}}$$

in the fourth order case.

The existence of solutions in general and minimizing solutions in particular is often a delicate question. In the classical case of the Yamabe problem (*i.e.* $k = 1$), the search started with Yamabe's work [54] and continued with the important contributions of Trudinger [52] and Aubin [5]. Schoen [48] finally resolved the Yamabe problem, showing that any Riemannian metric on a compact manifold without boundary is conformal to a constant scalar curvature metric. Schoen's solution uses the Green's function of the conformal Laplacian in a fundamental way, highlighting the important connection between the Green's function and scalar curvature. Since then, a sizeable community has sought to understand the solution and minimizing sets in all possible scenarios; we do not attempt to summarize the extensive literature here. We only mention some results characterizing the solution set in various situations. Combining the work of Trudinger and Aubin, one sees that if $\mathcal{Y}_{1,+}(M,[g]) < 0$ then there exists a unique solution, whereas in the positive case the existence of many solutions is possible. Schoen conjectured for many years the set of solutions is compact unless (M,g) is conformally equivalent to the round sphere. Eventually, Khuri, Marques and Schoen [31] verified this conjecture under positive mass theorem in the case that $n \leq 24$, while Marques and Brendle [40] demonstrated noncompactness in higher dimensions.

The search for solutions and/or minimizers is, as expected, more complicated in the higher order case. Here we highlight only some relatively recent results. Gursky and Malchiodi [27] showed that if $R_g \geq 0$ and $Q_{g,2} \geq 0$ but not identically zero, then the Green's function of $P_{g,2}$ is positive and the solution set $\mathcal{CQC}_{g,2}$ is nonempty. Later Hang and Yang [28] weakened the hypotheses of Gursky and Malchiodi, showing that it suffices to assume $\mathcal{Y}_{g,1}(M,[g]) > 0$. Most recently, Mazumdar and Vétois [42] showed the minimizing set $\mathcal{M}_{g,k}$ is non-empty under the following conditions. First they assume $\mathcal{Y}_{k,+}(M,[g]) > 0$. Next they assume that for every $\xi \in M$ the Green's function $G_{g,k,\xi}$ (which is the unique function such that distributionally $P_{g,k}(G_{g,k,\xi})(x) = \delta_\xi$, for δ_ξ the Dirac delta function at ξ) is positive everywhere on M.

Assuming $G_{g,k,\xi} > 0$, one can apply the analysis of Gursky and Malchiodi [27] (see also [28] and, for the general case $2 \leq k < \frac{n}{2}$, [43]) to see that

$$G_{g,k,\xi}(x) = b_{n,k}(\mathrm{dist}_g(x,\xi))^{2k-n} + m(\xi) + \mathrm{o}(1). \tag{1.7}$$

The quantity $m(\xi) = m_g(\xi)$ is usually called the mass of the GJMS operator $P_{g,k}$ at the point ξ. The final hypothesis of [42] is that if either $2k+1 \leq n \leq 2k+3$ or (M,g) is locally conformally flat, then $m(\xi) > 0$ for some $\xi \in M$.

Following the existence result of [42], we define the following space of admissible metrics. Let $k \in \mathbb{N}$ with $n > 2k$ and let $\alpha \in (0,1)$. Observe that the space of $\mathcal{C}^{k,\alpha}$-Riemannian metrics on M, denoted by $\mathrm{Met}^{k,\alpha}(M)$, is a convex cone in the space of all symmetric, rank-two covariant tensor fields over M whose coefficients are $\mathcal{C}^{k,\alpha}$ functions, and we equip all these spaces of tensor fields with the topology induced by convergence in the $\mathcal{C}^{k,\alpha}$ norm. We say a metric $g \in \mathrm{Met^{k,\alpha}(M)}$ is **admissible** if it satisfies all the existence hypotheses of Mazumdar and Vétois [42], as described in the previous two paragraphs. We denote the space of admissible metrics on M by $\mathfrak{A}_{k,\alpha}(M)$. That is, we let

$$\begin{aligned}\mathfrak{A}_{k,\alpha}(M) := \big\{ g \in \mathrm{Met}^{k,\alpha}(M) \,:\, &\mathcal{Y}_{k,+}(M,[g]) > 0,\, G_{g,k,\xi} > 0 \text{ for every } \xi \in M, \\ &m_g(\xi) > 0 \text{ for some } \xi \in M \big\}.\end{aligned}$$

1.2. **Quantitative stability estimates near minimizing metrics.** We are primarily interested in the stability of the minimizing set and in estimating the difference between $\mathcal{Q}_{g,k}(u)$ and its infimum in terms of the distance between u and the minimizing set $\mathcal{M}_{g,k}$. We define

$$d(u,\mathcal{M}_{g,k}) = \frac{\inf\{\|u-v\|_{W^{k,2}(M)} : v \in \mathcal{M}_{g,k}\}}{\|u\|_{W^{k,2}(M)}}. \tag{1.8}$$

Notice that this distance is well-defined whenever the minimizing set $\mathcal{M}_{g,k}$ is non-empty, which in turn implies the solution set $\mathcal{CQC}_{g,k}$ is non-empty, because $\mathcal{M}_{g,k} \subset \mathcal{CQC}_{g,k}$. We also interpret (1.3) weakly. In other words, we say that $u \in W^{k,2}(M)$ satisfies (1.3) in the weak sense if

$$\int_M u P_{g,k}(\phi)\,\mathrm{d}\mu_g = 0 \quad \text{for all} \quad \phi \in \mathcal{C}^\infty(M).$$

Our first theorem in this manuscript is the following general stability estimate.

Theorem 1. *Let $n,k \in \mathbb{N}$ with $n > 2k$ and let (M,g) be a smooth, closed, n-dimensional Riemannian manifold. If $g \in \mathfrak{A}_{k,\alpha}(M)$ for some $\alpha \in (0,1)$, then there exists $\gamma = \gamma(g) \geq 0$ such that*

$$d(u,\mathcal{M}_{g,k})^{2+\gamma} \lesssim \mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]) \quad \text{for all} \quad u \in W^{k,2}(M). \tag{1.9}$$

Furthermore, there exists a subset $\mathcal{G} \subset \mathfrak{A}_{k,\alpha}(M)$ of the space of admissible Riemannian metrics on M which is open and dense with respect to the $\mathcal{C}^{k,\alpha}$-topology such that if $g \in \mathcal{G}$, then (1.9) holds with $\gamma(g) = 0$.

We interpret (1.9) as saying there exists a constant c depending only on g such that

$$cd(u,\mathcal{M}_{g,k})^{2+\gamma} \leq \mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]).$$

When $\gamma = 0$ we refer to (1.9) as **quadratic stability**, and when $\gamma > 0$ we call (1.9) **degenerate stability** or **higher-order stability**. The last part of the statement of Theorem 1 may be phrased as saying that quadratic stability happens **generically**.

Remark 2. *The ideas used to prove the genericity part of our main theorem for $2 \leqslant k \leqslant \frac{n}{2}$ are in contrast with the ones in the case $k = 1$. On the one hand, our techniques rely on some results of Case, Lin and Yuan [12] combined with the transversality method inspired by [15, 22, 44]. On the other hand, the argument for the scalar curvature inspired by the ones given by Anderson [3] relies on some facts that are far from being known for higher-order curvatures such as a classification result by Obata [46].*

Previous work, which we summarize in this paragraph, proved quadratic stability in the case that (M,g) is conformally equivalent to the round sphere $(\mathbb{S}^n, \mathring{g})$. Bianchi and Egnell [9] proved this in the case $k = 1$, then Lu and Wei [39] proved it in the case $k = 2$, Bartsch, Weth and Willem [6] for integers $1 \leq k < \frac{n}{2}$, and finally Chen, Frank and Weth [14] for each $k \in (0, n/2)$, including non-integers. Notice that when the background manifold is conformally equivalent to the round sphere $(\mathbb{S}^n, \mathring{g})$, through the stereographic projection Eq. (1.9) is a refined version of the classical Sobolev inequality on the standard Euclidean space $(\mathbb{R}^n, \delta)$, namely

$$\|u\|_{W^{k,2}(\mathbb{R}^n)} \lesssim \|u\|_{L^{2^*_k}(\mathbb{R}^n)}.$$

In the setting of generic manifolds, Engelstein, Neumayer, and Spolaor [17] proved the stability estimate (1.9) in the case $k = 1$.

From a geometric point of view, one drawback of our first main result is that the distance $d(u,\mathcal{M}_{g,k})$ may depend on the choice of background metric $g \in [g]$. This is because the $W^{k,2}$-norm, with which $d(u,\mathcal{M}_{g.k})$ is defined, is not conformally invariant. However, we can modify our distance function to obtain the following conformally invariant stability estimate.

We define the following conformally invariant norm for metrics $h = u^{4/(n-2k)}g$:

$$\|h\| := \left(\int_M |u|^{\frac{2n}{n-2k}}\, \mathrm{d}\mu_g\right)^{\frac{n-2k}{2n}}. \tag{1.10}$$

The definition appears to depend on the choice of the background metric g, but the following computation shows $\|\cdot\|$ depends only on the conformal class $[g]$. If $\widehat{g} = \phi^{4/(n-2k)}g$ is a conformal metric, then $h = u^{4/(n-2k)}g = \widehat{u}^{4/(n-2k)}\widehat{g}$ where $u = \widehat{u}\phi$, and so

$$\|h\| = \int_M |u|^{\frac{2n}{n-2k}}\, \mathrm{d}\mu_g = \int_M |\widehat{u}|^{\frac{2n}{n-2k}} \phi^{\frac{2n}{n-2k}}\, \mathrm{d}\mu_g = \int_M |\widehat{u}|^{\frac{2n}{n-2k}}\, \mathrm{d}\mu_{\widehat{g}}.$$

Similarly, in the case when $\mathcal{Y}_{k,+}(M,[g]) \geq 0$, for $h = u^{4/(n-2k)}g$ we define

$$\|h\|_* = \left(\int_M uP_{g,k}(u)\, \mathrm{d}\mu_g\right)^{1/2} \tag{1.11}$$

for any $g \in \mathrm{Met}^\infty(M)$ with $\mathrm{vol}_g(M) = 1$. Again, although $\|\cdot\|_*$ is defined with respect to a fixed conformal representative, it turns out that the definition is independent of this choice. Namely that for any $\hat{g} \in [g]$ and $h = u^{4/(n-2k)}g = \hat{u}^{4/(n-2k)}\hat{g} \in [g]$, one has

$$\|h\|_* = \left(\int_M uP_{g,k}(u)\, \mathrm{d}\mu_g\right)^{1/2} = \left(\int_M \hat{u}P_{\hat{g},k}(\hat{u})\, \mathrm{d}\mu_{\hat{g}}\right)^{1/2}.$$

Corollary 3. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. If $g \in \mathfrak{A}_{k,\alpha}(M)$ for some $\alpha \in (0,1)$ is an admissible Riemannian metric, then there exists $\gamma(g) \geq 0$ such that*

$$\left(\frac{\inf\{\|h-\tilde{g}\| : \tilde{g} \in \mathcal{M}_{g,k}\}}{\mathrm{vol}_h(M)^{\frac{n-2k}{2n}}}\right)^{2+\gamma} \lesssim \mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]).$$

When $\mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]) \leqslant \delta_0$ for some $0 < \delta_0 \ll 1$ small enough, there exists $\gamma \geq 0$ depending on (M, g) such that

$$\left(\frac{\inf\{\|h-\tilde{g}\|_* : \tilde{g} \in \mathcal{M}_{g,k}\}}{\mathrm{vol}_h(M)^{1/2^*}}\right)^{2+\gamma} \lesssim \mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]) \quad \text{for all} \quad h \in [g].$$

Moreover, for an open dense subset in the $\mathcal{C}^2$ topology on the space of conformal classes of $\mathrm{Met}^\infty(M)$*, the above inequalities hold with $\gamma = 0$.*

Before going on, let us quickly sketch some heuristic ideas for the proof of Theorem 1. In a sense we make precise in our proofs below, quadratic stability is closely related to the nondegeneracy of u_0 as a minimizer of $\mathcal{Q}_{g,k}$. Indeed, if $g_0 = u_0^{4/(n-2k)} g$ is an element of the minimizing set $\mathcal{M}_{g,k}$ and we write a nearby metric in the conformal class as $g_v = (u_0 + v)^{4/(n-2k)} g$, we can write out a formal Taylor expansion

$$\begin{aligned}\mathcal{Q}_{g,k}(u_0+v) &= \mathcal{Q}_{g,k}(u_0) + D\mathcal{Q}_{g,k}(u_0)(v) + D^2\mathcal{Q}_{g,k}(u_0)(v,v) + \mathcal{O}(\|v\|^3)\\ &= \mathcal{Y}_{k,+}(M,[g]) + D^2\mathcal{Q}_{g,k}(u_0)(v,v) + \mathcal{O}(\|v\|^3).\end{aligned}$$

Here we used the fact that $u_0 \in \mathcal{M}_{g,k}$ and that g_0 is a constant Q-curvature metric, which, as we mentioned above, means u_0 is a critical point of $\mathcal{Q}_{g,k}$. If u_0 is a nondegenerate critical point, then the Hessian $D^2\mathcal{Q}_{g,k}(u_0)$ does not vanish on any v, implying in turn that the difference $\mathcal{Q}_{g,k}(u_0+v) - \mathcal{Y}_{k,+}(M,[g])$ is indeed quadratic in $\|v\|$. Furthermore, by a generalized version of Sard–Smale's theorem (see Lemma D), we expect nondegeneracy to happen generically. All of this is carried out in a rigorous way in Section 4.

1.3. **Examples for higher-order stability.** Given the description in the previous subsection, it is of great interest to produce examples of minimizing metrics satisfying a superquadratic stability estimate, i.e., manifolds (M, g) such that Theorem 1 holds for some $\gamma > 0$, but not for $\gamma = 0$. Motivated by a classical example of Schoen [50] and a newer one by Carlotto, Chodosh and Rubinstein [11] (based on work by Caffarelli, Gidas and Spruck [10] and by Schoen [49]) we produce two different families of constant curvature metrics, each of which satisfies a higher-order stability estimate. The degree of degeneracy of the metrics as minimizers, or more generally as critical points, is made precise by the Adams-Simon positivity condition (AS_p condition for short), for which, we refer to Definition 4.3 below.

Theorem 4. *Let $m, \ell \in \mathbb{N}$ with $\ell \geq 2$, let $\lambda \in \mathbb{R}$, and let (M, g, λ) be an m-dimensional Einstein manifold with Einstein constant λ. If $m \gg 1$ is sufficiently large, the function $u \equiv 1$ is a degenerate critical point of $\mathcal{Q}_{h,2}$ for certain values of λ, where $(X, h) = (M \times \mathbb{S}^\ell, g \oplus \mathring{g})$. Moreover, one can replace $(\mathbb{S}^\ell, \mathring{g})$ by $(\mathbb{CP}^\ell, g_{\mathrm{FS}})$, where g_{FS} is the Fubini-Study metric, in this example.*

Please see the statements of Propositions 5.1 and 5.2 below for more detailed statements. The reader will see that our proof of Theorem 4 is quite flexible. In particular, one should be able to prove a similar result for $k \geq 3$, but the computations quickly become unwieldy. While our proof is inspired by the example given in [11, Section 5.1], our techniques end up being somewhat different. Carlotto *et al* are able to choose the individual manifolds such that the product $h = g \oplus g_{\mathrm{FS}}$ is

Einstein on $M \times \mathbb{CP}^\ell$, whereas we cannot. This complicates our verification of the AS_3 condition, which in turn explains the requirement that the dimension of M is large enough. This is in contrast with the case $k = 1$, where no restrictions are imposed on $\dim(M)$.

We can also exhibit an explicit example of a manifold for which Theorem 1 holds with $\gamma = 2$. Our analysis covers every integer order $1 \le k < \frac{n}{2}$, thus extending the corresponding example for $k = 1$ discussed in [11], whose degenerate stability was recently analyzed in detail in [19]. We fix $n > 2k$ and consider the manifold

$$M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1} \tag{1.12}$$

with the standard (non-normalized) product metric h. Here $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and $\mathbb{S}^1(\tau_0)$ denotes the unit sphere in $\mathbb{R}^2$ of an appropriately chosen radius $\tau_0 > 0$, see (6.14) below.

Theorem 5. *Let $n, k \in \mathbb{N}$ with $n > 2k$. There exists a specific $\tau_0 > 0$, defined below as the unique number satisfying (6.13), such that for $(\mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}, h)$ the constant function $u \equiv 1$ is a degenerate minimizer of the functional $\mathcal{Q}_{h,k}$ which satisfies the AS_4 condition. Moreover, the function $u \equiv 1$ satisfies (1.9) with $\gamma = 2$, and does not satisfy (1.9) for any $\gamma < 2$.*

We point out some complications we encounter in the proof of Theorem 5, as compared to its second-order counterparts in [11,49]. First, in the case that $k = 1$ one only encounters polynomials of degree at most two, whose roots are relatively easy to find. In our analysis, we must find roots of higher-order polynomials, which we can do only through extremely careful and systematic accounting. More significantly, a fundamental ingredient in [11, 49] is the phase-plane analysis of the second-order ODE arising from (1.3) when $k = 1$. This phase-plane analysis allows one to quickly show that all ODE solutions are, up to translations, uniquely characterized by their period and thus occur in a one-parameter family. More refined arguments (see [11, Appendix B] and references therein) show that the period length is actually a monotone function of the parameter. Using this, the authors of [11, 49] are able to conclude the crucial fact that the only minimizers of $Q_{h,1}$ are the constants for $k = 1$. However, for integers $k \ge 2$ the classification of solutions to the corresponding higher-order ODE is only known for $k = 2, 3$ [4, 20] and even in those cases the monotonicity of the period length is an open problem. To overcome this difficulty, inspired by a remark in [19, p. 1463], we succeed in adapting an argument due to Beckner [7, Theorem 4]: see Lemma 6.6 and Step 3 in the proof of Lemma 6.7.

Remark 6. *The examples we describe in Theorems 4 and 5 are not only interesting for their novelty. Additionally, they could also provide important examples illustrating the slow convergence of the geometric flow towards a constant Q-curvature metric. We plan to address the convergence of this flow in a future paper.*

We close this introduction with a brief outline of the rest of the paper. We begin with some preliminaries in Section 3, proving that the total Q-curvature functional is regular in Section 3.1 and listing some auxiliary lemmas from elsewhere in Section 3.2. We prove Theorem 1 and Corollary 3 in Section 4. In Section 5 we prove Theorem 4, specifically discussing products with spheres in Section 5.1 and products with complex projective space in Section 5.2. Finally we prove Theorem 5 in Section 6. We include Juhl's general recursion formulas for the GJMS operators in Appendix A for the reader's reference, even though we do not use them in the main text.

2. Notation

Let us establish some standard terminology and definitions. In what follows, we will always be using Einstein's summation convention. Furthermore, we omit the subscript g, in the section the metric is fixed.

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

- $k \in \mathbb{N}$ and $n > 2k$
- (M, g) is a smooth closed n-dimensional Riemannian manifold;
- $\delta = g_{\mathbb{R}^n}$ denotes the standard Euclidean metric;
- $\mathring{g} = g_{\mathbb{S}^n}$ denotes the standard round metric;
- ω_n denotes the volume of the Euclidean n-sphere;
- $\mathrm{Met}^{k,\alpha}(M)$ denotes the space of $\mathcal{C}^{k,\alpha}$-metrics in M, when $\alpha = 0$, we simply denote $\mathrm{Met}^k(M)$;
- $\mathrm{Met}^\infty(M) = \cup_{j\in\mathbb{N}}\mathrm{Met}^j(M)$ denotes the space of smooth metrics in M;
- $(e_i)_{i=1}^n$ denotes a local coordinate frame;
- $\mathfrak{T}^r_s(M)$ denotes the set of (r,s)-type tensor over M with $\mathfrak{T}^0_0(M) = \mathcal{C}^\infty(M)$;
- $\mathrm{Rm}_g \in \mathfrak{T}^4_0(M)$ (or $\mathrm{Rm}_g \in \mathfrak{T}^3_1(M)$) denotes the (or covariant) Riemannian curvature tensor,
- $\mathrm{Ric}_g = \mathrm{tr}_g \,\mathrm{Rm}_g \in \mathfrak{T}^2_0(M)$, traced over the first and last indices, *i.e.* $(\mathrm{Ric}_g)_{ij} = g^{k\ell}\mathrm{Rm}_{\mathrm{kij}\ell}$;
- $R_g = \mathrm{tr}_g\mathrm{Ric}_g \in \mathfrak{T}^0_0(M)$ denotes the scalar curvature given by $R_g = g^{ij}\mathrm{Ric}_{ij}$;
- $\Delta_g = g^{ij}\nabla_i\nabla_j$ denotes the Laplace–Beltrami operator;
- $\delta_g = \mathrm{div}_g$ denotes the metric divergence;
- ∇_g denotes the Levi–Civita connection;
- $\mathrm{tr}_g : \mathfrak{T}^r_s(M) \to \mathfrak{T}^{r-2}_s(M)$ denotes a trace operator;
- $a_1 \lesssim a_2$ if $a_1 \leqslant Ca_2$, $a_1 \gtrsim a_2$ if $a_1 \geqslant Ca_2$, and $a_1 \simeq a_2$ if $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$;
- $u = \mathcal{O}(f)$ as $x \to x_0$ for $x_0 \in \mathbb{R}\cup\{\pm\infty\}$, if $\limsup_{x\to x_0}(u/f)(x) < \infty$ is the Big-O notation;
- $u = \mathrm{o}(f)$ as $x \to x_0$ for $x_0 \in \mathbb{R}\cup\{\pm\infty\}$, if $\lim_{x\to x_0}(u/f)(x) = 0$ is the little-o notation;
- $u \simeq \widetilde{u}$, if $u = \mathcal{O}(\widetilde{u})$ and $\widetilde{u} = \mathcal{O}(u)$ as $x \to x_0$ for $x_0 \in \mathbb{R}\cup\{\pm\infty\}$;
- $\mathcal{C}^{j,\alpha}(M)$, where $j \in \mathbb{N}$ and $\alpha \in (0,1)$, is the classical Hölder space over M; we simply denote $\mathcal{C}^j(M)$ when $\alpha = 0$;
- $W^{j,q}_g(M)$ is the Sobolev space over M, where $j \in \mathbb{N}$ and $q \in [1,\infty]$; when $j = 0$ we simply denote $L^q_g(M)$;
- $2^*_k = \frac{2n}{n-2k}$ is the critical exponent of the Sobolev embedding $W^{k,2}_g(M) \hookrightarrow L^{2^*_k}_g(M)$;
- $Q_{g,k}$ is the $2k$-th order Q-curvature of g;
- $P_{g,k}$ is the $2k$-th order GJMS operator of g;
- $\mathcal{Q}_{g,k}$ is the total $2k$-th order Q-curvature functional of g;
- $\mathcal{M}_{g,k}$ is the set of minimizers for $\mathcal{Q}_{g,k}$;
- $G_{g,k,\xi}$ is the Green's function of $P_{g,k}$ with pole at ξ.

3. Preliminaries

In this section we first establish the regularity of the total Q-curvature functional and compute its first two derivatives. Then we list some auxiliary lemmas from other papers which we will need.

3.1. **Regularity of the total Q-curvature functional.** We fix a background metric g and recall that the normalized total Q-curvature functional on the conformal class $[g]$ is given by

$$\mathcal{Q}_{g,k}(u) = \frac{2}{n-2k}\frac{\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g}{\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{n-2k}{n}}}.$$

Since $\mathcal{Q}_{g,k}(cu) = \mathcal{Q}_{g,k}(u)$ for any $c > 0$, it will often be easier to work with functions having $L^{\frac{2n}{n-2k}}$-norm equal to 1. To that end, we introduce the following Banach manifold

$$\mathcal{B} = \left\{u \in W^{k,2}_+(M) : \int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g = 1\right\}$$

and observe that if $u \in \mathcal{B} \cap \mathcal{C}^\infty(M)$ then $\widetilde{g} = u^{4/(n-2k)} g$ is a smooth metric in the conformal class $[g]$ with unit volume.

Lemma 3.1. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. The tangent space of $\mathcal{B}$ at u is given by*

$$T_u\mathcal{B} = \left\{ v \in W^{k,2}(M) : \int_M u^{\frac{n+2k}{n-2k}} v \, \mathrm{d}\mu_g = 0 \right\}.$$

Moreover, for each $v \in T_u\mathcal{B}$ the mappings

$$v \mapsto \pi_{T_u\mathcal{B}} : T_u\mathcal{B} \to \mathcal{L}(W^{k,2}(M), W^{k,2}(M))$$

and

$$v \mapsto \pi_{T_u\mathcal{B}} : T_u\mathcal{B} \to \mathcal{L}(\mathcal{C}^{2k,\alpha}(M), \mathcal{C}^{2k,\alpha}(M))$$

are both continuous.

Proof. By density, it suffices to take $v \in \mathcal{C}^\infty(M)$. Indeed, $v \in T_u\mathcal{B}$ precisely when

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \int_M (u + \varepsilon v)^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g = \frac{2n}{n-2k} \int_M u^{\frac{n+2k}{n-2k}} v \, \mathrm{d}\mu_g.$$

The proof of continuity of $\pi_{\mathcal{B}}$ is the same as the one at the end of the proof of [17, Lemma 2.1]. □

Lemma 3.2. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. The mapping $u \mapsto \mathcal{Q}_{g,k}(u)$ is $\mathcal{C}^2$. If $u \in \mathcal{B}$ and $v, w \in T_u\mathcal{B}$ then*

$$\begin{aligned} D\mathcal{Q}_{g,k}(u)(v) &= \frac{2}{n-2k} \int_M [uP_{g,k}(v) + vP_{g,k}(u)] \, \mathrm{d}\mu_g \qquad (3.1) \\ D^2\mathcal{Q}_{g,k}(u)(v,w) &= \frac{2}{n-2k} \int_M [vP_{g,k}(w) + wP_{g,k}(v)] \, \mathrm{d}\mu_g - 2\left(\frac{n+2k}{n-2k}\right) \mathcal{Q}_{g,k}(u) \int_M u^{\frac{4k}{n-2k}} vw \, \mathrm{d}\mu_g. \end{aligned}$$

Proof. We begin with the expansion

$$\begin{aligned} \left(\int_M (u+\varepsilon v)^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g\right)^{\frac{2k-n}{n}} &= \left(\int_M u^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g\right)^{\frac{2k-n}{n}} \qquad (3.2) \\ &\quad -2\varepsilon \left(\int_M u^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}} \int_M u^{\frac{n+2k}{n-2k}} v \, \mathrm{d}\mu_g \\ &\quad +\varepsilon^2 \left(\frac{2n-2k}{n}\right) \left(\int_M u^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g\right)^{\frac{2k-3n}{n}} \left(\int_M u^{\frac{n+2k}{n-2k}} v \, \mathrm{d}\mu_g\right)^2 \\ &\quad -\varepsilon^2 \left(\frac{n+2k}{n-2k}\right) \left(\int_M u^{\frac{2n}{n-2k}} \, \mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}} \int_M u^{\frac{4k}{n-2k}} v^2 \, \mathrm{d}\mu_g + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Combining this expansion with

$$\int_M (u+\varepsilon v) P_{g,k}(u+\varepsilon v) \, \mathrm{d}\mu_g = \int_M uP_{g,k}(u) \, \mathrm{d}\mu_g + \varepsilon \int_M [uP_{g,k}(v) + vP_{g,k}(u)] \, \mathrm{d}\mu_g + \varepsilon^2 \int_M vP_{g,k}(v) \, \mathrm{d}\mu_g,$$

we obtain

$$\begin{aligned}
\mathcal{Q}_{g,k}(u+\varepsilon v) &= \frac{2}{n-2k}\left(\int_M (u+\varepsilon v)^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-n}{n}}\int_M (u+\varepsilon v)P_{g,k}(u+\varepsilon v)\,\mathrm{d}\mu_g \\
&= \mathcal{Q}_{g,k}(u)+\frac{2\varepsilon}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-n}{n}}\int_M [vP_{g,k}(u)+uP_{g,k}(v)]\,\mathrm{d}\mu_g \\
&\quad -\frac{4\varepsilon}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}}\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g \\
&\quad -\frac{2\varepsilon^2}{n-2k}\left(\frac{n+2k}{n-2k}\right)\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}}\int_M u^{\frac{4k}{n-2k}}v^2\,\mathrm{d}\mu_g\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g \\
&\quad +\frac{2\varepsilon^2(2n-2k)}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-3n}{n}}\left(\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\right)^2\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g \\
&\quad +\frac{2\varepsilon^2}{n-2k}\left(\int_M u^{\frac{2k-n}{n}}\,\mathrm{d}\mu_g\right)^{\frac{2k-n}{n}}\int_M vP_{g,k}(v)\,\mathrm{d}\mu_g \\
&\quad -\frac{4\varepsilon^2}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}}\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\int_M [vP_{g,k}(u)+uP_{g,k}(v)]\,\mathrm{d}\mu_g+\mathcal{O}(\varepsilon^3) \\
&= \mathcal{Q}_{g,k}(u)+\varepsilon D\mathcal{Q}_{g,k}(u)(v)+\frac{1}{2}\varepsilon^2D^2\mathcal{Q}_{g,k}(u)(v,v)+\mathcal{O}(\varepsilon^3),
\end{aligned}$$

which implies $\mathcal{Q}_{g,k}$ is a $\mathcal{C}^2$ functional.

We can read off from this last expansion that

$$\begin{aligned}
D\mathcal{Q}_{g,k}(u)(v) &= \frac{2}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-n}{n}}\left(\int_M [vP_{g,k}(u)+uP_{g,k}(v)]\,\mathrm{d}\mu_g\right) \\
&\quad -\frac{4}{n-2k}\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}}\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{n-2k}{2}D^2\mathcal{Q}_{g,k}(u)(v,w) \\
=\ &\int_M [vP_{g,k}(w)+wP_{g,k}(v)]\,\mathrm{d}\mu_g\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-n}{n}} \\
&-2\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}} \\
&\quad\times\left(\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\int_M [wP_{g,k}(u)+uP_{g,k}(w)]\,\mathrm{d}\mu_g+\int_M u^{\frac{n+2k}{n-2k}}w\,\mathrm{d}\mu_g\int_M [vP_{g,k}(u)+uP_{g,k}(v)]\,\mathrm{d}\mu_g\right) \\
&-2\left(\frac{n+2k}{n-2k}\right)\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-2n}{n}}\int_M u^{\frac{4k}{n-2k}}vw\,\mathrm{d}\mu_g\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g \\
&+\left(\frac{2n-2k}{n}\right)\left(\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g\right)^{\frac{2k-3n}{n}}\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g \\
&\quad\times\left(\left(\int_M u^{\frac{n+2k}{n-2k}}(v+w)\,\mathrm{d}\mu_g\right)^2-\left(\int_M u^{\frac{n+2k}{n-2k}}v\,\mathrm{d}\mu_g\right)^2-\left(\int_M u^{\frac{n+2k}{n-2k}}w\,\mathrm{d}\mu_g\right)^2\right).
\end{aligned} \tag{3.4}$$

In this last expression, we have used the polarization identity

$$D^2\mathcal{Q}_{g,k}(u)(v,w) = \frac{1}{2}\left(D^2\mathcal{Q}_{g,k}(u)(v+w,v+w) - D^2\mathcal{Q}_{g,k}(u)(v,v) - D^2\mathcal{Q}_{g,k}(u)(w,w)\right).$$

Restricting to the case $u \in \mathcal{B}$ and $v, w \in T_u\mathcal{B}$, we impose the constraints

$$\int_M u^{\frac{2n}{n-2k}}\,\mathrm{d}\mu_g = 1 \quad \text{and} \quad \int_M u^{\frac{n+2k}{n-2k}} v\,\mathrm{d}\mu_g = 0 = \int_M u^{\frac{n+2k}{n-2k}} w\,\mathrm{d}\mu_g, \tag{3.5}$$

we see that for $u \in \mathcal{B}$ and $v, w \in T_u\mathcal{B}$ the expressions (3.3) and (3.4) reduce to

$$D\mathcal{Q}_{g,k}(u)(v) = \frac{2}{n-2k}\int_M [vP_{g,k}(u) + uP_{g,k}(v)]\,\mathrm{d}\mu_g$$

and

$$\begin{aligned} D^2\mathcal{Q}_{g,k}(u)(v,v) = {} & \frac{2}{n-2k}\int_M [vP_{g,k}(w) + wP_{g,k}(v)]\,\mathrm{d}\mu_g \\ & - \frac{4}{n-2k}\left(\frac{n+2k}{n-2k}\right)\int_M u^{\frac{4k}{n-2k}} vw\,\mathrm{d}\mu_g \int_M uP_{g,k}(u)\,\mathrm{d}\mu_g. \end{aligned}$$

The first of these formulas is exactly the first derivative as listed in (3.1). We obtain the listed formula for the second derivative after using the identity

$$\mathcal{Q}_{g,k}(u) = \frac{2}{n-2k}\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g.$$

□

We obtain modulus of continuity estimates from the structure of the linearized operator. For a given $u \in \mathcal{B}$ we consider the linearization of (1.3) about u, which is the operator

$$L_u = P_{g,k} - \left(\frac{n+2k}{n-2k}\right)\mathcal{Q}_{g,k}(u)u^{\frac{4k}{n-2k}}.$$

Lemma 3.3. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. The mappings*

$$u \mapsto \frac{D^2(\mathcal{Q}_{g,k})(u)(v,w)}{\|v\|_{W^{k,2}(M)}\|w\|_{W^{k,2}(M)}}$$

and

$$u \mapsto \frac{D^2(\mathcal{Q}_{g,k})(u)(v,\cdot)}{\|v\|_{\mathcal{C}^{2k,\alpha}(M)}}$$

are both continuous with moduli of continuity that are uniformly bounded with respect to v and w.

Proof. Let $u_0, u_1 \in \mathcal{B}$. Using the fundamental theorem of calculus we see

$$\begin{aligned} L_{u_1}(v) - L_{u_0}(v) \quad = \quad & -\frac{n+2k}{n-2k}\int_0^1 D(\mathcal{Q}_g^2)(tu_1 + (1-t))(u_1-u_0)((1-t)u_0 + tu_1)^{\frac{4k}{n-2k}} v\,\mathrm{d}t \\ & -\frac{4k(n+2k)}{(n-2k)^2}\int_0^1 \mathcal{Q}_{g,k}((1-t)u_0 + tu_1)((1-t)u_0+tu_1)^{\frac{6k-n}{n-2k}}(u_1-u_0)v\,\mathrm{d}t, \end{aligned}$$

which we can in turn integrate to obtain the following estimate

$$\|(L_{u_1} - L_{u_0})(v)\|_{W^{k,2}(M)} \lesssim \|u_1 - u_0\|_{W^{k,2}(M)}\|v\|_{W^{k,2}(M)},$$

uniformly on u and v. It follows from the second formula in (3.1) that

$$u \mapsto \frac{D^2(\mathcal{Q}_{g,k})(u)(v,w)}{\|v\|_{W^{k,2}(M)}\|w\|_{W^{k,2}(M)}}$$

is a continuous map whose modulus of continuity is uniform over $v, w \in W^{k,2}(M)$. One can similarly show

$$u \mapsto \frac{D^2(\mathcal{Q}_{g,k})(u)(v,\cdot)}{\|v\|_{\mathcal{C}^{2k,\alpha}(M)}} \in \mathcal{C}^{2k,\alpha}(M)$$

is also a continuous map. □

3.2. **Some auxiliary results.** In this section, we present some auxiliary results that will be used in the proofs of our main results.

First, we establish the variational setting we will use for the remainder of our analysis. This is a version of the Lyapunov-Schmidt reduction, which appears in many applications. Heuristically, it divides a variation of our functional into a component that changes the functional by an amount we can estimate and an orthogonal component that lies in a finite-dimensional space.

Let (M, g) be a closed Riemannian manifold with $n > 2k$ and $k \in \mathbb{N}$ such that $g \in \mathrm{Met}^3(M)$. Let $u \in \mathcal{M}^*_{g,k} = \mathcal{B} \cap \mathcal{M}_{g,k}$ correspond to a minimizing metric in the conformal class $[g]$ with unit volume and let

$$K = \ker(D^2\mathcal{Q}_{g,k}(u)(\cdot,\cdot)) = \{v \in W^{k,2}(M) \cap T_u\mathcal{B} : D^2\mathcal{Q}_{g,k}(u)(v,v) = 0\}.$$

This second variation operator is elliptic, so K is finite-dimensional (see, for instance, [45, Section 10.4]). We let $\ell = \dim(K)$ and denote by $K^\perp$ the orthogonal complement of K inside $W^{k,2}(M)$ with respect to the L^2 inner product.

One can find proof of the following lemma [17, Appendix A].

Lemma A. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. Assume that $u \in \mathcal{CQC}^*_{g,k}$. Then there is an open neighborhood $U \subset K$ of 0 in K and a map $F : U \to K^\perp$ with $F(0) = 0$ and $\nabla F(0) = 0$ satisfying the following properties:*

(i) *Firstly*
$$\mathcal{L} := \{u + \varphi + F(\varphi) : \varphi \in U\} \subset \mathcal{B}.$$
(ii) *If we define $q : U \to \mathbb{R}$ by $q(\varphi) = \mathcal{Q}_{g,k}(u + \varphi + F(\varphi))$ then we have*
$$\nabla_{\mathcal{B}}\mathcal{Q}_{k,g}(v + \varphi + F(\varphi)) = \pi_K \nabla_{\mathcal{B}}\mathcal{Q}_{g,k}(v + \varphi + F(\varphi)) = \nabla q(\varphi).$$
Furthermore, $\varphi \mapsto q(\varphi)$ is real analytic.
(iii) *There exists $\delta > 0$ depending on u such that for any $\widetilde{u} \in \mathcal{B}$ with $\|u - \widetilde{u}\|_{W^{k,2}(M)} \leq \delta$ we have $\pi_K(u - \widetilde{u}) \in U$. Furthermore, if $\widetilde{u} \in \mathcal{CQC}^*_{g,k} = \mathcal{CQC}_{g,k} \cap \mathcal{B}$ with $\|u - \widetilde{u}\|_{W^{k,2}(M)} \leq \delta$ then*
$$\widetilde{u} = u + \pi_K(\widetilde{u} - u) + F(\pi_K(\widetilde{u} - u)).$$
(iv) *For all $\varphi \in U$ and $\eta \in K$, we have*
$$\|\nabla F(\varphi)[\eta]\|_{\mathcal{C}^{2k,\alpha}(M)} \lesssim \|\eta\|_{\mathcal{C}^{0,\alpha}(M)}.$$

Proof. See [17, Appendix A] □

Second, we need the following compactness result for minimizing sequences.

Lemma B. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. If $g \in \mathfrak{A}_k$ is admissible and $\{u_m\}_{m\in\mathbb{N}} \subset \mathcal{B}$ is a sequence such that $\lim_{m\to\infty} \mathcal{Q}_{g,k}(u_m) = \mathcal{Y}_{k,+}(M, [g])$, then there exists $v \in \mathcal{M}_{g,k} \cap \mathcal{B}$ such that $\lim_{m\to\infty} \|u_m - v\|_{W^{k,2}(M)} = 0$ up to a subsequence.*

Proof. For $k = 1$, this is a classical result due to Aubin [5]. For $k \geq 1$, the lemma follows from [41, Theorem 3] and its proof is in the spirit of Lions' concentration-compactness [38, Theorem 4.1]. For the analogous statement in a dual formulation when $k = 2$, see also [28, Proposition 2.6]. □

We remark that the statement of [41, Theorem 3] allows for singular limits, but the additional hypotheses we've listed here preclude the development of singularities.

Third, we introduce the so-called finite-dimensional "distance" Łojasiewicz inequality.

Lemma C. *Let $q : \mathbb{R}^\ell \to \mathbb{R}$ with $\ell \in \mathbb{N}$ be a real analytic function and assume that $\nabla q(\varphi_0) = 0$. There exist $\tilde{\delta} > 0$ and $\gamma \geq 0$, depending on q and φ_0 such that*

$$|q(\varphi) - q(\varphi_0)| \gtrsim \inf\{|\varphi - \bar{\varphi}| : \bar{\varphi} \in B(\varphi_0, \tilde{\delta}) \text{ and } q(\bar{\varphi}) = q(\varphi_0)\}^{2+\gamma} \quad \text{for all} \quad \varphi \in B(\varphi_0, \tilde{\delta}).$$

Proof. See [35, Théorème 2]. □

Finally, we state a generalized version by Henry [30] of Smale's version of Sard's theorem [51], which we use to prove the genericity part of our main results.

Lemma D. *Let X, Y and Z be Banach spaces, let $U \subset X$ and $V \subset Y$ be open subsets and let $\mathcal{F} : V \times U \to Z$ be a map of class $\mathcal{C}^1$ with $z_0 \in \mathrm{Im}(\mathcal{F})$. Suppose that*

- (i) *For each $y \in V$ the map $x \mapsto \mathcal{F}(y, x) =: \mathcal{F}_y(x)$ is Fredholm of index $\ell < 1$, i.e. we have $\partial_x \mathcal{F}_y : X \to Z$ is Fredholm of index $\ell < 1$ for any $x \in U$;*
- (ii) *z_0 is a regular value of $\mathcal{F}$;*
- (iii) *Let $\iota : \mathcal{F}^{-1}(z_0) \to Y \times X$ be the canonical embedding and let $\pi_1 : Y \times X \to Y$ be projection onto the first factor. Then, $\pi_1 \circ \iota$ is σ-proper, i.e., $\mathcal{F}^{-1}(z_0) = \bigcup_{j=1}^\infty C_j$, where C_j is a closed subset of $\mathcal{F}^{-1}(z_0)$ and $\pi_1 \circ \iota|_{C_j}$ is proper for all $j \in \mathbb{N}$.*

Then, the set $\{y \in V : z_0$ is a regular value of $\mathcal{F}(y, \cdot)\}$ is an open and dense subset of V.

Proof. See [30, Theorem 5.4] □

4. Generic stability estimates (proofs of Theorem 1 and Corollary 3)

We prove Theorem 1 in stages. Like in [9, 17] the main point is to prove a local version of the quantitative stability (Proposition 4.5 below), which can then be extended to the global case. Finally we prove the genericity statement, the arguments for which go beyond those in [17].

4.1. **Local stability estimate.**

In this section, we prove that if u lies in a small neighborhood of the minimizing set $\mathcal{M}^*_{g,k}$ then it satisfies (1.9). As a first step, we introduce the notion of integrability and prove that the function q defined in Lemma A is constant in the integrable case, which we will see implies stability.

Definition 4.1. *Let $k \in \mathbb{N}$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold with $n > 2k$. A function $v \in \mathcal{CQC}^*_{g,k}$ is said to be integrable if for all $\varphi \in K$ there exists a one-parameter family of functions $\{v_t\}_{t \in (-\delta, \delta)}$, with $v_0 = v$, $\partial_t|_{t=0} v_t = \varphi$, and $v_t \in \mathcal{CQC}^*_{g,k}$ for all $0 < t \ll 1$ sufficiently small.*

With this definition in mind, we have the following auxiliary result.

Lemma 4.2. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. If $v \in \mathcal{M}^*_{g,k}$, then v is integrable if and only if q is constant in a neighborhood of $0 \in K$. In particular, if $v \in \mathcal{M}^*_{g,k}$ is an integrable minimizer, then there is $\delta > 0$ such that*

$$\mathcal{M}^*_{g,k} \cap \mathcal{B}(v, \delta) = \mathcal{L} \cap \mathcal{B}(v, \delta), \tag{4.1}$$

where $\mathcal{L}$ is as described in Lemma A (i).

Proof. In this proof, we identify K with $\mathbb{R}^\ell$. We may assume that $\ell \geq 1$, for otherwise there is nothing to show.

First, suppose that $q \equiv q(0)$ in a neighborhood $V \subset \mathbb{R}^\ell$ of 0, and let $\varphi \in K$. Let $s \in \mathbb{R}$ be small enough so that $s\varphi \in U$ and consider

$$v_s = v + s\varphi + F(s\varphi)$$

where U and F are as in Lemma A. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} v_s = \varphi$$

because $\nabla F(0) = 0$ by Lemma A. Still by that lemma, we have $v_s \in \mathcal{B}$ and

$$\nabla_{\mathcal{B}} \mathcal{Q}_{g,k}(v_s) = \nabla q(s\varphi) = 0$$

for s small enough, because $\nabla q \equiv 0$ on V by assumption. Hence $v_s \in \mathcal{CQC}^*_{g,k}$. Since $\varphi \in K$ was arbitrary, v is integrable.

Conversely, assume now that v is integrable. By contradiction, suppose that q is non-constant on every neighborhood of 0 in $\mathbb{R}^\ell$. Since q is analytic, we then have, for φ in some neighborhood V of 0,

$$q(\varphi) = q(0) + q_{k_0}(\varphi) + q_R(\varphi).$$

Here q_{k_0} is a non-trivial homogeneous polynomial of some degree $k_0 \in \mathbb{N}$ and the remainder term q_R is a sum of homogeneous polynomials of degree greater than k_0.

We fix a $\varphi \in U$ such that $\nabla q_{k_0}(\varphi) \neq 0$. Since v is integrable, there exists $(v_s)_{s\in(-\delta,\delta)} \subset \mathcal{CQC}^*_{g,k}$ such that $v_0 = v$ and $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} v_s = \varphi$ for all $s \in (-\delta, \delta)$. By Lemma A.(ii), after possibly choosing δ and U to be smaller, there are $\varphi_s \in U$ such that

$$v_s = v + \varphi_s + F(\varphi_s) \quad \text{for every} \quad s \in (-\delta, \delta), \tag{4.2}$$

where F is the map from Lemma A. By that lemma, we have

$$\begin{aligned} 0 &= \nabla_{\mathcal{B}} \mathcal{Q}_{g,k}(v_s) = \nabla q(\varphi_s) = \nabla q_{k_0}(\varphi_s) + \nabla q_R(\varphi_s) \\ &= |\varphi_s|^{k_0-1} \left(\nabla q_{k_0}\left(\frac{\varphi_s}{|\varphi_s|}\right) + |\varphi_s|^{-k_0+1} \nabla q_R(\varphi_s) \right). \end{aligned} \tag{4.3}$$

Here we used that the vector $\nabla q_{k_0}(\varphi)$ consists of homogeneous polynomials of degree $k_0 - 1$. Moreover, since q_R is a sum of homogeneous polynomials of degree strictly greater than k_0, we have $|\varphi_s|^{-k_0+1} \nabla q_R(\varphi_s)$ as $s \to 0$. On the other hand, $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} v_s = \varphi$ and (4.2) combined with $\nabla F(0) = 0$ imply $\frac{\varphi_s}{|\varphi_s|} \to \frac{\varphi}{|\varphi|}$ as $s \to 0$. Thus the sum inside the parentheses on the right side of (4.3) is non-zero for s small enough. This contradiction finishes the proof of the converse implication.

It remains to prove that (4.1) holds provided v is integrable. The inclusion $\subset$ is given by Lemma A.(ii), because $\mathcal{M}^*_{g,k} \subset \mathcal{CQC}^*_{g,k}$. For the reverse inclusion, we have already proved that q is locally constant on K near 0 if $v \in \mathcal{M}^*_{g,k}$ is integrable. By the definition of q, this is the same thing as saying that $\mathcal{Q}_{g,k}$ is constant on $\mathcal{L} \cap \mathcal{B}(v, \delta)$ for some $\delta > 0$. Hence, for $u \in \mathcal{L} \cap \mathcal{B}(v, \delta)$ it follows $\mathcal{Q}_{g,k}(u) = \mathcal{Q}_{g,k}(v)$ and hence $u \in \mathcal{M}^*_{g,k}$. Thus (4.1) is proved. □

We recall the notion of the Adams-Simon positivity condition as in [2] (see also [11]).

Definition 4.3. *Let $k \in \mathbb{N}$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold with $n > 2k$. Suppose that $u_0 \in \mathcal{CQC}^*_{g,k}$ is nonintegrable and that $q : U \to \mathbb{R}$, where*

$U \subset \ker \nabla^2_{\mathcal{B}} \mathcal{Q}_{g,k}(u_0) \cong \mathbb{R}^\ell$, *is the function defined in Lemma A. Since q is analytic, we can expand it in a power series*

$$q(v) = q(0) + \sum_{j \geq p} q_j(v),$$

where each q_j is a homogeneous polynomial of degree j and p is chosen so that $q_p \not\equiv 0$. We call p the order of integrability of u_0. We say that u_0 satisfies the Adams-Simon positivity condition of order p, denoted by AS_p, if p is the order of integrability of u_0 and $q_p|_{\mathbb{S}^{\ell-1}}$ attains a positive maximum at some $v \in \mathbb{S}^{\ell-1}$.

Remark 4.4. *When the order of integrability p is odd, then $q_p \not\equiv 0$ is an odd function, and hence the Adams–Simon positivity condition is automatically satisfied in this case. Moreover, by arguing as in [11, Appendix A] one finds that the order of integrability always satisfies $p \geqslant 3$, and that $q_3(v)$ can be explicitly expressed*

$$q_3(v) = C_{n,k} \mathcal{Q}_{g,k}(u_0) \int_M v^3 \, \mathrm{d}\mu_g \tag{4.4}$$

for some dimensional constant $C_{n,k} > 0$.

We establish the local version of Theorem 1. We need a localized measure of how far u is from being a minimizer close to some given minimizer v. Given $\delta > 0$ and $v \in \mathcal{M}^*_{g,k}$, we let

$$d_{v,\delta}\left(u, \mathcal{M}^*_{g,k}\right) = \frac{\inf\left\{\|u - \tilde{v}\|_{W^{k,2}(M)} : \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta)\right\}}{\|u\|_{W^{k,2}(M)}}. \tag{4.5}$$

Proposition 4.5. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let (M, g) be a smooth, closed, n-dimensional Riemannian manifold. For any $v \in \mathcal{M}^*_{g,k}$, there exist constants $c > 0$, $\gamma \geq 0$ and $\delta > 0$ depending on v such that*

$$\mathcal{Q}_{g,k}(u) - \mathcal{Y}_{k,+}(M,[g]) \geq c d_{v,\delta}\left(u, \mathcal{M}^*_{g,k}\right)^{2+\gamma} \quad \text{for all} \quad u \in \mathcal{B}(v,\delta).$$

Proof. Let $v \in \mathcal{M}^*_{g,k}$. We will use the notation of Lemma A without further comment throughout this proof. Also, we will abbreviate $\mathcal{Y} := \mathcal{Y}_{k,+}(M,[g])$ to make notation lighter. To start with, let $u \in \mathcal{B}(v,\delta)$.

We divide the proof into some steps as follows:

Step 1. Decomposing u.

We now decompose u according to the Lyapunov-Schmidt reduction from Lemma A by letting

$$u_{\mathcal{L}} := v + \pi_K(u - v) + F(\pi_K(u - v))$$

and setting $u^\perp := u - u_{\mathcal{L}}$. Note that, since F maps into $K^\perp$, we have

$$u^\perp = (u - v) - \pi_K(u - v) - F(\pi_K(u - v)) \in K^\perp.$$

It will be convenient to write

$$\mathcal{Q}_{g,k}(u) - \mathcal{Y} = (\mathcal{Q}_{g,k}(u) - \mathcal{Q}_{g,k}(u_{\mathcal{L}})) + (\mathcal{Q}_{g,k}(u_{\mathcal{L}}) - \mathcal{Y}) \tag{4.6}$$

and bound the two brackets on the right side separately.

Step 2. The non-degenerate and the integrable case.

We first bound the first term of (4.6). Here is where we crucially use the decomposition $u = u_{\mathcal{L}} + u^\perp$. Using Taylor's theorem with the mean-value formula for the remainder term, we can express

$$\begin{aligned}\mathcal{Q}_{g,k}(u) - \mathcal{Q}_{g,k}(u_{\mathcal{L}}) &= D_{\mathcal{B}}\mathcal{Q}_{g,k}(u_{\mathcal{L}})(u^{\perp}) + \frac{1}{2}D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(\zeta)(u^{\perp}, u^{\perp}) \\ &= \frac{1}{2}D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(v)(u^{\perp}, u^{\perp}) + \mathrm{o}(1)\|u^{\perp}\|^2_{W^{k,2}(M)},\end{aligned} \tag{4.7}$$

where ζ lies on a geodesic in $\mathcal{B}$ between u and $u_{\mathcal{L}}$. For the second equality we used that $u^{\perp} \in K^{\perp}$ implies $D_{\mathcal{B}}\mathcal{Q}_{g,k}(u_{\mathcal{L}})(u^{\perp}) = 0$ by Lemma A, as well as the fact that $D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(\cdot)$ is continuous.

To estimate the right-hand-side of (4.7) we use the fact that v minimizes $\mathcal{Q}_{g,k}$, and so $D^2\mathcal{Q}_{g,k}(v) \geq 0$. Since $u^{\perp} \in K^{\perp}$, we have the lower bound

$$D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(v)[u^{\perp}, u^{\perp}] \geq \lambda_1 \|u^{\perp}\|_{L^2(M)} \geq c\int_M v^{\frac{4k}{n-2k}}(u^{\perp})^2 \,\mathrm{d}\mu_g, \tag{4.8}$$

where $\lambda_1 > 0$ is the smallest positive eigenvalue of $D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(v)$. Since $v > 0$ on the compact manifold M, we may choose $c = \lambda_1(\max_M v)^{-4k/(n-2k)}$ in the second inequality. Recalling from Lemma 3.2 that

$$D^2\mathcal{Q}_{g,k}(v)(w, w) = \frac{4}{n-2k}\left(\int_M wP_{g,k}(w)\,\mathrm{d}\mu_g + \frac{n+2k}{n-2k}\mathcal{Y}\int_M v^{\frac{4k}{n-2k}}w^2\,\mathrm{d}\mu_g\right)$$

some elementary manipulations give us

$$D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(v)[u^{\perp}, u^{\perp}] \geq c_1\int_M u^{\perp}P_{g,k}(u^{\perp})\,\mathrm{d}\mu_g \geq c_2\|u\|^2_{W^{k,2}(M)},$$

where $c_1 = c\left(\frac{n+2k}{n-2k}\mathcal{Y} - c\right)^{-1} > 0$, $c > 0$ is given in (4.8), and $c_2 > 0$ exists since $P_{g,k}$ is coercive.

Inserting this last expression into (4.7), we obtain

$$\mathcal{Q}_{g,k}(u) - \mathcal{Q}_{g,k}(u_{\mathcal{L}}) \geq \frac{1}{4}c_2\|u^{\perp}\|^2_{W^{k,2}(M)} \tag{4.9}$$

whenever $0 < \|u^{\perp}\|_{W^{k,2}(M)} \ll 1$ is sufficiently small.

On the other hand, the second term in (4.6) trivially satisfies

$$\mathcal{Q}_{g,k}(u_{\mathcal{L}}) - \mathcal{Y} \geq 0 \tag{4.10}$$

by the definition of $\mathcal{Y}$.

If $K = 0$ (i.e., if v is non-degenerate), then it follows directly from the definitions that $u_{\mathcal{L}} = v$, and hence

$$\|u^{\perp}\|^2_{W^{k,2}(M)} = \|u - v\|^2_{W^{k,2}(M)} \geq \inf\{\|u - \tilde{v}\|^2_{W^{k,2}(M)} \,:\, \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v, \delta)\} \geq c d_{v,\delta}(u, \mathcal{M}^*_{g,k})^2, \tag{4.11}$$

where in the last inequality, we used

$$\|u\|^2_{W^{k,2}(M)} \geq \frac{1}{2}\|v\|^2_{W^{k,2}(M)} \gtrsim \int_M vP_{g,k}(v)\,\mathrm{d}\mu_g = \mathcal{Y} > 0.$$

Thus the assertion in case $K = 0$ follows by putting together (4.6), (4.9), (4.10) and (4.11).

More generally, if v is integrable, then by Lemma 4.2 we have $u_{\mathcal{L}} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v, \delta)$ if u is close enough to v. Thus, similarly to the above,

$$\|u^{\perp}\|^2_{W^{k,2}(M)} = \|u - u_{\mathcal{L}}\|^2_{W^{k,2}(M)} \geq \inf\{\|u - \tilde{v}\|^2_{W^{k,2}(M)} \,:\, \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v, \delta)\} \geq c d_{v,\delta}(u, \mathcal{M}^*_{g,k})^2. \tag{4.12}$$

Thus the assertion in case v is integrable follows by putting together (4.6), (4.9), (4.10) and (4.12).

We emphasize that in both of these cases (v non-degenerate or v integrable) the inequality (1.9) restricted to $\mathcal{B}(v, \delta)$ holds with $\gamma = 0$.

Step 3. The non-integrable case.

It remains to discuss the third and hardest case, namely the case when v is non-integrable. The complication is that the estimates (4.11) resp. (4.12) are not applicable because $u_{\mathcal{L}}$ is not necessarily in $\mathcal{M}^*_{g,k}$. Consequently we require a better lower bound on $\mathcal{Q}_{g,k}(u_{\mathcal{L}}) - \mathcal{Y}$. To obtain it, we invoke the Łojasiewicz inequality from Lemma C, which applies because q is analytic by Lemma A. We conclude that there exist $\gamma > 0$ such that

$$\mathcal{Q}_{g,k}(u_{\mathcal{L}}) - \mathcal{Y} = q(\varphi) - q(0) \gtrsim \inf\{|\varphi - \bar{\varphi}| : \bar{\varphi} \in K \cap B(0,\delta) \text{ and } q(\bar{\varphi}) = 0\}^{2+\gamma}, \tag{4.13}$$

where $\varphi = \pi_K(u - v)$. (Note that we could drop the absolute value of $|q(\varphi) - q(0)|$ because 0 is a local minimum of q.) To turn this into the desired bound, we observe that for any $\bar{\varphi} \in K \cap B(0,\delta)$ with $q(\bar{\varphi}) = q(0)$, the function $\bar{v} = v + \bar{\varphi} + F(\bar{\varphi})$ satisfies $\mathcal{Q}_{g,k}(\bar{v}) = q(\bar{\varphi}) = 0$ by Lemma A. Hence, $\bar{v} \in \mathcal{M}^*_{g,k}$. Moreover, for such $\bar{v}$, we have (still writing $\varphi = \pi_K(u - v)$)

$$\begin{aligned}
\|u_{\mathcal{L}} - \bar{v}\|_{W^{k,2}(M)} &= \|\varphi + F(\varphi) - \bar{\varphi} - F(\bar{\varphi})\|_{W^{k,2}(M)} \\
&\leq \|\varphi - \bar{\varphi}\|_{W^{k,2}(M)} + \|F(\varphi) - F(\bar{\varphi})\|_{W^{k,2}(M)} \\
&\lesssim \|\varphi - \bar{\varphi}\|_{W^{k,2}(M)} + \|\varphi - \bar{\varphi}\|_{\mathcal{C}^{0,\alpha}(M)} \\
&\lesssim \|\varphi - \bar{\varphi}\|_{W^{k,2}(M)}.
\end{aligned}$$

Here, the $\mathcal{C}^{0,\alpha}$ estimate follows from the corresponding one in Lemma A. The last inequality simply uses the equivalence of any two norms on the finite-dimensional space K. From this chain of inequalities we get

$$\inf\{|\varphi - \bar{\varphi}| : \bar{\varphi} \in K \cap B(0,\delta), \nabla q(\bar{\varphi}) = 0\}^{2+\gamma} \gtrsim \inf\left\{\|u_{\mathcal{L}} - \tilde{v}\|_{W^{k,2}(M)} : \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta)\right\}^{2+\gamma}$$

Together with (4.13) and (4.9), since $u^\perp = u - u_{\mathcal{L}}$ satisfies $\|u^\perp\|_{W^{k,2}} \leq 1$, we thus get

$$\begin{aligned}
\mathcal{Q}_{g,k}(u) - \mathcal{Y} &\gtrsim \frac{1}{4}\|u - u_{\mathcal{L}}\|^2_{W^{k,2}(M)} + \inf\left\{\|u_{\mathcal{L}} - \tilde{v}\|_{W^{k,2}(M)} : \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta)\right\}^{2+\gamma} \\
&\gtrsim \inf\left\{\lambda_1 \|u - u_{\mathcal{L}}\|^{2+\gamma}_{W^{k,2}(M)} + \|u_{\mathcal{L}} - \tilde{v}\|^{2+\gamma}_{W^{k,2}(M)} : \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta)\right\} \\
&\gtrsim \inf\left\{\|u - \tilde{v}\|^{2+\gamma}_{W^{k,2}(M)} : \tilde{v} \in \mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta)\right\}
\end{aligned}$$

and the proof is complete. □

If v is integrable or non-degenerate (*i.e.* the kernel $K = \ker(D^2_{\mathcal{B}}\mathcal{Q}_{g,k}(v)(\cdot,\cdot))$ is trivial) then we may take $\gamma = 0$.

4.2. **Global stability estimate.** In the following, we again abbreviate $\mathcal{Y} = \mathcal{Y}_{k,+}(M,[g])$.

Proof of Theorem 1. Since both sides of inequality (1.9) are zero-homogeneous, it suffices to prove (1.9) for every $u \in \mathcal{B}$. By contradiction, suppose that for every $\eta \geq 0$ there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset \mathcal{B}$ such that

$$\frac{\mathcal{Q}_{g,k}(u_m) - \mathcal{Y}}{d(u_m, \mathcal{M}_{g,k})^{2+\eta}} \to 0 \qquad \text{as} \qquad m \to \infty. \tag{4.14}$$

By diagonal extraction of a subsequence, we may actually assume that (4.14) holds for every $\eta \geq 0$ with the *same* sequence (u_m).

Since $d(u_m, \mathcal{M}_{g,k}) \leq 1$, (4.14) implies that $\mathcal{Q}_{g,k}(u_m) - \mathcal{Y} \to 0$ as $m \to \infty$. By Lemma B, there exists $v \in \mathcal{M}^*_{g,k}$ such that, up to extracting a further subsequence, one has $u_m \to v$ strongly in $W^{k,2}(M)$.

Now, let $\delta = \delta(v) > 0$ and $\gamma = \gamma(v) \geq 0$ be as in Proposition 4.5. Since $\mathcal{M}^*_{g,k} \cap \mathcal{B}(v,\delta) \subset \mathcal{M}_{g,k}$, the definitions (1.8) and (4.5) of d and $d_{v,\delta}$ directly imply $d_{v,\delta}(u_m, \mathcal{M}^*_{g,k}) \geq d(u_m, \mathcal{M}_{g,k})$. Hence, by (4.14) applied with $\eta = \gamma$, it follows

$$\frac{\mathcal{Q}_{g,k}(u_m) - \mathcal{Y}}{d_{v,\delta}(u_m, \mathcal{M}^*_{g,k})^{2+\gamma}} \leq \frac{\mathcal{Q}_{g,k}(u_m) - \mathcal{Y}}{d(u_m, \mathcal{M}_{g,k})^{2+\gamma}} \to 0 \quad \text{as} \quad m \to \infty.$$

But since $u_m \in \mathcal{B}(v,\delta)$ for every $m \gg 1$ large enough, this contradicts Proposition 4.5. Hence (4.14) is false, and the theorem is proven. □

Proof of Corollary 3. By Theorem 1 and the definition (1.10) of $\|\cdot\|$ we have

$$\begin{aligned}
\mathcal{Q}_{g,k}(u) - \mathcal{Y} &\geq c\left(\frac{\inf_{v\in\mathcal{M}_{g,k}} \|u-v\|_{W^{k,2}(M)}}{\|u\|_{W^{k,2}(M)}}\right)^{2+\gamma} \\
&\geq \tilde{c}\left(\frac{\inf_{v\in\mathcal{M}_{g,k}} \|u-v\|_{L^{\frac{2n}{n-2k}}(M)}}{\|u\|_{L^{2^*_k}(M)}}\right)^{2+\gamma} \\
&= \left(\frac{\inf\{\|h-\tilde{g}\| : \tilde{g} \in \mathcal{M}_{g,k}\}}{\mathrm{vol}_h(M)^{\frac{n-2k}{2n}}}\right)^{2+\gamma}.
\end{aligned}$$

For the second inequality, note that if $d(u, \mathcal{M}_{g,k}) \leq \delta_0$ (for some appropriately small $\delta_0 > 0$), then $\|u\|_{W^{k,2}(M)} \simeq \|u\|_{L^{2^*_k}(M)}$, so the inequality follows from Sobolev's inequality. If $d(u, \mathcal{M}_{g,k}) > \delta_0$ on the other hand, it suffices to take $\tilde{c}$ sufficiently small because

$$\inf_{v\in\mathcal{M}_{g,k}} \|u-v\|_{L^{2^*_k}(M)} \|u\|^{-1}_{L^{2^*_k}(M)} \leq 1.$$

Similarly, by Theorem 1 and the definition of $\|\cdot\|_*$ in (1.11) we have

$$\begin{aligned}
\mathcal{Q}_{g,k}(u) - \mathcal{Y} &\geq c\left(\frac{\inf_{v\in\mathcal{M}_{g,k}} \|u-v\|_{W^{k,2}(M)}}{\|u\|_{W^{k,2}(M)}}\right)^{2+\gamma} \\
&\gtrsim \left(\frac{\inf_{v\in\mathcal{M}_{g,k}} (\int_M (u-v)P_{g,k}(u-v)\,\mathrm{d}\mu_g)^{1/2}}{\|u\|_{L^{2^*_k}(M)}}\right)^{2+\gamma} \\
&= \left(\frac{\inf\{\|h-\tilde{g}\|_* : \tilde{g} \in \mathcal{M}_{g,k}\}}{\mathrm{vol}_h(M)^{\frac{n-2k}{2n}}}\right)^{2+\gamma}.
\end{aligned}$$

Here we have used $\int_M (u-v)P_{g,k}(u-v)\,\mathrm{d}\mu_g \lesssim \|u-v\|^2_{W^{k,2}(M)}$, which holds because $P_{g,k}$ is a k-th order elliptic differential operator. Moreover, since by assumption $\mathcal{Q}_{g,k}(u) - \mathcal{Y} \leq \delta_0$, Theorem 1 yields

$$d(u, \mathcal{M}_{g,k}) \lesssim \delta_0^{\frac{1}{2+\gamma}}.$$

Thus, for $0 < \delta_0 \ll 1$, one has $\|u\|_{W^{k,2}(M)} \simeq \|u\|_{L^{2^*_k}(M)}$, which proves that the inequality above holds. □

4.3. **Generic nondegeneracy.** Now, let us prove the genericity part of our main result. Following ideas in [15, 22, 44, 47], our strategy is to verify hypotheses (i), (ii), and (iii) of the abstract transversality result in Lemma D. To apply Lemma D we must examine the second variation of the functional $\mathcal{Q}_{g,k}$ among *all* admissible metrics, not just those conformal to a fixed metric g_0. To this end, we recall the machinery developed by Case, Lin and Yuan in [12]. In their language, the

k-th-order Q-curvature is an example of a conformal variational Riemannian invariant of weight $-2k$.

Lemma 4.6. *Let $n \in \mathbb{N}$ with $n > 2k$. Let M be a smooth, closed, n-dimensional manifold. The set*

$$\mathcal{G} := \{g \in \mathfrak{A}_{k,\alpha}(M) : \gamma(g) = 0\} \subset \mathfrak{A}_{k,\alpha}(M)$$

is open and dense with respect to the $\mathcal{C}^{k,\alpha}-$topology for some $\alpha \in (0, 1)$.

Proof. In Step 2 of the proof of Proposition 4.5, we have seen that $\gamma(g) = 0$ if $\ker D^2\mathcal{Q}_{g,k}(v) = \{0\}$ for every $v \in \mathcal{M}^*_{g,k}$. Thus it suffices to show that the (potentially) smaller set

$$\{g \in \mathfrak{A}_{k,\alpha}(M) \, : \ker D^2\mathcal{Q}_{g,k}(v) = \{0\} \text{ for every } v \in \mathcal{M}^*_{g,k}\} \subset \mathcal{G}$$

is open and dense.

We recall that $G_{g,k,\xi}$ is the Green's function of the GJMS operator $P_{g,k}$ with its pole at $\xi \in M$, and that $G_{g,k,\xi} > 0$ if g is an admissible metric. Now we fix $g_0 \in \mathfrak{A}_{k,\alpha}(M)$ and define

$$\mathcal{F} : \mathfrak{A}_{k,\alpha}(M) \times W^{k,2}_{g_0}(M) \to W^{k,2}_{g_0}(M),$$

by

$$\mathcal{F}(g, u)(\xi) = u(\xi) - \int_M G_{g,k,\xi}(y) f_{n,k}(u(y)) \, \mathrm{d}\mu_g(y),$$

where $f_{n,k}(u) = c_{n,k} u^{\frac{n+2k}{n-2k}}$. Observe that the zero set of $\mathcal{F}$ consists of the pairs (g, u) such that $\widetilde{g} = u^{4/(n-2k)} g$ has constant k-th-order Q-curvature, *i.e.* u solves the PDE

$$P_{g,k}(u) - f_{n,k}(u) = 0 \quad \text{on} \quad M. \tag{$\mathcal{P}_{g,k}$}$$

These solutions are also critical points of the functional

$$\mathcal{E}_{g,k}(u) = \frac{1}{2} \int_M u P_{g,k}(u) \, \mathrm{d}\mu_g - F_{n,k}(u),$$

where $F_{n,k}(u) = \int_0^u f_{n,k}(s) \, \mathrm{d}s$. Furthermore, the set of metrics such that $\ker D^2\mathcal{Q}_{g,k} = \{0\}$ are those admissible metrics such that every conformal constant Q-curvature metric $\widetilde{g} = u^{4/(n-2k)} g$ is nondegenerate in the sense that if $\mathcal{F}(g, u) = 0$, then any $v \in W^{k,2}_{g_0}(M)$ such that

$$L_{g,k,u}(v) = P_{g,k}(v) - f'_{n,k}(u) v = 0 \quad \text{on} \quad M. \tag{$\mathcal{L}'_{g,k,u}$}$$

must be the zero function itself.

We have now reduced our problem to showing that the set

$$\widetilde{\mathcal{G}} = \{g \in \mathfrak{A}_{k,\alpha}(M) : \mathcal{F}(g, u) = 0 \text{ and } \ker(L_{g,k,u}) = \{0\}\}$$

is open and dense. It is helpful now to discuss the regularity of $\mathcal{F}$ and describe some of its partial derivatives. We've already shown that for each $g \in \mathfrak{A}_{k,\alpha}(M)$ the mapping $u \mapsto \mathcal{F}(g, u)$ is $\mathcal{C}^2$, and we computed the second derivative explicitly in Lemma 3.2. Case, Lin and Yuan [12] proved that for each $u > 0$ the mapping $g \mapsto \mathcal{F}(g, u)$ is also $\mathcal{C}^2$. In their language, the map

$$g \mapsto Q_{g,k} = \frac{2}{n - 2k} P_{g,k}(1)$$

is a conformally variational Riemannian invariant of weight $-2k$ with a conformal primitive $\mathcal{E}_{g,k}(1)$. Furthermore, they denote the linearization of $g \mapsto Q_{g,k}$ by

$$\Gamma_g(h) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{2}{n - 2k} P_{g+th,k}(1). \tag{4.15}$$

This operator has a formal adjoint Γ_g^*, determined by

$$\int_M \langle \Gamma_g^*(f), h\rangle \, \mathrm{d}\mu_g = \int_M f\Gamma_g(h) \, \mathrm{d}\mu_g;$$

it is this formal adjoint that is the (Frechet) derivative of $\mathcal{F}$ with respect to g, namely

$$\Gamma_g^* = \partial_g \mathcal{F}.$$

In [12, Section 8.1] they show this functional is variationally stable, which in the notation established above means

$$\ker(D^2(g \mapsto \mathcal{E}_{g,k}(1)) =: T_g\mathfrak{K} = \left\{ Y \in \mathcal{C}^\infty(M) : Y = \frac{n}{2}\operatorname{div}(X) \ \text{ for some } X \in \mathcal{K}(g) \right\}. \tag{4.16}$$

Here $\mathcal{L}_X g$ is the Lie derivative of the metric tensor g in the direction of X and $\mathcal{K}(g)$ is the space of conformal Killing fields on (M, g), *i.e.* the space of vector fields whose flows act as one-parameter families of global conformal diffeomorphisms. Recall that $X \in \mathcal{K}(g)$ if and only if $\mathcal{L}_X g = \psi g$ for some function ψ. Evaluating the trace of both sides of this equation shows that multiple ψ must be $\frac{2}{n}\operatorname{div}(X)$. In summary, the only variations of $\mathcal{F}$ with respect to the metric preserving the functional $\mathcal{E}_{g,k}(1)$ at a critical point are those arising from global conformal diffeomorphisms. Given that solution space of the PDE ($\mathcal{P}_{g,k}$) is conformally invariant this is the smallest one asks for the kernel to be. In any case, this kernel is always finite-dimensional and by [8, Theorem 7.4] it is generically trivial.

We complete our proof by verifying the hypotheses of Lemma D. It follows from the analysis in [25, Section 2.1] that the transversality map $\mathcal{F} : \mathfrak{A}_{k,\alpha}(M) \times W^{k,2}_{g0}(M) \to W^{k,2}_{g0}(M)$ is of class $\mathcal{C}^1$. We must work slightly more to show $\mathcal{F}(g, \cdot)$ is Fredholm and has index zero for each $g \in \mathfrak{A}_{k,\alpha}(M)$. Let $H^k_g(M)$ be the completion of $\mathcal{C}^\infty(M)$ with respect to the inner product

$$((u, v))_g = \int_M u P_{g,k}(v) \, \mathrm{d}\mu_g. \tag{4.17}$$

Standard computations show that the Hilbert subspaces $H^k_g(M)$ and $W^{k,2}_g(M)$ of $\mathcal{C}^\infty(M)$ are equivalent, and so the canonical inclusion $W^{k,2}_g(M) \hookrightarrow H^k_g(M)$ is an isomorphism of Banach spaces. The same holds for the inclusion $W^{k,2}_{g'}(M) \hookrightarrow H^k_g(M)$ for any $g' \in \mathfrak{A}_{k,\alpha}(M)$ (see [29, Proposition 2.2]). By the Kondrakov theorem, the canonical inclusion $i_g : H^k_g(M) \to L^{2^*_k}_g(M)$ is compact, and we let i^*_g be its adjoint with respect to the canonical isomorphism $(L^p_g(M))' \simeq L^{p'}_g(M)$ with $p' = \frac{p}{p-1}$. In fact, we can specify i^*_g by the relation

$$((i^*_g u, v))_g = \int_M uv \, \mathrm{d}\mu_g \quad \text{for each} \quad v \in L^{2^*_k}_g(M).$$

Now define the Nemytskii operator

$$\mathcal{N}_{n,k} : W^{k,2}_g(M) \to L^{\frac{2n}{n+2k}}_g(M),$$

by

$$\mathcal{N}_{n,k}(u) = f_{n,k}(u) = c_{n,k} u^{\frac{n+2k}{n-2k}},$$

which is a compact operator because $g \in \mathfrak{A}_{k,\alpha}(M)$ is an admissible metric. The inclusion $W^{k,2}_g(M) \hookrightarrow L^{\frac{2n}{n+2k}}_g(M)$ given by the Sobolev embedding shows $\mathcal{N}_{n,k}$ is $\mathcal{C}^1$ and its Frechet derivative is

$$\mathrm{d}\mathcal{N}_{n,k}[u](v) = \frac{n+2k}{n-2k} c_{n,k} u^{\frac{4k}{n-2k}} v.$$

In addition, direct computation now yields

$$\partial_u\mathcal{F}(v) = v - \left(i_g^* \circ \mathrm{d}\mathcal{N}_{n,k}[u]\right)(v).$$

Since the composition $i_g^* \circ \mathrm{d}\mathcal{N}_{n,k}[u]$ is compact, we have just demonstrated that $\mathcal{F}(g,\cdot)$ is Fredholm and has index zero. We now conclude that the following four conditions are equivalent:

(i) 0 is a regular value of $\mathcal{F}(g,\cdot)$;
(ii) $\partial_u\mathcal{F}(g,u) = \mathrm{id} - (i_g^* \circ f'_{n,k})(u)$ is injective for every solution u to ($\mathcal{P}_{g,k}$);
(iii) $v \equiv 0$ is the only solution to ($\mathcal{L}'_{g,k,u}$) for every solution u to ($\mathcal{P}_{g,k}$);
(iv) every solution $u \in W^{k,2}_{g_0}(M)$ to ($\mathcal{P}_{g,k}$) is non-degenerate.

We now verify that 0 is a regular value of $\mathcal{F}$, which is one of the assumptions of Lemma D. To do so, let $(g,u) \in \mathfrak{A}_{k,\alpha}(M) \times W^{k,2}_{g_0}(M)$ such that $\mathcal{F}(g,u) = 0$ and let $v \in W^{k,2}_{g_0}(M)$. We need to find a symmetric 2-tensor $h \in \Sigma^2(TM)$ and a function $w \in W^{k,2}_{g_0}(M)$ such that

$$\partial_g\mathcal{F}|_{(g,u)}(h) + \partial_u\mathcal{F}|_{(g,u)}(w) = v. \tag{4.18}$$

Since $\partial_u\mathcal{F}(g,u)$ is a self-adjoint compact perturbation of the identity, we have the orthogonal decomposition

$$W^{k,2}_{g_0}(M) = \ker \partial_u\mathcal{F}|_{(g,u)} \oplus \mathrm{Im}\, \partial_u\mathcal{F}|_{(g,u)},$$

and that $\dim\ker \partial_u\mathcal{F}|_{(g,u)} < \infty$.

Let $\Pi : W^{k,2}_{g_0}(M) \to \ker \partial_u\mathcal{F}|_{(g,u)}$ be the projection onto $\ker \partial_u\mathcal{F}|_{(g,u)}$. We first claim that $\Pi\, \partial_g\mathcal{F}|_{(g,u)}$ is surjective onto $\ker \partial_u\mathcal{F}|_{(g,u)}$. Indeed, if this is not the case, there is $0 \not\equiv \psi \in \ker \partial_u\mathcal{F}|_{(g,u)}$ such that

$$0 = ((\partial_g\mathcal{F}(\cdot,u)[h]),\psi))_g = ((\Gamma_g^*(h),\psi))_g \quad \text{for all} \quad h \in \mathrm{Sym}^{k,\alpha}(M). \tag{4.19}$$

where $((,))_g$ is the inner product given by (4.17). A direct computation shows that (4.19) can be reformulated as

$$\int_M \left[\Gamma_g^*(h)(\nabla u, \nabla\psi) - \frac{1}{2}\left(\mathrm{tr}_g\, h\right) f_{n,k}(u)\psi\right] \mathrm{d}\mu_g = 0,$$

where $\Gamma_g^*(h)$ is given by (4.15).

Following [22, Lemma 12] and using normal coordinates centered at arbitrary $x \in M$ and specific perturbations of g, one can prove that this last displayed equation implies

$$\langle\nabla u, \nabla\psi\rangle_g = 0 \quad \mu_g\text{-a.e.} \quad \text{for each } h \in \mathrm{Sym}^{k,\alpha}(M). \tag{4.20}$$

By taking $h = \varphi g$ for arbitrary $\varphi \in \mathcal{C}^\infty(M)$ we find that (4.19) and (4.20) imply $f_{n,k}(u)\psi = 0$ almost everywhere on M with respect to μ_g. However, since $f_{n,k}(u) = c_{n,k}u^{\frac{n+2k}{n-2k}} > 0$ on M, this implies that $\psi \equiv 0$ on M, which is a contradiction. Thus we have shown that $\Pi\, \partial_g\mathcal{F}|_{(g,u)}$ is surjective onto $\ker \partial_u\mathcal{F}|_{(g,u)}$. Consequently, there is $h \in \Sigma^2(TM)$ such that $\partial_g\mathcal{F}|_{(g,u)}(h) = v_1 + z$, for some $z \in \mathrm{Im}\, \partial_u\mathcal{F}|_{(g,u)}$. Moreover, since $v_2 - z \in \mathrm{Im}\, \partial_u\mathcal{F}|_{(g,u)}$, there is $w \in W^{k,2}_{g_0}(M)$ such that $\partial_u\mathcal{F}|_{(g,u)}(w) = v_2 - z$. Thus, h and w satisfy (4.18). We conclude that 0 is a regular value of $\mathcal{F}$.

Moreover, the same argument as in [22, Lemma 11] and [44, Lemma 4.2] shows that the map $\pi_1 \circ \iota : \mathcal{F}^{-1}(0) \to W^{k,2}_{g_0}(M)$ is σ-proper. Here $\iota : \mathcal{F}^{-1}(0) \to W^{k,2}_{g_0}(M) \times \mathfrak{A}_{k,\alpha}(M)$ is the canonical embedding and $\pi_1 : W^{k,2}_{g_0}(M) \times \mathfrak{A}_{k,\alpha}(M) \to W^{k,2}_{g_0}(M)$ is the projection onto the first factor.

Finally, putting all this information together, we see the hypotheses of Lemma D all hold, and so

$$\{g \in \mathfrak{A}_{k,\alpha}(M) : 0 \text{ is a regular value of } \mathcal{F}(g,\cdot)\} = \mathcal{G}$$

is an open, dense set subset $\mathfrak{A}_{k,\alpha}(M)$ with respect to the $\mathcal{C}^{k,\alpha}$-topology. □

5. Cubic degenerate stability for $k = 2$ (proof of Theorem 4)

In this section, we construct examples of manifolds satisfying the Adams–Simon integrability condition in Definition 4.3. Our approach follows closely the ones in [11, Section 5].

5.1. **Products with spheres.** In this section we prove that certain products of Einstein manifolds with a standard round sphere are degenerate critical points of $\mathcal{Q}_{h,2}$ satisfying the AS_3 condition.

Proposition 5.1. *Let $\ell, m \in \mathbb{N}$ with $\ell \geq 2$, let $\lambda \in \mathbb{R}$, let (M, g) be an m-dimensional λ-Einstein manifold, and let $(X, h) = (M \times \mathbb{S}^\ell, g \oplus \mathring{g})$. If $m \gg 1$ is sufficiently large, then there exist two real numbers $\lambda_- < \lambda_+$ such that the function $u \equiv 1$ is a degenerate critical point of the functional $\mathcal{Q}_{h,2}$ satisfying the AS_3 condition. If $\ell \in \{2, 3, 4, 5\}$ then $\lambda_- < 0 < \lambda_+$, but if $\ell \geq 6$ then $\lambda_- < \lambda_+ < 0$.*

Proof. Following (4.4) and the analysis in [11, Section 5], our goal now is to produce a nontrivial test function $v \in W^{2,2}_h(X)$ such that

$$v \in \Lambda_0 = \ker \left(P_{h,2} - \frac{m + \ell + 4}{2} Q_{h,2} \right). \tag{5.1}$$

Recall that the round metric on the sphere is an Einstein metric with Einstein constant of the round sphere $\ell - 1$. Direct computation shows that since $\mathrm{Ric}_h = \lambda g \oplus (\ell - 1)\mathring{g}$, one has

$$R_h = m\lambda + \ell(\ell - 1), \quad \text{and} \quad |\mathrm{Ric}_h|^2 = m\lambda^2 + \ell(\ell - 1)^2,$$

which we substitute into (1.5) to see

$$\begin{aligned} Q_{h,2} &= -\frac{2}{(m + \ell - 2)^2} |\mathrm{Ric}_h|^2 + \frac{(m + \ell)^3 - 4(m + \ell)^2 + 16(m + \ell) - 16}{8(m + \ell - 1)^2(m + \ell - 2)^2} R_h^2 \\ &= -\frac{2(m\lambda^2 + \ell(\ell - 1)^2)}{(m + \ell - 2)^2} + \frac{((m + \ell)^3 - 4(m + \ell)^2 + 16(m + \ell) - 16)(m\lambda + \ell(\ell - 1))^2}{8(m + \ell - 1)^2(m + \ell - 2)^2} \\ &= \left(\frac{m}{8} - \frac{1}{2} \right) \lambda^2 + \frac{\ell(\ell - 1)}{4} \lambda + \mathcal{O}\left(\frac{1}{m} \right). \end{aligned} \tag{5.2}$$

Using these expressions for Ric_h, R_h and $Q_{h,2}$ in (1.6) we find

$$\begin{aligned} \left(P_{h,2} - \frac{n + 4}{2} Q_{h,2} \right) &= \Delta_h^2 + \frac{4}{m + \ell - 2} (\lambda \Delta_g + (\ell - 1)\Delta_{\mathring{g}}) \\ &\quad - \frac{((m + \ell - 2)^2 + 4)(m\lambda + \ell(\ell - 1))}{2(m + \ell - 1)(m + \ell - 2)} (\Delta_g + \Delta_{\mathring{g}}) \\ &\quad - \left(\left(\frac{m}{2} - 2 \right) \lambda^2 + \ell(\ell - 1)\lambda + \mathcal{O}\left(\frac{1}{m} \right) \right) \\ &= \Delta_g^2 + \Delta_g \circ \Delta_{\mathring{g}} + \Delta_{\mathring{g}} \circ \Delta_g + \Delta_{\mathring{g}}^2 \\ &\quad + \left(-\frac{m\lambda}{2} + 2\lambda - \frac{\ell(\ell - 1)}{2} + \mathcal{O}\left(\frac{1}{m} \right) \right) (\Delta_g + \Delta_{\mathring{g}}) \\ &\quad - \left(\left(\frac{m}{2} - 2 \right) \lambda^2 + \ell(\ell - 1)\lambda + \mathcal{O}\left(\frac{1}{m} \right) \right). \end{aligned} \tag{5.3}$$

For our test function we choose $v = 1 \otimes \widetilde{v}$, where

$$\widetilde{v} = x_1 x_2 + x_2 x_3 + x_3 x_1$$

is a homogeneous, harmonic polynomial of degree two restricted to the sphere. This is an eigenfunction of $-\Delta_{\mathring{g}}$ with eigenvalue $2(\ell+1)$. As in [33, p.5], observe that

$$\int_X v^3 \, \mathrm{d}\mu_h = \mathrm{vol}_g(M) \int_{\mathbb{S}^n} \widetilde{v}^3 \, \mathrm{d}\mu_{\mathring{g}} \neq 0.$$

Substituting this choice of v into (5.3) we obtain

$$\left(P_{h,2} - \frac{n+4}{2} Q_{h,2} \right) v = (a(m,\ell)\lambda^2 + b(m,\ell)\lambda + c(m,\ell))v,$$

where

$$\begin{aligned} a(m,\ell) &= -\frac{m}{2} + 2 + \mathcal{O}\left(\frac{1}{m}\right) \\ b(m,\ell) &= -m(\ell+1) - \ell^2 + 5\ell + 4 + \mathcal{O}\left(\frac{1}{m}\right) \\ c(m,\ell) &= -\ell(\ell^2-1) + 4(\ell+1)^2 + \mathcal{O}\left(\frac{1}{m}\right) \\ &= -\ell^3 + 4\ell^2 + 9\ell + 4 + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \tag{5.4}$$

Thus we complete our proof by showing that, provided $m \gg 1$ is sufficiently large, the quadratic polynomial $\mathfrak{p}(\lambda) = a\lambda^2 + b\lambda + c$ has two real roots. The discriminant is

$$\mathrm{disc}_\lambda(\mathfrak{p}) = b^2 - 4ac = m^2(\ell+1)^2 + \mathcal{O}(m),$$

which is indeed positive for m sufficiently large, proving the existence of the two real roots. To see that one root is positive and the other is negative when $\ell \in \{2,3,4,5\}$, observe that $a(m,\ell) < 0$ for m sufficiently large, whereas $c(m,\ell) > 0$ for m sufficiently large and $\ell \in \{2,3,4,5\}$. This in turn implies the quadratic is positive when $\lambda = 0$ and negative when $|\lambda| \gg 1$ is sufficiently large, giving us one positive root and one negative root by the intermediate value theorem. ☐

5.2. **Products with complex projective space.** We repeat the analysis of the previous section using complex projective space instead of a sphere.

Proposition 5.2. *Let $\ell, m \in \mathbb{N}$ with $\ell \geq 2$, let $\lambda \in \mathbb{R}$, let (M,g) be an m-dimensional λ-Einstein manifold, and let $(\widetilde{X}, \widetilde{h}) = (M \times \mathbb{CP}^\ell, g \oplus g_{\mathrm{FS}})$, where g_{FS} is the Fubini-Study metric. If $m \gg 1$ is sufficiently large, then there exist two real numbers $\lambda_- < \lambda_+$ such that the function $u \equiv 1$ is a degenerate critical point of the functional $\mathcal{Q}_{h,2}$ satisfying the* AS_3 *condition. If $\ell \in \{2,3\}$ then $\lambda_- < 0 < \lambda_+$, but if $\ell \geq 4$ then $\lambda_- < \lambda_+ < 0$.*

We recall that the Fubini-Study metric is the unique metric on $\mathbb{CP}^\ell$ making the natural quotient $\mathbb{S}^{2\ell+1} \to \mathbb{CP}^\ell$ a Riemannian submersion.

Proof. In this setting, we use (4.4) to see that our goal is to produce a nontrivial test function v with

$$v \in \ker \left(P_{\widetilde{h},2} - \frac{m+2\ell+4}{2} Q_{\widetilde{h},2} \right). \tag{5.5}$$

The Fubini-Study metric on $\mathbb{CP}^\ell$ is an Einstein metric with Einstein constant $\lambda = 2(\ell+1)$. Once again, since $\mathrm{Ric}_{\widetilde{h}} = \lambda g \oplus 2(\ell+1) g_{\mathrm{FS}}$, direct computations implies

$$R_{\widetilde{h}} = m\lambda + 4\ell(\ell+1) \quad \text{and} \quad |\,\mathrm{Ric}_{\widetilde{h}}|^2 = m\lambda^2 + 8\ell(\ell+1)^2,$$

and so (1.5) yields

$$\begin{aligned} Q_{\widetilde{h},2} &= -\frac{2}{(m+2\ell-2)^2}|\operatorname{Ric}_{\widetilde{h}}|^2 + \frac{(m+2\ell)^3-4(m+2\ell)^2+16(m+2\ell)-16)}{8(m+2\ell-1)^2(m+2\ell-2)^2} R_{\widetilde{h}}^2 \\ &= \frac{-2m\lambda^2-16\ell(\ell+1)^2}{(m+2\ell-2)^2} + \frac{((m+2\ell)^3-4(m+2\ell)^2+16(m+2\ell)-16)(m\lambda+4\ell(\ell+1))^2}{8(m+2\ell-1)^2(m+2\ell-2)^2} \\ &= \left(\frac{m}{8}-\frac{1}{2}\right)\lambda^2 + \ell(\ell+1)\lambda + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned}$$

Substituting this into (1.6) we then obtain

$$\begin{aligned} P_{\widetilde{h},2} - \frac{n+4}{2}Q_{\widetilde{h},2} &= \Delta_{\widetilde{h}}^2 + \frac{4}{m+2\ell-2}(\lambda\Delta_g + 2(\ell+1)\Delta_{g_{\mathrm{FS}}}) \\ &\quad - \frac{((m+2\ell-2)^2+4)(m\lambda+4\ell(\ell+1))}{2(m+2\ell-1)(m+2\ell-2)}(\Delta_g + \Delta_{\mathrm{g}}) - 4Q_{\widetilde{h},2} \\ &= \Delta_g^2 + \Delta_g \circ \Delta_{g_{\mathrm{FS}}} + \Delta_{g_{\mathrm{FS}}} \circ \Delta_g + \Delta_{g_{\mathrm{FS}}}^2 \\ &\quad + \left(-\frac{m\lambda}{2} + 2\lambda - 2\ell(\ell+1) + \mathcal{O}\left(\frac{1}{m}\right)\right)(\Delta_g + \Delta_{g_{\mathrm{FS}}}) \\ &\quad + \left(-\frac{m}{2}+2\right)\lambda^2 - 4\ell(\ell+1)\lambda + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \tag{5.6}$$

Once again we choose a test function of the form $v = 1 \otimes \widetilde{v}$, where $\widetilde{v}$ is an eigenfunction of $\Delta_{g_{\mathrm{FS}}}$. In this case we choose

$$\widetilde{v}(z,\overline{z}) = z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_3} + z_3\overline{z_2} + z_3\overline{z_1} + z_1\overline{z_3},$$

which satisfies the equation $-\Delta_{g_{\mathrm{FS}}}\widetilde{v} = (8\ell+16)\widetilde{v}$. Furthermore, by [34, page 25] we have

$$\int_{\widetilde{X}} v^3 \, d\mu_{\widetilde{h}} = \operatorname{vol}_g(M) \int_{\mathbb{CP}^\ell} \widetilde{v}^3 \, d\mu_{g_{\mathrm{FS}}} \neq 0,$$

so v is indeed nontrivial. Substituting this choice of v into (5.6) we obtain

$$\left(P_{\widetilde{h},2} - \frac{n+4}{2}Q_{\widetilde{h},2}\right) v = (\widetilde{a}(m,\ell)\lambda^2 + \widetilde{b}(m,\ell)\lambda + \widetilde{c}(m,\ell))v =: \mathfrak{p}(\lambda),$$

where

$$\begin{aligned} \widetilde{a}(m,\ell) &= -\frac{m}{2} + 2 + \mathcal{O}\left(\frac{1}{m}\right) \\ \widetilde{b}(m,\ell) &= -4m(\ell+2) - 4(\ell^2 - 3\ell - 8) + \mathcal{O}\left(\frac{1}{m}\right) \\ \widetilde{c}(m,\ell) &= -16\ell^3 + 16\ell^2 + 224\ell + 256 + \mathcal{O}\left(\frac{1}{m}\right). \end{aligned} \tag{5.7}$$

We see directly that the discriminant is

$$\operatorname{disc}(\mathfrak{p}) = \widetilde{b}^2 - 4\widetilde{a}\widetilde{c} = 16m^2 + \mathcal{O}(m),$$

which is once again positive so long as m is sufficiently large. This once more proves that the quadratic polynomial $\mathfrak{p}(\lambda) = \widetilde{a}\lambda^2 + \widetilde{b}\lambda + \widetilde{c}$ has two real roots. We also see directly from (5.7) that $\widetilde{a} < 0$ provided m is sufficiently large, and that $\widetilde{c} > 0$ for $\ell = 2,3$. By the same argument as in the previous case, we see that when $\ell = 2,3$ we obtain one positive and one negative root, while when $\ell \geq 4$ we have two negative roots. ∎

6. Quartic degenerate stability for arbitrary integer $1 \leq k < \frac{n}{2}$ (proof of Theorem 5)

In this section, we give the proof of Theorem 5. Before we start with the main argument, several preparations are needed.

6.1. **Preliminaries: Explicit formulas for $P_{g,k}$ and τ_0.** It is crucial for our purpose to provide and expression for the operator $P_{g,k}$ efficiently and explicitly for arbitrary order $k \geq 1$. This is done in the following lemma when $M_\tau := \mathbb{S}^1(\tau) \times \mathbb{S}^{n-1}$ with $\tau > 0$ is furnished with the product metric $g \in \mathrm{Met}^k(M)$ defined as

$$g = g_{\mathbb{S}^1(\tau)} \oplus g_{\mathbb{S}^{n-1}}. \tag{6.1}$$

Notice that the radius $\tau > 0$ of the circle $\mathbb{S}^1(\tau)$ is allowed to be arbitrary and the resulting formulas (6.3) and (6.5) do not depend on τ.

Lemma 6.1. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $u \in \mathcal{C}^k(M_\tau)$ be of the form $u(t, \omega) = f(t)Y_j(\omega)$. Then, one has*

$$P_{g,k}u(t, \omega) = (\mathcal{L}_{k,j} f)(t)Y_j(\omega) \quad \text{on} \quad M_\tau, \tag{6.2}$$

where

$$\mathcal{L}_{k,j} = \prod_{\ell=1}^{k} \left(-\partial_t^2 + \left(j + \frac{n}{2} + k - 2\ell\right)^2\right). \tag{6.3}$$

Here $(t, \omega) \in \mathbb{S}^1(\tau) \times \mathbb{S}^{n-1}$ and Y_j is a spherical harmonic of $\mathbb{S}^{n-1}$, i.e. it is an eigenfunction of the Laplace-Beltrami operator $(-\Delta)_{\mathbb{S}^{n-1}}$ with eigenvalue $\mu_j = j(j + n - 2)$. Moreover, for any $u \in \mathcal{C}^{2k}(M_\tau)$, let $\tilde{u} \in \mathcal{C}^{2k}(\mathbb{R} \times \mathbb{S}^{n-1})$ be its $2\pi\tau$-periodic extension and let $\hat{u} \in \mathcal{C}^{2k}(\mathbb{R}^n \setminus \{0\})$ be defined by

$$\hat{u}(x) = |x|^{-\frac{n-2k}{2}} \tilde{u}\left(\ln |x|, \omega\right) \quad \text{with} \quad \omega = \frac{x}{|x|}. \tag{6.4}$$

Then, it holds

$$P_{g,k}u(t, \omega) = e^{\frac{n+2k}{2}t}((-\Delta)^k \hat{u})(e^t \omega) \quad \text{for} \quad (t, \omega) \in M_\tau. \tag{6.5}$$

Finally, the Green's function $G_\tau : M_\tau \times M_\tau \to \mathbb{R}$ is given by

$$G_\tau(t, \omega, s, \eta) = c_{n,k} \sum_{m \in \mathbb{Z}} |\cosh(t - s - 2\pi m\tau) - \langle \omega, \eta \rangle|^{-\frac{n-2k}{2}}, \tag{6.6}$$

where $c_{n,k} > 0$ is a normalizing dimensional constant and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

Since $L^2(M_\tau)$ is spanned by functions of the form $u(t, \omega) = f(t)Y_j(\omega)$, (6.2) completely describes the action of $P_{g,k}$ on $L^2(M_\tau)$. Indeed, fix an orthonormal basis $(Y_{j,l})_{l \in \{1,\ldots,N_j\}}$ of the N_j-dimensional space of spherical harmonics of degree j on $\mathbb{S}^{n-1}$, namely

$$N_j = \frac{(2j + n - 2)(j + n - 3)!}{(n - 2)!j!}.$$

Then, the functions $\mathfrak{a}_{m,j,l}(t) = \cos(\frac{mt}{\tau})Y_{j,l}(\omega)$ and $\mathfrak{b}_{m,j,l} \sin(\frac{mt}{\tau})Y_{j,l}(\omega)$ form an orthogonal basis

$$(\mathfrak{a}_{m,j,l}, \mathfrak{b}_{m,j,l})_{(m,j,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \in \{1,\ldots,N_j\}} \tag{6.7}$$

of $L^2(M_\tau)$. Through Lemma 6.1, we can express the action of $P_{g,k}$ on this basis.

Corollary 6.2. *Let $n, k \in \mathbb{N}$ with $n > 2k$. For every $\tau > 0$ and $m, j \in \mathbb{N}_0$, one has*

$$P_{g,k}(\mathfrak{a}_{m,j}) = \alpha_{m,j}(\tau)\mathfrak{a}_{m,j} \quad \text{and} \quad P_{g,k}(\mathfrak{b}_{m,j}) = \alpha_{m,j}(\tau)\mathfrak{b}_{m,j} \quad \text{on} \quad M_\tau$$

for the eigenvalues

$$\alpha_{m,j}(\tau) = \prod_{\ell=1}^{k} \left(m^2\tau^{-2} + (j + \frac{n}{2} + k - 2\ell)^2 \right). \tag{6.8}$$

Proof. This is an immediate consequence of (6.2) and (6.3) together with the fact that $-\partial_t^2 \cos(\frac{mt}{\tau}) = m^2\tau^{-2}\cos(\frac{mt}{\tau})$ and $-\partial_t^2 \sin(\frac{mt}{\tau}) = m^2\tau^{-2}\sin(\frac{mt}{\tau})$. □

Proof of Lemma 6.1. We let $u(t,\omega) = f(t)Y_j(\omega)$ and write $\mu_j = j(j+n-2)$ for the j-th eigenvalue of the Laplace-Beltrami operator $(-\Delta)_{\mathbb{S}^{n-1}}$.

Furthermore, by [13, Theorem 1.1], we have

$$P_{g,k}u(t,\omega) = (\mathcal{L}_{k,j}f)(t)Y_j(\omega),$$

where the operator $\mathcal{L}_{k,j}$ is given by

$$\mathcal{L}_{k,j} = \begin{cases} \prod_{s=0}^{\frac{k-2}{2}} D_{k-1-2s,j} & \text{if } k \text{ is even}, \\ (-\partial_t^2 + \mu_j + \frac{(n-2)^2}{4}) \prod_{s=0}^{\frac{m-3}{2}} D_{k-1-2s,j} & \text{if } k \text{ is odd}. \end{cases} \tag{6.9}$$

Here, for any $L \in \mathbb{N}_0$, the operator $D_{L,j}$ is given by

$$D_{L,j} := (\partial_t^2 - \mu_j)^2 - \frac{(n-2)^2}{2}(\partial_t^2 - \mu_j) - 2L^2(\partial_t^2 + \mu_j) - \left(\frac{(n-2)^2}{4} - L^2 \right)^2.$$

Recalling $\mu_j = j(j+n-2)$, we can check by direct computation that

$$D_{L,j} = \left(-\partial_t^2 + (j + \frac{n-2}{2} - L)^2 \right) \left(-\partial_t^2 + (j + \frac{n-2}{2} + L)^2 \right). \tag{6.10}$$

Let $k \in \mathbb{N}$ be an even number. Then, as the number $L = k - 1 - 2s$ in (6.9) runs through $\{k-1, k-3, \dots, 3, 1\}$, the number $\frac{n-2}{2} - L$ in (6.10) runs through $\{\frac{n}{2}-k, \frac{n}{2}-k+2, \dots, \frac{n}{2}-4, \frac{n}{2}-2\}$, and the number $\frac{n-2}{2} + L$ in (6.10) runs through $\{\frac{n}{2}+k-2, \frac{n}{2}+k-4, \dots, \frac{n}{2}+2, \frac{n}{2}\}$. Thus, for $k \in \mathbb{N}_0$ even, (6.3) follows from (6.9). An analogous reasoning, together with the fact that $\mu_j + \frac{(n-2)^2}{4} = (j + \frac{n}{2} - 1)^2$, gives the conclusion if $k \in \mathbb{N}_0$ is odd.

The claimed transformation formula (6.5) is now a direct consequence of (6.2) and (6.3) combined with [16, Lemma 2]. Indeed, the cited result shows that the transformation of $(-\Delta)^k$ through (6.4) coincides exactly with the action of $P_{g,k}$ given by (6.2) and (6.3).

Using (6.5), we can also derive the expression (6.6) of the Green's function G_τ. To do so, suppose that $u \in \mathcal{C}^{2k}(M_\tau)$ and $f \in \mathcal{C}(M_\tau)$ are such that $P_{g,k}u = f$ on M_τ. By (6.5), the functions

$$\hat{u}(x) := |x|^{-\frac{n-2k}{2}}\tilde{u}(\ln|x|, \omega) \quad \text{and} \quad \hat{f}(x) := |x|^{-\frac{n+2k}{2}}\tilde{f}(\ln|x|, \omega)$$

(where $\tilde{u}$ and $\tilde{f}$ again denote the τ-periodic extensions of u and f to $\mathbb{R} \times \mathbb{S}^{n-1}$) then satisfy $(-\Delta)^k\hat{u} = \hat{f}$ in $\mathbb{R}^n \setminus \{0\}$. Since the Green's function on $\mathbb{R}^n$ of $(-\Delta)^k$ is $\tilde{c}_{n,k}|x-y|^{-n+2k}$ (for a certain constant $\tilde{c}_{n,k} > 0$), this implies

$$\hat{u}(x) = \tilde{c}_{n,k} \int_{\mathbb{R}^n} |x-y|^{-n+2k} \hat{f}(y)\, \mathrm{d}y.$$

Using this, for any $(t,\omega)\in M_\tau$, we compute

$$\begin{aligned} u(t,\omega)=\tilde{u}(t,\omega) &= e^{\frac{n-2k}{2}t}\hat{u}(e^t\omega) \\ &= \tilde{c}_{n,k}e^{\frac{n-2k}{2}t}\int_{\mathbb{R}^n}|e^t\omega-y|^{-n+2k}\hat{f}(y)\,\mathrm{d}y \\ &= \tilde{c}_{n,k}e^{\frac{n-2k}{2}t}\int_0^\infty \mathrm{d}r\int_{\mathbb{S}^{n-1}}\mathrm{d}\eta\, r^{n-1}|e^t\omega-r\eta|^{-n+2k}r^{-\frac{n+2k}{2}}\tilde{f}(\ln r,\eta) \\ &= \tilde{c}_{n,k}\int_{\mathbb{R}}\mathrm{d}s\int_{\mathbb{S}^{n-1}}\mathrm{d}\eta\, e^{(t-s)\frac{n-2k}{2}}|e^{t-s}\omega-\eta|^{-n+2k}\tilde{f}(s,\eta) \\ &= \tilde{c}_{n,k}2^{-\frac{n-2k}{2}}\int_{\mathbb{R}}\mathrm{d}s\int_{\mathbb{S}^{n-1}}\mathrm{d}\eta\,|\cosh(t-s)-\langle\omega,\eta\rangle|^{-\frac{n-2k}{2}}\tilde{f}(s,\eta) \\ &= \int_0^{2\pi\tau}\mathrm{d}s\int_{\mathbb{S}^{n-1}}\mathrm{d}\eta\left(c_{n,k}\sum_{m\in\mathbb{Z}}|\cosh(t-s-2\pi m\tau)-\langle\omega,\eta\rangle|^{-\frac{n-2k}{2}}\right)f(s,\eta), \end{aligned}$$

where in the last step we have set $c_{n,k}:=\tilde{c}_{n,k}2^{-\frac{n-2k}{2}}$ and used the τ-periodicity of $\tilde{f}$. Since u was arbitrary, it follows that

$$(t,\omega,s,\eta)\mapsto c_{n,k}\sum_{m\in\mathbb{Z}}|\cosh(t-s-2\pi m\tau)-\langle\omega,\eta\rangle|^{-\frac{n-2k}{2}}$$

is the Green's function of $P_{g,k}$ on M_τ. □

We now turn to the proper choice of the radius τ_0 in (1.12). For this purpose, by (6.3), let us write

$$\mathsf{P}_k(X):=X^{2k}+p_{k,k-1}X^{2k-2}+...+p_{k,1}X^2+p_{k,0}:=\prod_{\ell=1}^k\left(X^2+(\frac{n}{2}+k-2\ell)^2\right) \tag{6.11}$$

for the polynomial such that $\mathsf{P}_k(-\partial_t^2)=\mathcal{L}_{k,0}$. Notice that the coefficients $p_{k,m}$ are positive for all $m=0,...,k-1$ because $\frac{n}{2}+k-2\ell>0$ for all $\ell\in\{1,\dots,k\}$ (as a consequence of $n>2k$). For completeness, we define $p_{k,k}:=1$.

We choose $\tau_0>0$ such that $\varphi(t,\omega)=\sin(t/\tau_0)$ is in the kernel of the linearization of the equation

$$P_{g,k}u=p_{k,0}u^{\frac{n+2k}{n-2k}}\quad\text{on}\quad M_{\tau_0}$$

about the constant solution $u=1$. In other words, we require $\varphi(t,\omega)=\sin(t/\tau_0)$ to solve the linear equation

$$P_{g,k}\varphi=p_{k,0}\frac{n+2k}{n-2k}\varphi\quad\text{on}\quad M_{\tau_0}. \tag{6.12}$$

This will lead to the desired degeneracy, as we check in Lemma 6.8. Let us check that this requirement determines $\tau_0>0$ uniquely. Indeed, using that $\mathcal{L}_{k,0}=\mathsf{P}_k(-\partial_t^2)$, we have, for every $\tau>0$,

$$P_{g,k}\sin\left(\frac{t}{\tau}\right)=\mathsf{P}_k(\tau^{-2})\sin\left(\frac{t}{\tau}\right)\quad\text{on}\quad M_\tau. \tag{6.13}$$

Since all the $p_{k,m}$ are positive, $\mathsf{P}_k(\tau^{-2})$ is strictly decreasing as a function of $\tau\in(0,\infty)$, with $\lim_{\tau\to 0}\mathsf{P}_k(\tau^{-2})=\infty$ and $\lim_{\tau\to\infty}\mathsf{P}_k(\tau^{-2})=p_{k,0}$. Since $\frac{n+2k}{n-2k}>1$, we can pick $\tau_0>0$ as the unique number satisfying

$$\mathsf{P}_k(\tau_0^{-2})=p_{k,0}\frac{n+2k}{n-2k}. \tag{6.14}$$

Thus, by (6.13), $\varphi(t,\omega)=\sin(t/\tau_0)$ solves (6.12) as desired.

Remark 6.3. *By a similar argument one can check more generally that for every $m \in \mathbb{N}$, there is precisely one $\tau_0^{(m)} > 0$ such that $\alpha_{m,0}(\tau_0^{(m)}) = \frac{n+2k}{n-2k} p_{k,0}$. Consequently, the functions*

$$\varphi(t,\omega) = \sin\left(\frac{mt}{\tau_0^{(m)}}\right) \quad \text{or} \quad \varphi(t,\omega) = \cos\left(\frac{mt}{\tau_0^{(m)}}\right)$$

solve (6.12) on $\mathbb{S}^1(\tau_0^{(m)}) \times \mathbb{S}^{n-1}$. On the other hand, if $j \geq 1$, then for every $m \in \mathbb{N}_0$ and $\tau > 0$ we have

$$\alpha_{m,j}(\tau) \geq \alpha_{0,1}(\tau) = \alpha_{0,1}(\tau_0) > \alpha_{1,0}(\tau_0) = \frac{n+2k}{n-2k} p_{k,0},$$

by the inequality (6.36) below.

Since the $\alpha_{m,j}(\tau)$ are decreasing functions of τ, the value $\tau = \tau_0$ from (6.14) is actually the smallest *value of τ such that the linearized equation (6.12) has a non-zero solution.*

6.2. **Bounding P_k and τ_0.** We will find the following bounds regarding τ_0 useful in later computations.

Lemma 6.4. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $\tau_0 = \tau_0(n,k)$ be defined by (6.14). Then, one has*

$$\frac{1}{\sqrt{n+2k-4}} \leq \tau_0(n,k) \leq \frac{1}{\sqrt{n-2k}}.$$

Notice that for $k = 1$ the upper and lower bounds coincide and we recover $\tau_0(n,1) = \frac{1}{\sqrt{n-2}}$.

Proof. Since P_k is strictly increasing on $[0,\infty)$, in view of (6.14), the inequality $\tau_0 \leq \frac{1}{\sqrt{n-2k}}$ follows if we can show that $\mathsf{P}_k(n-2k) \leq \frac{n+2k}{n-2k} p_{k,0}$.

By writing

$$\frac{n+2k}{n-2k} p_{k,0} = \prod_{\ell=1}^{k} \left(\frac{n}{2} + k - 2\ell + 2\right)\left(\frac{n}{2} + k - 2\ell\right)$$

and

$$\mathsf{P}_k(n-2k) = \prod_{\ell=1}^{k} \left(n - 2k + \left(\frac{n}{2} + k - 2\ell\right)^2\right)$$

the desired inequality $\mathsf{P}_k(n-2k) \leq \frac{n+2k}{n-2k} p_{k,0}$ follows from the fact that

$$\begin{aligned}\left(\frac{n}{2} + k - 2\ell + 2\right)\left(\frac{n}{2} + k - 2\ell\right) - \left(n - 2k + \left(\frac{n}{2} + k - 2\ell\right)^2\right) &= 2\left(\frac{n}{2} + k - 2\ell\right) - (n-2k) \\ &= 4k - 4\ell \geq 0\end{aligned}$$

for every $\ell \in \{1, \ldots, k\}$.

Analogously, from the fact that

$$\begin{aligned}\left(\frac{n}{2} + k - 2\ell + 2\right)\left(\frac{n}{2} + k - 2\ell\right) - \left(n + 2k - 4 + \left(\frac{n}{2} + k - 2\ell\right)^2\right) &= 2\left(\frac{n}{2} + k - 2\ell\right) - (n+2k-4) \\ &= -4\ell + 4 \leq 0\end{aligned}$$

for every $\ell \in \{1, \ldots, k\}$, we deduce $\mathsf{P}_k(n+2k-4) \geq \frac{n+2k}{n-2k} p_{k,0}$, and hence $\tau_0(n,k) \geq \frac{1}{\sqrt{n+2k-4}}$. ☐

Lemma 6.5. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). Then, one has*

$$\mathcal{Q}_{g,k}(1) < \mathcal{S}_{n,k}, \tag{6.15}$$

where $\mathcal{S}_{n,k} := \omega_n^{2k/n}\Gamma(\frac{n+2k}{2})\Gamma(\frac{n-2k}{2})^{-1}$ *is the best constant for the* k*-th order Sobolev inequality on* $\mathbb{S}^n$ *with* $\omega_n = 2\pi^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})^{-1}$ *its volume measure.*

Proof. We have $\mathcal{Q}_{g,k}(1) = \mathrm{vol}_g(M)^{\frac{2}{n}} p_{k,0}$, with

$$p_{k,0} = \prod_{\ell=1}^{k}\left(\frac{n}{2}+k-2\ell\right)^2$$

the 0-th order coefficient of the polynomial $\mathsf{P}_k(X) = \prod_{\ell=1}^{k}(X + (\frac{n}{2}+k-2\ell)^2)$ defined in (6.11). Using that $\mathrm{vol}_g(M) = 2\pi\tau_0\omega_{n-1}$ and since $\omega_n = 2\pi^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})^{-1}$, inequality (6.15) is equivalent to

$$p_{k,0}\frac{\Gamma(\frac{n}{2}-k)}{\Gamma(\frac{n}{2}+k)} < \left(\frac{1}{2\sqrt{\pi}\tau_0}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\right)^{\frac{2}{n}}. \tag{6.16}$$

By Lemma 6.4, we know that $\tau_0 = \tau_0(n,k) \le \frac{1}{\sqrt{n-2k}}$.

Consequently, in view of (6.16), it suffices to prove

$$\Psi_k(n) := \left(p_{k,0}\frac{\Gamma(\frac{n}{2}-k)}{\Gamma(\frac{n}{2}+k)}\right)^{\frac{n}{2}} < \frac{\sqrt{n-2k}}{2\sqrt{\pi}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} =: \Phi_k(n). \tag{6.17}$$

By Stirling's formula, $\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z(1+\mathrm{o}(1))$ as $z\to\infty$. It follows

$$\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} = \sqrt{\frac{2}{n}}(1+\mathrm{o}(1)), \qquad \text{and so} \qquad \Phi_k(n) \to \frac{1}{\sqrt{2\pi}} \quad \text{as} \quad n\to\infty.$$

Moreover, we have

$$\Psi_k(n) = \left(p_{k,0}\frac{\Gamma(\frac{n}{2}-k)}{\Gamma(\frac{n}{2}+k)}\right)^{\frac{n}{2}} = \prod_{\ell=1}^{k}\left(\frac{\frac{n}{2}+k-2\ell}{\frac{n}{2}+k-2\ell+1}\right)^{\frac{n}{2}}$$
$$\prod_{\ell=1}^{k}\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right)^{\frac{n}{2}+k-2\ell+1}\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right)^{-k+2\ell-1} \to e^{-k}$$

as $n\to\infty$. Since $e^{-k} < \frac{1}{\sqrt{2\pi}}$ for every $k\ge 1$, we can conclude the proof of (6.17), by showing that (treating n as a real variable)

$$(\log\Psi_k)'(n) \ge (\log\Phi_k)'(n) \qquad \text{for all} \qquad n > 2k. \tag{6.18}$$

On the one hand, we have

$$(\log\Phi_k)'(n) = \frac{1}{2}\left(\frac{1}{n-2k} + \psi\left(\frac{n}{2}\right) - \psi\left(\frac{n+1}{2}\right)\right) \le \frac{1}{2}\left(\frac{1}{n-2k} - \frac{1}{n}\right) = \frac{k}{n(n-2k)}, \tag{6.19}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Digamma function. The claimed inequality follows from the concavity of ψ together with the functional equation $\psi(z+1) = \psi(z) + \frac{1}{z}$ (used with $z = n/2$).

On the other hand,

$$\log\Psi_k(n) = \sum_{\ell=1}^{k}\left(\frac{n}{2}+k-2\ell+1\right)\log\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right) \tag{6.20}$$
$$+\sum_{\ell=1}^{k}(-k+2\ell-1)\log\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right).$$

To treat the first summand of (6.20), we estimate, for every $m > 1$,

$$\log\left(1-\frac{1}{m}\right) = -\sum_{j=1}^{\infty}\frac{1}{jm^j} \geq -\frac{1}{m}-\frac{1}{2}\sum_{j=2}^{\infty}\frac{1}{m^j} = -\frac{1}{m}-\frac{1}{2}\left(-1-\frac{1}{m}+\frac{1}{1-\frac{1}{m}}\right) = \frac{1}{2}-\frac{1}{2m}-\frac{m}{2(m-1)},$$

using the series expansion of $\log(1+z)$ and the geometric series. As a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}m}\left(m\log\left(1-\frac{1}{m}\right)\right) = \log\left(1-\frac{1}{m}\right)+\frac{1}{m-1} \geq \frac{1}{2}-\frac{1}{2m}-\frac{m}{2(m-1)}+\frac{1}{m-1} = \frac{1}{2m(m-1)}$$

Applying this with $m = m_{k,\ell} = \frac{n}{2}+k-2\ell+1$ (note that $m > 1$ because $\ell \leq k < \frac{n}{2}$) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}n}\left(\sum_{\ell=1}^{k}(\frac{n}{2}+k-2\ell+1)\log\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right)\right) &\geq \frac{1}{4}\sum_{\ell=1}^{k}\frac{1}{m_{k,\ell}(m_{k,\ell}-1)} \qquad (6.21)\\ &= \sum_{\ell=1}^{k}\frac{1}{(n+2k-4\ell)(n+2k-4\ell+2)}.\end{aligned}$$

Next, direct computation gives that the derivative of the second summand in (6.20) is

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\sum_{\ell=1}^{k}(-k+2\ell-1)\log\left(1-\frac{1}{\frac{n}{2}+k-2\ell+1}\right)\right) = -\sum_{\ell=1}^{k}\frac{2k-4\ell+2}{(n+2k-4\ell+2)(n+2k-4\ell)}. \qquad (6.22)$$

By combining (6.19), (6.21) and (6.22) we obtain

$$\begin{aligned}(\log\Psi_k-\log\Phi_k)'(n) &\geq \sum_{\ell=1}^{k}\frac{-2k+4\ell-1}{(n+2k-4\ell)(n+2k-4\ell+2)}-\frac{k}{n(n-2k)}\\ &= \sum_{\ell=1}^{k-1}\frac{-2k+4\ell-1}{(n+2k-4\ell)(n+2k-4\ell+2)}+\left(\frac{2k-1}{(n-2k)(n-2k+2)}-\frac{k}{n(n-2k)}\right)\\ &\geq \sum_{\ell=1}^{k-1}\frac{-2k+4\ell-1}{(n+2k-4\ell)(n+2k-4\ell+2)}+\frac{k-1}{(n-2k)(n-2k+2)}.\end{aligned}$$

In the k summands on the right side, the denominators form a decreasing sequence of positive numbers. Moreover, the numerators form an increasing sequence of numbers which sum to zero:

$$\sum_{\ell=1}^{k-1}(-2k+4\ell-1)+(k-1) = \left(-2k(k-1)+4\frac{(k-1)k}{2}-(k-1)\right)+(k-1) = 0.$$

From these facts it is elementary to conclude that the right side is non-negative. Thus, (6.18) follows, and the proof is complete. ☐

Lemma 6.6. *Let $n, k \in \mathbb{N}$ and $n > 2k$ and let τ_0 be defined by (6.14). For any $\ell \geq 0$, one has*

$$\frac{\Gamma\left(\frac{n-2k}{2n}\right)\Gamma\left(\frac{n+2k}{2n}+\ell\right)}{\Gamma\left(\frac{n+2k}{2n}\right)\Gamma\left(\frac{n-2k}{2n}+\ell\right)} \leq p_{k,0}^{-1}\mathsf{P}_k({\tau_0}^{-2}\ell^2).$$

Furthermore, equality holds if and only if $\ell = 0$ or $\ell = 1$.

Proof. Let us denote

$$\Phi_{n,k}(\ell) := \frac{\Gamma\left(\frac{n-2k}{2n}\right)\Gamma\left(\frac{n+2k}{2n}+\ell\right)}{\Gamma\left(\frac{n+2k}{2n}\right)\Gamma\left(\frac{n-2k}{2n}+\ell\right)} \quad \text{and} \quad \Lambda_{n,k}(\ell) := p_{k,0}^{-1}\mathsf{P}_k({\tau_0}^{-2}\ell^2).$$

The equality $\Phi_{n,k}(0) = \Lambda_{n,k}(0)$ is immediate. Using the definition (6.14) we moreover have $\mathsf{P}_k(\tau_0^{-2}\ell^2) = \frac{n+2k}{n-2k} p_{k,0}$, so that

$$\Lambda_{n,k}(1) = p_{k,0}^{-1}\mathsf{P}_k(\tau_0^{-2}) = \frac{n+2k}{n-2k} = \Phi_{n,k}(1).$$

To prove the strict inequality $\Phi_{n,k}(\ell) < \Lambda_{n,k}(\ell)$ for every $n > 2k$ and $\ell \geq 2$, we argue similarly to Beckner [7, proof of Theorem 4]. Since $\Phi_{n,k}(1) = \Lambda_{n,k}(1)$, it is enough to show

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\ln\Phi_{n,k}(\ell) < \frac{\mathrm{d}}{\mathrm{d}\ell}\ln\Lambda_{n,k}(\ell) \quad \text{for every} \quad \ell > 1 \tag{6.23}$$

(where we treat ℓ as a real variable). To prove (6.23), we write $p = \frac{2n}{n-2k} = \frac{2}{1-2s}$ with $s = \frac{m}{n} \in (0, \frac{1}{2})$ thanks to $n > 2m$. The left side of (6.23) then is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\ell}\ln\Phi_{n,k}(\ell) &= \psi\left(\frac{1}{p'}+\ell\right) - \psi\left(\frac{1}{p}+\ell\right) = \psi\left(\frac{1}{2}+s+\ell\right) - \psi\left(\frac{1}{2}-s+\ell\right)\\ &= \sum_{m=0}^{\infty} \frac{1}{\frac{1}{2}-s+\ell+m} - \frac{1}{\frac{1}{2}+s+\ell+m} = 2s\sum_{m=0}^{\infty}\frac{1}{(\frac{1}{2}+\ell+m)^2 - s^2}\\ &< 2s\sum_{m=0}^{\infty}\frac{1}{(\frac{1}{2}+\ell+m)^2-(\frac{1}{2})^2}.\end{aligned}$$

Here $\psi = \frac{\Gamma'}{\Gamma}$ denotes the Digamma function. The claimed series expansion follows, e.g., from integrating the expansion [1, eq. (6.4.10)]

$$\psi'(z) = \sum_{m=0}^{\infty}\frac{1}{(z+m)^2}.$$

But now it is easy to directly evaluate

$$\sum_{m=0}^{\infty}\frac{1}{(\frac{1}{2}+\ell+m)^2-(\frac{1}{2})^2} = \sum_{m=0}^{\infty}\frac{1}{\ell+m} - \frac{1}{1+\ell+m} = \frac{1}{\ell}$$

as a telescopic sum. Recalling $s = \frac{m}{n}$, in conclusion we have shown

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\ln\Phi_{n,k}(\ell) < \frac{2m}{n\ell}. \tag{6.24}$$

In view of (6.24) it remains to show that

$$\frac{2m}{n\ell} \leq \frac{\mathrm{d}}{\mathrm{d}\ell}\ln\Lambda_{n,k}(\ell) = \frac{\frac{\mathrm{d}}{\mathrm{d}\ell}\Lambda_{n,k}(\ell)}{\Lambda_{n,k}(\ell)}$$

Since $\Lambda_{n,k}(\ell) = p_{k,0}^{-1}\mathsf{P}_k(\tau_0^{-2}\ell^2) = p_{k,0}^{-1}\sum_{k=0}^{k} p_{k,m}\tau_0^{-2m}\ell^{2m}$, some elementary manipulations show that this is equivalent to

$$m p_{k,0} \leq \sum_{m=1}^{k} p_{k,m}(nk-m)\tau_0^{-2m}\ell^{2m}. \tag{6.25}$$

Since $\ell > 1$ and $p_{k,m} > 0$ for all $k \in \{1,\dots,m\}$, the right side of (6.25) is estimated by

$$\sum_{m=1}^{k} p_{k,m}(nk-m)\tau_0^{-2m}\ell^{2m} \geq (n-m)\sum_{m=1}^{k} p_{k,m}\tau_0^{-2m} = (n-m)(\mathsf{P}_k(\tau_0^{-2}) - p_{k,0}) = (n-m)\frac{4m}{n-2k}p_{k,0}.$$

Since $(n-m)\frac{4m}{n-2k} > m$, the proof of (6.25) is complete. As explained above, this concludes the proof of the lemma. □

6.3. **The minimizers of $\mathcal{Y}_{k,+}(M,[g])$ and their degeneracy.** From now on, the value $\tau = \tau_0$ from (6.14) and the manifold $M = M_{\tau_0} = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ (furnished with the product metric $g \in \mathrm{Met}^k(M)$ defined as (6.1)) are fixed. We will write $\alpha_{m,j} := \alpha_{m,j}(\tau_0)$ for the eigenvalues of $P_{g,k}$ from (6.8).

Lemma 6.7. *Let $n, k \in \mathbb{N}$ with $2k < n$ and let $M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). Then $\mathcal{Y}_{k,+}(M,g)$ is uniquely minimized by the constant functions.*

Proof. The proof will be divided into three steps.
Step 1: Existence of minimizers.

Once we prove coercivity of $P_{g,k}$, the existence of a minimizer $u > 0$ follows directly from [41, Theorem 3] combined with Lemma 6.5 and Lemma 6.1 (which guarantees the needed positivity of the Green's function). It only remains to show that there exists $c > 0$ such that

$$\int_M u P_{g,k}(u)\,\mathrm{d}\mu_g \geq c\|u\|^2_{W^{k,2}(M)}$$

Write

$$u = \sum_{(m,j,l)\in\mathcal{I}} u_{m,j,l}\mathfrak{a}_{m,j,l} + \tilde{u}_{m,j,l}\mathfrak{b}_{m,j,l}$$

for certain coefficients $u_{m,j,l}, \tilde{u}_{m,j,l} \in \mathbb{R}$, where $\mathfrak{a}_{m,j,l}$ and $\mathfrak{b}_{m,j,l}$ are the basis functions from (6.7) and $\mathcal{I} := \mathbb{N}_0 \times \mathbb{N}_0 \times \{1, \dots, N_j\}$. Then it can be deduced from the orthogonality properties of the $\mathfrak{a}_{m,j,l}$, $\mathfrak{b}_{m,j,l}$ that

$$\|u\|^2_{W^{k,2}(M)} \lesssim \sum_{(m,j,l)\in\mathcal{I}} (u^2_{m,j,l} + \tilde{u}^2_{m,j,l})(1 + m^{2k} + j^{2k}).$$

On the other hand, the expression (6.8) of the eigenvalues $\alpha_{m,j}$ of $P_{g,k}$ implies the estimate

$$\alpha_{m,j} \gtrsim 1 + m^{2k} + j^{2k},$$

Hence, the estimate below holds

$$\int_M u P_{g,k}(u)\,\mathrm{d}\mu_g = \sum_{(m,j,l)\in\mathcal{I}} (u^2_{m,j,l} + \tilde{u}^2_{m,j,l})\alpha_{m,j} \gtrsim \sum_{(m,j,l)\in\mathcal{I}} (u^2_{m,j,l} + \tilde{u}^2_{m,j,l})(1 + m^{2k} + j^{2k}),$$

which proves coercivity.
Step 2: Minimizers are radial.

Let $0 < u \in \mathcal{C}^{2k}(M)$ be a minimizer of $\mathcal{Y}_{k,+}(M,[g])$ on M, which exists by Step 1. Then, up to multiplying it by a suitable scalar factor,u satisfies the Euler-Lagrange equation

$$P_{g,k}u = c_{n,k}u^{\frac{n+2k}{n-2k}} \quad \text{on} \quad M, \tag{6.26}$$

where $c_{n,k} > 0$ is a normalizing dimensional constant. Let $\tilde{u} \in \mathcal{C}^{2k}(\mathbb{R} \times \mathbb{S}^{n-1})$ be the $2\pi\tau_0$-periodic extension of u to $\mathbb{R} \times \mathbb{S}^{n-1}$ and $v(x) = |x|^{-\frac{n-2k}{2}}\tilde{u}(\ln|x|, \omega)$ be its logarithmic transform on $\mathbb{R}^n \setminus \{0\}$. By (6.26) and (6.5), v satisfies

$$(-\Delta)^k v = c_{n,k}v^{\frac{n+2k}{n-2k}} \quad \text{on} \quad \mathbb{R}^n \setminus \{0\}.$$

Since $v > 0$, by applying the moving planes method as in [37, 53] it follows that $v(x)$ only depends on $|x|$. Equivalently, $\tilde{v}(t,\omega)$, and hence $v(t,\omega)$, only depends on t.
Step 3: Constants are the unique minimizers.

While Steps 1 and 2 still are valid for arbitrary $\tau > 0$, in this step we will make crucial use of the expression (6.14) for τ_0.

By Steps 1 and 2, we only need to prove that constant functions uniquely minimize $\mathcal{Q}_{g,k}$ among functions u only depending on $t \in \mathbb{S}^1(\tau_0)$. By Lemma 6.1, the inequality we need to show is thus

$$\int_0^{2\pi\tau_0} u\mathsf{P}_k(-\partial_t^2)u\,\mathrm{d}t \geq p_{k,0}(2\pi\tau_0)^{\frac{2k}{n}}\left(\int_0^{2\pi\tau_0}|u|^{\frac{2n}{n-2k}}\,\mathrm{d}t\right)^{\frac{n-2k}{n}} \tag{6.27}$$

(where $p_{k,0}$ is defined through (6.11)) with equality if and only if u is constant.

For the following argument, it will be convenient to rescale $\mathbb{S}^1(\tau_0)$ back to the unit sphere $\mathbb{S}^1 \simeq (0, 2\pi)$. For $v(t) = u(t\tau_0)$ the inequality we need to show then reads as

$$\int_{\mathbb{S}^1} v\mathsf{P}_k(-\tau_0^{-2}\partial_t^2)v\,\mathrm{d}\sigma \geq p_{k,0}(2\pi)^{\frac{2k}{n}}\left(\int_{\mathbb{S}^1}|v|^{\frac{2n}{n-2k}}\,\mathrm{d}\sigma\right)^{\frac{n-2k}{n}}. \tag{6.28}$$

where $\mathrm{d}\sigma$ denotes the standard measure on the unit circle $\mathbb{S}^1$. We will now prove that (6.28) holds, with equality if and only if v is constant, through an argument inspired by Beckner [7, proof of Theorem 4]. For this purpose, we decompose a given $v \in W^{k,2}(\mathbb{S}^1)$ into its Fourier eigenmodes

$$v(t) = \sum_{\ell=0}^{\infty} Y_\ell(t),$$

where, for every $\ell \geq 0$, $Y_\ell(t) = c_\ell \cos(\ell t) + d_\ell \sin(\ell t)$ for some $c_\ell, d_\ell \in \mathbb{R}$.

Then the left side of (6.28) reads as

$$\int_{\mathbb{S}^1} u\mathsf{P}_k(-\tau_0^{-2}\partial_t^2)u\,\mathrm{d}t = \sum_{\ell=0}^{\infty}\mathsf{P}_k(\tau_0^{-2}\ell^2)\int_{\mathbb{S}^1}|Y_\ell|^2\,\mathrm{d}\sigma.$$

By the 'dual-spectral' version of the Hardy–Littlewood–Sobolev inequality [7, eq. (19)] we can estimate the right side of (6.28) as

$$(2\pi)^{1-\frac{n-2k}{n}}\left(\int_{\mathbb{S}^1}|v|^{\frac{2n}{n-2k}}\,\mathrm{d}\sigma\right)^{\frac{n-2k}{n}} \leq \sum_{\ell=0}^{\infty}\frac{\Gamma\left(\frac{n-2k}{2n}\right)\Gamma\left(\frac{n+2k}{2n}+\ell\right)}{\Gamma\left(\frac{n+2k}{2n}\right)\Gamma\left(\frac{n-2k}{2n}+\ell\right)}\int_{\mathbb{S}^1}|Y_\ell|^2\,\mathrm{d}\sigma, \tag{6.29}$$

In Lemma 6.6 we have proved that

$$\frac{\Gamma\left(\frac{n-2k}{2n}\right)\Gamma\left(\frac{n+2k}{2n}+\ell\right)}{\Gamma\left(\frac{n+2k}{2n}\right)\Gamma\left(\frac{n-2k}{2n}+\ell\right)} \leq p_{k,0}^{-1}\mathsf{P}_k(\tau_0^{-2}\ell^2). \tag{6.30}$$

for every $n > 2k$, $\ell \geq 0$, and so (6.30) follows.

Moreover, still by Lemma 6.6, equality in (6.30) occurs precisely for $\ell = 0, 1$.

On the other hand, by the classification of HLS optimizers [36], equality in (6.29) holds if and only if v is a conformal factor, i.e.,

$$v(s) = c(1 + a\cos s + b\sin s)^{\frac{n-2k}{2n}} \tag{6.31}$$

for some $c \in \mathbb{R}\setminus\{0\}$, $a, b \in \mathbb{R}$ with $a^2 + b^2 < 1$.

Hence, if equality holds in (6.28) for some v, then by the equality conditions for (6.29) and (6.30) we must have

$$v(s) = C(1 + A\cos s + B\sin s) = c(1 + a\cos s + b\sin s)^{\frac{n-2k}{2n}}.$$

It is easy to see that this implies that v must be constant. This ends the proof. □

In what follows, it will be convenient to write $\|u\|_{2_k^*} = \|u\|_{L^{2_k^*}(M)}$ with $2_k^* = \frac{2n}{n-2k}$, as well as $\|u\|_{k,2} = \|u\|_{W^{k,2}(M)}$ and $\mathcal{Y} := \mathcal{Y}_{k,+}(M,[g])$. Here, M is given by (1.12) furnished with the standard product metric, denoted by g. Moreover, we abbreviate

$$\mathcal{E}(u) := \frac{2}{n-2k}\int_M uP_{g,k}(u)\,\mathrm{d}\mu_g, \tag{6.32}$$

so that

$$\mathcal{Q}_{g,k}(u) = \frac{\mathcal{E}(u)}{\|u\|_{2_k^*}^2}.$$

Lemma 6.7 yields that

$$\mathcal{F}(u) := \mathcal{E}(u) - \mathcal{Y}\|u\|_{2_k^*}^2 \tag{6.33}$$

is uniquely minimized by constant functions (with minimal value 0).

We now check that, by our special choice of τ_0, the kernel of $D^2\mathcal{F}(1)$ contains a non-constant function.

Lemma 6.8. *Let $n,k \in \mathbb{N}$ with $n > 2k$ and let $M = \mathbb{S}^1(\tau_0)\times\mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). The kernel of $D^2\mathcal{F}(1)$ is spanned by 1, $\cos(t/\tau_0)$ and $\sin(t/\tau_0)$.*

Proof. Clearly, 1 is in the kernel of $D^2\mathcal{F}(1)$ because $\mathcal{F}(c) = 0$ for all $c \in \mathbb{R}$.

On the other hand, for any $\rho \in W^{k,2}(M)$ with $\int_M \rho\,\mathrm{d}\mu_g = 0$, we have

$$\mathcal{E}(1+\varepsilon\rho) = \mathcal{E}(1) + \varepsilon^2\mathcal{E}(\rho)$$

and

$$\begin{aligned}\|1+\varepsilon\rho\|_{2_k^*}^2 &= \left[\int_M \left(1+\varepsilon^2\frac{2_k^*(2_k^*-1)}{2}\rho^2 + \mathrm{o}(\varepsilon^2)\right)\mathrm{d}\mu_g\right]^{2/2_k^*} \\ &= \mathrm{vol}_g(M)^{2/2_k^*} + \varepsilon^2(2_k^*-1)\mathrm{vol}_g(M)^{2/2_k^*-1}\int_M \rho^2\,\mathrm{d}\mu_g + \mathrm{o}(\varepsilon^2).\end{aligned}$$

Thus, we find

$$\begin{aligned}D^2\mathcal{F}(1)[\rho,\rho] &= \mathcal{E}(\rho) - (2_k^*-1)\mathrm{vol}_g(M)^{2/2_k^*-1}\mathcal{Y}\int_M \rho^2\,\mathrm{d}\mu_g \\ &= \mathcal{E}(\rho) - \frac{n+2k}{n-2k}\cdot\frac{2}{n-2k}p_{k,0}\int_M \rho^2\,\mathrm{d}\mu_g,\end{aligned} \tag{6.34}$$

For the last equality, we used that by Lemma 6.7, it holds $\mathcal{Y} = \mathcal{Q}_{g,k}(1) = \frac{2}{n-2k}\mathrm{vol}_g(M)^{1-2/2_k^*}p_{k,0}$.

Thus, recalling (6.32), $\varphi \in W^{k,2}(M)$ with $\int_M \varphi\,\mathrm{d}\mu_g = 0$ is in the kernel of $D^2\mathcal{F}(1)$ if and only if

$$P_{g,k}(\varphi) = \frac{n+2k}{n-2k}p_{k,0}\varphi. \tag{6.35}$$

We have already checked in (6.13) that the definition (6.14) of τ_0 ensures that $\varphi(t) = \sin(t/\tau_0)$ solves (6.35). By exactly the same argument, $\varphi(t) = \cos(t/\tau_0)$ solves (6.35).

It remains to justify that there can be no functions in the kernel of $D^2\mathcal{F}(1)$ which are linearly independent of 1, $\sin(t/\tau_0)$ and $\cos(t/\tau_0)$. In view of (6.35), it therefore remains to check that $\alpha_{m,j} \neq \alpha_{1,0}$ for all $(m,j) \neq (1,0)$. Since, by (6.8), the eigenvalues $\alpha_{m,j}$ are strictly increasing in m and j, this follows if we can show

$$\alpha_{0,1} > \alpha_{1,0}. \tag{6.36}$$

But since

$$\alpha_{1,0} = \frac{n+2k}{n-2k} p_{k,0} = 2^{-2k} \frac{n+2k}{n-2k} \prod_{\ell=1}^{k} (n+2k-4\ell)^2 = (n-2k)(n-2k+4)^2 \cdots (n+2k-4)^2 (n+2k)$$

and

$$\alpha_{0,1} = 2^{-2k} \prod_{\ell=1}^{k} (2+n+2k-4\ell)^2,$$

we obtain

$$\begin{aligned} \frac{\alpha_{0,1}}{\alpha_{1,0}} &= \frac{(n-2k+2)^2}{(n-2k)(n-2k+4)} \cdot \frac{(n-2k+6)^2}{(n-2k+4)(n-2k+8)} \cdot \dots \cdot \frac{(n+2k-2)^2}{(n+2k-4)(n+2k)} \\ &= \prod_{\ell=1}^{k} \frac{(n+2k+2-2\ell)^2}{(n+2k-4\ell)(n+2k-4\ell+4)} > 1, \end{aligned}$$

because of the inequality $\frac{N^2}{(N+2)(N-2)} = \frac{N^2}{N^2-4} > 1$, applied with $N := n+2k+2-2\ell$ for every $\ell \in \{1, \dots, k\}$. Thus (6.36) is shown, and the proof is complete. □

6.4. **The secondary non-degeneracy condition.** We can now start to give the core argument for the proof of Theorem 5.

As in [19], our goal is to verify a 'secondary nondegeneracy condition' using an iterative refinement of Bianchi and Egnell's classical strategy. This strategy consists in decomposing a candidate sequence into a main part and a remainder part orthogonal to it. The orthogonality then implies an improved spectral estimate which can be used to conclude in the classical (i.e., non-degenerate) setting. Because of the degeneracy given by Lemma 6.8, in our setting, we need to further decompose the remainder into a main part which turns out to be in $\ker D^2\mathcal{F}(1)$, and a secondary remainder. At this point only, one has precise enough information to conclude by spectral estimates.

The first and more standard step of this strategy is contained in the following lemma.

Lemma 6.9. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). Assume that $\{u_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ is a sequence such that $\mathcal{Q}_{g,k}(u_m) \to \mathcal{Y}$ and $\|u_m\|_{2_k^*} = \mathrm{vol}_g(M)^{1/2_k^*}$. Then, there are sequences $\{\lambda_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ and $\{\rho_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ such that $\lambda_m \to \pm 1$, $\int_M \rho_m = 0$, $\mathcal{E}(\rho_m) \to 0$, and up to extracting a subsequence, one has*

$$u_m = \lambda_m (1 + \rho_m).$$

Proof. By Lemma 6.7, the only minimizers of $\mathcal{Y}$ are the constants. By Lemma B, up to the extraction of a subsequence every minimizing sequence converges to a minimizer of $\mathcal{Y}$, strongly in $W^{k,2}(M)$. Notice that M satisfies the assumptions of Lemma B, as we have checked in the proof of the first step of Lemma 6.7.

In view of the normalization of u_m, we thus must have $u_m \to \pm 1$ in $W^{k,2}(M)$. Setting $\lambda_m := \overline{u}_m := \mathrm{vol}_g(M)^{-1} \int_M u_m \, \mathrm{d}\mu_g$ and $\rho_m := \frac{u_m}{\overline{u}_m} - 1$, the assertion of the lemma follows. □

Now we derive a more precise expansion. For sequences $\{u_m\}_{m\in\mathbb{N}}$ such that $\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2$ goes to zero superquadratically in $\mathcal{E}(u_m - \overline{u}_m)$, we show that the subleading term is necessarily proportional to the function

$$\phi(t, \omega) := \cos\left(\frac{t}{\tau_0}\right) \in \ker D^2\mathcal{F}(1), \tag{6.37}$$

up to a rotation in the t-coordinate.

Lemma 6.10. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). Let $\{u_m\}_{m\in\mathbb{N}}, \{\rho_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ be sequences such that $u_m = 1 + \rho_m$, where $\int_M \rho_m \, \mathrm{d}\mu_g = 0$ and $\mathcal{E}(\rho_m) \to 0$ as $m \to \infty$. Suppose that*

$$\frac{\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|^2_{2^*_k}}{\mathcal{E}(\rho_m)} \to 0.$$

Then, up to extracting a subsequence and a rotation in the t-coordinate, one has

$$u_m = 1 + \rho_m = 1 + \xi_m(\phi + R_m), \tag{6.38}$$

where $\xi_m \to 0$, $\phi \in \ker D^2\mathcal{F}(1)$ is defined by (6.37), and $\{R_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ satisfies $\int_M R_m \, \mathrm{d}\mu_g = \int_M R_m \sin(t/\tau_0) \, \mathrm{d}\mu_g = \int_M R_m \cos(t/\tau_0) \, \mathrm{d}\mu_g = 0$, and $\mathcal{E}(R_m) \to 0$ as $m \to \infty$.

Proof. By $\int_M \rho_m \, \mathrm{d}\mu_g = 0$, we clearly have

$$\mathcal{E}(u_m) = \mathcal{E}(1) + \mathcal{E}(\rho_m).$$

Moreover, for any $a \in \mathbb{R}$ and $\nu > 2$ the pointwise expansion below holds

$$|1 + a|^\nu = 1 + \nu a + \frac{\nu(\nu-1)}{2} a^2 + \mathcal{O}(|a|^{\min\{3,\nu\}}),$$

which yields

$$\int_M |u_m|^{2^*_k} \, \mathrm{d}\mu_g = \int_M \mathrm{d}\mu_g + \frac{2^*_k(2^*_k - 1)}{2} \int_M \rho_m^2 \, \mathrm{d}\mu_g + \mathrm{o}(\mathcal{E}(\rho_m)),$$

and hence

$$\|u_m\|^2_{2^*_k} = \left(\int_M \mathrm{d}\mu_g\right)^{2/2^*_k} + (2^*_k - 1)\left(\int_M \mathrm{d}\mu_g\right)^{2/2^*_k - 1} \int_M \rho_m^2 \, \mathrm{d}\mu_g + \mathrm{o}(\mathcal{E}(\rho_m)).$$

Since

$$\mathcal{Y} = \frac{2}{n-2k}\mathcal{E}(1)\left(\int_M \mathrm{d}\mu_g\right)^{-2/2^*_k} = \left(\int_M \mathrm{d}\mu_g\right)^{1-2/2^*_k} \alpha_{0,0}$$

and $\alpha_{1,0} = (2^*_k - 1)\alpha_{0,0}$, it follows

$$\mathrm{o}(1) = \frac{\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|^2_{2^*_k}}{\mathcal{E}(\rho_m)} = \frac{\mathcal{E}(\rho_m) - \frac{2}{n-2k}\alpha_{1,0}\int_M \rho_m^2 \, \mathrm{d}\mu_g}{\mathcal{E}(\rho_m)}. \tag{6.39}$$

The quadratic form $\rho \mapsto \mathcal{E}(\rho) - \frac{2}{n-2k}\alpha_{1,0}\int_M \rho^2 \, \mathrm{d}\mu_g$ vanishes on the subspace spanned by $\cos(t/\tau_0)$ and $\sin(t/\tau_0)$. Since $\alpha_{m,j} > \alpha_{1,0}$ for $m \geq 2$ or $j \geq 1$, it is positive definite and equivalent to $\mathcal{E}$ on the orthogonal complement of 1, $\cos(t/\tau_0)$ and $\sin(t/\tau_0)$. Together with $\int_M \rho_m \, \mathrm{d}\mu_g = 0$, it is easy to deduce (6.38) from (6.39) by arguing as in [19, proof of Lemmas 5 and 9]. □

With the refined expansion from Lemma 6.10 at hand, we can now expand $\mathcal{E}(u_m)$ up to fourth order and derive the following 'second-order' stability inequality:

Lemma 6.11. *Let $n, k \in \mathbb{N}$ with $n > 2k$ and let $M = \mathbb{S}^1(\tau_0) \times \mathbb{S}^{n-1}$ with τ_0 defined by (6.14) furnished with the product metric $g \in \mathrm{Met}^k(M)$ given by (6.1). Let $\{u_m\}_{m\in\mathbb{N}}, \{\rho_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ be sequences such that $u_m = 1 + \rho_m$, where $\int_M \rho_m \, \mathrm{d}\mu_g = 0$ and $\mathcal{E}(\rho_m) \to 0$ as $m \to \infty$. Then, one has*

$$\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|^2_{2^*_k} \geq (c + \mathrm{o}(1))\mathcal{E}(\rho_m)^2,$$

where

$$c := \frac{n-2k}{2} \cdot \frac{(2^*_k - 2)}{8\,\mathrm{vol}_g(M)} \alpha_{1,0}^{-1}\left((2^*_k + 1) - \frac{\alpha_{1,0}}{\alpha_{2,0} - \alpha_{1,0}}(2^*_k - 2)\right) > 0.$$

Proof. By Lemma 6.10, we can write $u_m = 1+\rho_m = 1+\xi_m(\phi+R_m)$ with $\xi_m \to 0$ and $\mathcal{E}(R_m)\to 0$ as $m\to\infty$.

Again we follow the strategy of [19], but we apply a technical simplification of the argument from [32] (see also [21]) to deal with the set where R_m has pointwise large values.

Let $M_m \subset M$ be the subset of points $x\in M$ such that $|\rho_m(x)| < \frac{1}{2}$. In particular, on M_m, the function $1+\rho_m$ takes values in $(1/2, 3/2)$, where $t\mapsto |t|^{2_k^*}$ is smooth. In particular, by Taylor's theorem, we can expand to fourth order to get

$$\begin{aligned}\int_{M_m} |1+\rho_m|^{2_k^*}\,\mathrm{d}\mu_g &= \int_{M_m} \mathrm{d}\mu_g + 2_k^*\int_{M_k}\rho_m\,\mathrm{d}\mu_g + \frac{2_k^*(2_k^*-1)}{2}\int_{M_m}\rho_m^2\,\mathrm{d}\mu_g\\ &\quad + \frac{2_k^*(2_k^*-1)(2_k^*-2)}{6}\int_{M_m}\rho_m^3\,\mathrm{d}\mu_g + \frac{2_k^*(2_k^*-1)(2_k^*-2)(2_k^*-3)}{24}\int_{M_m}\rho_m^4\,\mathrm{d}\mu_g + \mathrm{o}(\xi_m^4).\end{aligned}$$

For the complementary set, we directly have

$$\int_{M\setminus M_m}\left(\frac{1}{2}\right)^{2_k^*}\mathrm{d}\mu_g \le \int_{M\setminus M_m}|\rho_m|^{2_k^*}\,\mathrm{d}\mu_g \lesssim \mathcal{E}(\xi_m\phi)^{2_k^*/2} + \mathrm{o}(\xi_m^{2_k^*}) = \xi_m^{2_k^*}(\mathcal{E}(\phi)^{2_k^*/2}+\mathrm{o}(1)),$$

and hence $\mathrm{vol}_g(M\setminus M_m)\lesssim \xi_m^{2_k^*}$. Thus we can use a second-order Taylor expansion to deduce, by estimates analogous to those in [32, p.11], that

$$\begin{aligned}&\int_{C_m}|1+\rho_m|^{2_k^*}\,\mathrm{d}\mu_g\\ &= \int_{M\setminus M_m}\mathrm{d}\mu_g + 2_k^*\int_{M\setminus M_m}\rho_m\,\mathrm{d}\mu_g + \frac{2_k^*(2_k^*-1)}{2}\int_{M\setminus M_m}\rho_m^2\,\mathrm{d}\mu_g + \frac{2_k^*(2_k^*-1)(2_k^*-2)}{6}\int_{M\setminus M_m}\rho_m^3\,\mathrm{d}\mu_g\\ &\quad + \frac{2_k^*(2_k^*-1)(2_k^*-2)(2_k^*-3)}{24}\int_{M\setminus M_m}\rho_m^4\,\mathrm{d}\mu_g + \mathrm{o}(\xi_m^4 + \xi_m^2\mathcal{E}(R_m)).\end{aligned}$$

Adding up the two expansions above, we obtain

$$\begin{aligned}\|u_m\|_{2_k^*}^2 &= \mathrm{vol}_g(M)^{2/2_k^*}\\ &\quad + \mathrm{vol}_g(M)^{2/2_k^*-1}(2_k^*-1)\left(\int_M\rho_m^2\,\mathrm{d}\mu_g + \frac{2_k^*-2}{3}\int_M\rho_m^3\,\mathrm{d}\mu_g + \frac{(2_k^*-2)(v-3)}{12}\int_M\rho_m^4\,\mathrm{d}\mu_g\right)\\ &\quad - \mathrm{vol}_g(M)^{2/2_k^*-2}\frac{(q-2)(2_k^*-1)^2}{4}\left(\int_M\rho_m^2\right)^2 + o\left((\xi_m^4+\xi_m^2\mathcal{E}(R_m)\right),\end{aligned}$$

which with $\mathcal{E}(u_m) = \mathcal{E}(1)+\mathcal{E}(\rho_m)$, and again recalling

$$\mathcal{Y}\,\mathrm{vol}_g(M)^{2/2_k^*-1}(2_k^*-1) = \frac{2}{n-2k}\alpha_{0,0}(2_k^*-1) = \frac{2}{n-2k}\alpha_{1,0},$$

gives us

$$\begin{aligned}\mathcal{E}(u_m)-\mathcal{Y}\|u_m\|_{2_k^*}^2 &= \mathcal{E}(\rho_m) - \frac{2}{n-2k}\alpha_{1,0}\left(\int_M\rho_m^2\,\mathrm{d}\mu_g + \frac{2_k^*-2}{3}\int_M\rho_m^3\,\mathrm{d}\mu_g + \frac{(2_k^*-2)(2_k^*-3)}{12}\int_M\rho_m^4\,\mathrm{d}\mu_g\right)\\ &\quad + \frac{2}{n-2k}\alpha_{1,0}\mathrm{vol}_g(M)^{-1}\frac{(2_k^*-1)(2_k^*-2)}{4}\left(\int_M\rho_m^2\,\mathrm{d}\mu_g\right)^2 + \mathrm{o}\left((\xi_m^4+\xi_m^2\mathcal{E}(R_m)\right).\end{aligned} \tag{6.40}$$

Now we expand the terms on the right more precisely, according to the refined decomposition $\rho_m = \xi_m(\phi+R_m)$ from Lemma 6.10. Clearly, by orthogonality,

$$\mathcal{E}(\rho_m) - \frac{2}{n-2k}\alpha_{1,0}\int_M\rho_m^2\,\mathrm{d}\mu_g = \xi_m^2\mathcal{E}(R_m) - \frac{2}{n-2k}\alpha_{1,0}\xi_m^2\int_M R_m^2\,\mathrm{d}\mu_g$$

because $\phi \in W^{k,2}(M)$ is in the kernel of this quadratic form. Moreover, since $\int_M \phi^3 \, \mathrm{d}\mu_g = 0$,

$$\int_M \rho_m^3 \, \mathrm{d}\mu_g = 3\xi_m^3 \int_M \phi^2 R_m \, \mathrm{d}\mu_g + \mathrm{o}\left((\xi_m^4 + \xi_m^2 \mathcal{E}(R_m)\right).$$

Finally, in the power-four terms, only the leading term $\xi_m \phi$ is relevant in the sense that

$$\int_M \rho_m^4 \, \mathrm{d}\mu_g = \xi_m^4 \int_M \phi^4 \, \mathrm{d}\mu_g + \mathrm{o}(\xi_m^4) \quad \text{and} \quad \left(\int_M \rho_m^2 \, \mathrm{d}\mu_g\right)^2 = \xi_m^4 \left(\int_M \phi^2 \, \mathrm{d}\mu_g\right)^2 + \mathrm{o}(\xi_m^4).$$

In view of

$$\phi^2(s) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2s}{\tau_0}\right),$$

we further decompose

$$R_m = b_m \cos\left(\frac{2s}{\tau_0}\right) + S_k \quad \text{with} \quad \int_M S_k \cos\left(\frac{2s}{\tau_0}\right) \mathrm{d}\mu_g = 0.$$

Then, one has

$$\int_M \phi^2 R_m \, \mathrm{d}\mu_g = \frac{1}{2} b_m \int_M \cos^2\left(\frac{2s}{\tau_0}\right) \mathrm{d}\mu_g = b_m \frac{\mathrm{vol}_g(M)}{4}.$$

Moreover, we can estimate

$$\begin{aligned}
&\mathcal{E}(\xi_m R_m) - \frac{2}{n-2k}\alpha_{1,0} \int_M (\xi_m R_m)^2 \, \mathrm{d}\mu_g + \mathrm{o}(\xi_m^2 \mathcal{E}(R_m)) \\
&= \frac{2}{n-2k}(\alpha_{2,0} - \alpha_{1,0} + \mathrm{o}(1))\xi_m^2 b_m^2 \frac{\mathrm{vol}_g(M)}{2} + (1 + \mathrm{o}(1))\xi_m^2 \mathcal{E}(S_k) - \frac{2}{n-2k}\alpha_{1,0}\xi_m^2 \int_M S_k^2 \, \mathrm{d}\mu_g \\
&\geq \frac{2}{n-2k}(\alpha_{2,0} - \alpha_{1,0} + \mathrm{o}(1))\xi_m^2 b_m^2 \frac{\mathrm{vol}_g(M)}{2}.
\end{aligned}$$

Finally, we can compute explicitly the numerical values of

$$\int_M \phi^4 \, \mathrm{d}\mu_g = \frac{3}{8}\mathrm{vol}_g(M) \quad \text{and} \quad \left(\int_M \phi^2 \, \mathrm{d}\mu_g\right)^2 = \frac{\mathrm{vol}_g(M)^2}{4}.$$

Inserting all of this back into (6.40), we arrive at

$$\begin{aligned}
&\frac{n-2k}{2}\left(\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2\right) \\
&\geq (\alpha_{2,0} - \alpha_{1,0} + \mathrm{o}(1))\xi_m^2 b_m^2 \frac{\mathrm{vol}_g(M)}{2} - \alpha_{1,0}\frac{q-2}{4}\mathrm{vol}_g(M)\xi_m^3 b_m \\
&+ \alpha_{1,0}\mathrm{vol}_g(M)\xi_m^4 \left(\frac{(2_k^*-1)(2_k^*-2)}{16} - \frac{(2_k^*-2)(2_k^*-3)}{32}\right) + \mathrm{o}(\xi_m^4) \\
&= \frac{\mathrm{vol}_g(M)}{8}\xi_m^2 \left(\sqrt{4(\alpha_{2,0} - \alpha_{1,0} + \mathrm{o}(1))} b_m - \frac{\alpha_{1,0}(2_k^*-2)}{\sqrt{4(\alpha_{2,0} - \alpha_{1,0} + \mathrm{o}(1))}}\xi_m\right)^2 - \frac{\mathrm{vol}_g(M)}{8}\frac{\alpha_{1,0}^2(2_k^*-2)^2}{4(\alpha_{2,0} - \alpha_{1,0})}\xi_m^4 \\
&+ \frac{\alpha_{1,0}\mathrm{vol}_g(M)}{32}(2_k^*-2)(2_k^*+1)\xi_m^4 + \mathrm{o}(\xi_m^4) \qquad (6.41) \\
&\geq \frac{\mathrm{vol}_g(M)}{32}(2_k^*-2)\alpha_{1,0}\,\xi_m^4 \left((2_k^*+1) - \frac{\alpha_{1,0}}{\alpha_{2,0} - \alpha_{1,0}}(2_k^*-2) + \mathrm{o}(1)\right).
\end{aligned}$$

Here we completed a square in b_m and simplified the occurring terms. Since from (6.38), we have

$$\xi_m^4 = \mathcal{E}(\rho_m)^2(\mathcal{E}(\phi)^{-2} + \mathrm{o}(1)) = \mathcal{E}(\rho_m)^2 \left(\left(\frac{n-2k}{2}\right)^2 \frac{4}{\mathrm{vol}_g(M)^2}\alpha_{1,0}^{-2} + o(1)\right),$$

the proof of the lemma is complete if we can show that $(2_k^*+1) - \frac{\alpha_{1,0}}{\alpha_{2,0}-\alpha_{1,0}}(2_k^*-2)$ is strictly positive, or equivalently,

$$\alpha_{2,0} > \frac{2\cdot 2_k^* - 1}{2_k^*+1}\alpha_{1,0}. \tag{6.42}$$

Now recall that $\alpha_{k,0} = \mathsf{P}_k(k\tau_0^{-1})$ for all $k \in \mathbb{N}_0$ and that $\alpha_{1,0} = (2_k^*-1)\alpha_{0,0}$ by the choice of τ_0. Using that $\mathsf{P}_k(X)$ is a convex function of X (being a sum of even-degree monomials with positive coefficients), we find

$$\alpha_{2,0} = \mathsf{P}_k(2\tau_0^{-1}) \geq 2\mathsf{P}_k(\tau_0^{-1}) - \mathsf{P}_k(0) = 2\alpha_{1,0} - \alpha_{0,0} = \left(2 - \frac{1}{2_k^*-1}\right)\alpha_{1,0}.$$

A direct computation shows $2 - \frac{1}{\nu-1} > \frac{2\nu-1}{\nu+1}$ for any $\nu > 2$. In particular taking $\nu = 2_k^* > 2$, we conclude that (6.42) is proved. □

Proof of Theorem 5. Let us prove that there is $c > 0$ such that

$$\frac{(\mathcal{E}(u) - \mathcal{Y}\|u\|_{2_k^*}^2)\mathcal{E}(u)}{\mathcal{E}(u-\overline{u})^2} \geq c \quad \text{for all non-constant } W^{k,2}(M). \tag{6.43}$$

We argue by contradiction and assume that there is a sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ such that

$$\frac{(\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2)\mathcal{E}(u_m)}{\mathcal{E}(u_m-\overline{u}_m)^2} \to 0 \quad \text{as} m \to \infty. \tag{6.44}$$

By zero homogeneity of the quotient in (6.44), we may assume without loss that $\|u_m\|_{2_k^*} = \mathrm{vol}_g(M)^{1/2_k^*}$. Since $\mathcal{E}(u_m - \overline{u}_m) \leq \mathcal{E}(u_m)$, from (6.44) it follows that

$$0 = \lim_{m\to\infty} \frac{(\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2)\mathcal{E}(u_m)}{\mathcal{E}(u_m)^2} = 1 - \lim_{m\to\infty}\frac{\mathcal{Y}\|u_m\|_{2_k^*}^2}{\mathcal{E}(u_m)}.$$

Thus, $\{u_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ satisfies the assumptions of Lemma 6.9. As a consequence, we get that $\lambda_m^{-1}u_m = 1 + \rho_m$ satisfies the assumptions of Lemmas 6.10 and 6.11 and we conclude

$$0 < c \leq \frac{(\mathcal{E}(1+\rho_m) - \mathcal{Y}\|1+\rho_m\|_{2_k^*}^2)}{\mathcal{E}(\rho_m)^2} = \lambda_m^2 \frac{(\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2)}{\mathcal{E}(u_m-\overline{u}_m)^2} \lesssim \frac{(\mathcal{E}(u_m) - \mathcal{Y}\|u_m\|_{2_k^*}^2)\mathcal{E}(u_m)}{\mathcal{E}(u_m-\overline{u}_m)^2}.$$

This is a contradiction to (6.44) and so the proof of (6.43) is complete.

From (6.43), it is straightforward to deduce the stability estimate (1.9). Indeed, by (6.43) we can estimate

$$\mathcal{Q}_{g,k}(u) - \mathcal{Y} = \frac{(\mathcal{E}(u) - \mathcal{Y}\|u\|_{2_k^*}^2)\mathcal{E}(u)}{\mathcal{E}(u-\overline{u})^2}\frac{\mathcal{E}(u-\overline{u})^2}{\mathcal{E}(u)\|u\|_{2_k^*}^2} \geq c\frac{\mathcal{E}(u-\overline{u})^2}{\mathcal{E}(u)\|u\|_{2_k^*}^2}.$$

Using that $\|u\|_{2_k^*}^2 \leq \mathcal{Y}^{-1}\mathcal{E}(u)$, and that $\mathcal{E}(u) \lesssim \|u\|_{k,2}^2$ and $\mathcal{E}(u-\overline{u}) \gtrsim \|u-\overline{u}\|_{k,2}^2$ by norm equivalence, we get

$$\mathcal{Q}_{g,k}(u) - \mathcal{Y} \geq \tilde{c}\frac{\|u-\overline{u}\|_{k,2}^4}{\|u\|_{k,2}^4} \geq \tilde{c}\frac{\inf_{c>0}\|u-c\|_{k,2}^4}{\|u\|_{k,2}^4} = d(u,\mathcal{M}_{g,k})^4.$$

We recall that by Lemma 6.7 the minimizing set $\mathcal{M}_{g,k}$ consists precisely of the constants and that $d(u,\mathcal{M}_{g,k})$ is defined in (1.8).

By considering specifically the sequence $\{u_m\}_{m\in\mathbb{N}} \subset W^{k,2}(M)$ given by

$$u_m(s) = 1 + m^{-1}\left[\cos\left(\frac{s}{\tau_0}\right) + b_m\cos\left(\frac{2s}{\tau_0}\right)\right],$$

with b_m chosen so that the square in (6.41) vanishes, the inequalities in the above computations become (asymptotic) equalities, and we find

$$\mathcal{Q}_{g,k}(u_m) - \mathcal{Y} \sim d(u_m, \mathcal{M}_{g,k})^4.$$

Since $d(u_m, \mathcal{M}_{g,k}) \to 0$ as $m \to \infty$, we therefore cannot have $\mathcal{Q}_{g,k}(u_m) - \mathcal{Y} \gtrsim d(u_m, \mathcal{M}_{g,k})^{2+\gamma}$ for any $\gamma < 2$. This proves the sharpness of $\gamma = 2$ in the statement of Theorem 5. □

Appendix A. Recursive formulas for GJMS operators

Formulas for the GJMS operator are only known in a few cases, for instance when the background manifold is Einstein [23], or more generally a special Einstein product [13, 24]. Nevertheless, in [18] one can find a recursion formula for these operators. We follow the recent construction [42, Proposition 2.1] and define this operator using the following recursion formula.

Let A_g be the Schouten tensor defined as

$$\mathrm{A}_g := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R_g}{2(n-1)} g\right)$$

and B_g be the Bach tensor whose coordinates are given by

$$\mathrm{B}_j := \mathrm{A}_{m\ell}\mathrm{W}_i^{m\ell} + \mathrm{P}^m_{ij;m} - \mathrm{P}^m_{im;j},$$

where $\mathrm{W}_{imj\ell}$, $\mathrm{A}_{m\ell}$ and $\mathrm{A}_{ij;m\ell}$ are the coordinates of $\mathrm{W}_g, \mathrm{A}_g$ and $\nabla^2_g \mathrm{A}_g$, respectively. We let $(\cdot,\cdot)$ be the multiple inner product induced by the metric g for the tensors of the type $\mathfrak{T}^{r,s}(M)$.

For every $k \in \mathbb{N}$ such that $n > 2k$, we set

$$\begin{aligned} P_{g,k} =& (-\Delta_g)^k + k(-\Delta_g)^{k-1}\left(J_{g,1}\cdot\right) + k(k-1)(-\Delta_g)^{k-2}\left(J_{g,2}\cdot + (\mathrm{T}_{g,1}, \nabla) + \left(\mathrm{T}_{g,2}, \nabla_g^2\right)\right) \\ &+ k(k-1)(k-2)(-\Delta_g)^{k-3}\left(\left(\mathrm{T}_{g,3}, \nabla^2\right) + \left(\mathrm{T}_{g,4}, \nabla_g^3\right)\right) \\ &+ k(k-1)(k-2)(k-3)(-\Delta_g)k-4\left(\mathrm{T}_{g,5}, \nabla_g^4\right) + Z, \end{aligned}$$

where Z is a smooth linear operator of order less than $2k-4$ if $k \geq 3$ and $Z := 0$ if $k \leq 2$, the functions $J_{g,1}, J_{g,2} \in \mathcal{C}^\infty(M)$ are defined as

$$J_{g,1} := \frac{n-2}{4(n-1)} R_g$$

and

$$J_{g,2} := \frac{1}{6}\left(\frac{3n^2 - 12n - 4k + 8}{16(n-1)^2} R_g^2 - (k+1)(n-4)|A_g|^2 - \frac{3n+2k-4}{4(n-1)}(-\Delta_g) R_g\right)$$

with the tensors $\mathrm{T}_{g,1}, \mathrm{T}_{g,2}, \mathrm{T}_{g,3}, \mathrm{T}_{g,4}$ and $\mathrm{T}_{g,5}$ being defined as

$$\begin{aligned} \mathrm{T}_{g,1} :=& \frac{n-2}{4(n-1)}\nabla R_g - \frac{2}{3}(k+1)\delta_g \mathrm{A}_g, \\ \mathrm{T}_{g,2} :=& \frac{2}{3}(k+1)\mathrm{A}_g, \\ \mathrm{T}_{g,3} :=& \frac{n-2}{6(n-1)}\nabla_g^2 R_g + \frac{(k+1)(n-2)}{6(n-1)} R_g \mathrm{A}_g - \frac{k+1}{3}(\delta\nabla_g \mathrm{A}_g + 2\nabla_g\delta_g \mathrm{A}_g + 2\mathrm{Rm}_g \cdot \mathrm{A}_g) \\ &- \frac{2}{15}(k+1)(k+2)\left(3\mathrm{A}_g^{\#}\mathrm{A}_g + \frac{\mathrm{B}_g}{n-4}\right), \\ \mathrm{T}_{g,4} :=& \frac{2}{3}(k+1)\nabla_g \mathrm{A}_g, \end{aligned}$$

and

$$\mathrm{T}_{g,5} := \frac{2}{5}(k+1)\left(\frac{5k+7}{9}\mathrm{A}_g \otimes \mathrm{A}_g + \nabla_g^2 \mathrm{A}_g\right).$$

Here # stands for the musical isomorphism with respect to g and $\delta_g \nabla_g \mathrm{A}_g$, $\nabla_g \delta_g \mathrm{A}_g$ and $\mathrm{Rm}_g \cdot \mathrm{A}_g$ stand for the covariant tensors whose coordinates are given by

$$(\delta_g \nabla_g \mathrm{A})_{ij} := -\mathrm{A}^m_{ij;m}, \quad (\nabla_g \delta_g \mathrm{A}_g)_{ij} := -\mathrm{A}^m_{i;mj} \quad \text{and} \quad (\mathrm{Rm}_g \cdot \mathrm{A}_g)_{ij} := \mathrm{A}^m_{im\ell}\mathrm{A}^\ell_j + \mathrm{Rm}_{i\ell jm}\mathrm{A}^{m\ell},$$

where $\mathrm{Rm}_{i\ell jm}$ are the coordinates of the Riemann curvature tensor.

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(J.H. Andrade) Department of Mathematics, University of São Paulo
05508-090, São Paulo-SP, Brazil
Email address: andradejh@ime.usp.br

(T. König) Institut für Mathematik, Goethe-Universität Frankfurt,
60325 Frankfurt am Main, Germany
Email address: koenig@mathematik.uni-frankfurt.de

(J. Ratzkin) Department of Mathematics, Universität Würzburg
97070, Würzburg-BA, Germany
Email address: jesse.ratzkin@mathematik.uni-wuerzburg.de

(J. Wei) Department of Mathematics, University of British Columbia
V6T 1Z2, Vancouver-BC, Canada
and
Department of Mathematics, The Chinese University of Hong Kong
Shatin-NT, Hong Kong
Email address: jcwei@math.ubc.ca
Email address: wei@math.cuhk.edu.hk